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NOTE ON A THEOREM IN DETERMINANTS.

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It is well known that if 12, &c., denote the determinants formed with the matrix

$$\left\|\begin{matrix} \alpha, & \beta, & \gamma, & \delta \\ \alpha', & \beta', & \gamma', & \delta' \end{matrix}\right\|,$$

then, identically,

$$12 \,.\, 34 + 13 \,.\, 42 + 14 \,.\, 23 = 0.$$

The proper proof of the theorem is obtained by remarking that we have

$$0 = \left|\begin{matrix} \alpha, & \beta, & \gamma, & . \\ \alpha', & \beta', & \gamma', & . \\ \alpha, & \beta, & \gamma, & \delta \\ \alpha', & \beta', & \gamma', & \delta' \end{matrix}\right|,$$

as at once appears by subtracting the first and second lines from the third and fourth lines respectively; and, this being so, the development of the determinant gives the theorem. The theorem might, it is clear, have been obtained in four different forms according as in the determinant the missing terms were taken to be as above (δ, δ'), or to be (α, α'), (β, β'), or (γ, γ'); but the four results are equivalent to each other.

There is obviously a like theorem for the sums of products of determinants formed with the matrix

$$\left\|\begin{matrix} \alpha, & \beta, & \gamma, & \delta, & \epsilon, & \zeta \\ \alpha', & \beta', & \gamma', & \delta', & \epsilon', & \zeta' \\ \alpha'', & \beta'', & \gamma'', & \delta'', & \epsilon'', & \zeta'' \end{matrix}\right\|,$$

viz. the theorem is obtained by development of the determinant in an identical equation, such as

$$0 = \left| \begin{array}{cccccc} \alpha\,, & \beta\,, & \gamma\,, & \delta\,, & .\,, & . \\ \alpha', & \beta', & \gamma', & \delta', & .\,, & . \\ \alpha'', & \beta'', & \gamma'', & \delta'', & .\,, & . \\ \alpha\,, & \beta\,, & \gamma\,, & \delta\,, & \epsilon\,, & \zeta \\ \alpha', & \beta', & \gamma', & \delta', & \epsilon', & \zeta' \\ \alpha'', & \beta'', & \gamma'', & \delta'', & \epsilon'', & \zeta'' \end{array} \right| ;$$

but we thus obtain 15 results which are not all equivalent.

If, for shortness, we write

$$\begin{aligned} A &= 123\,.\,456, \\ -B &= 124\,.\,356, \\ -C &= 125\,.\,346, \\ D &= 126\,.\,345, \\ -E &= 134\,.\,256, \\ -F &= 135\,.\,246, \\ G &= 136\,.\,245, \\ -H &= 145\,.\,236, \\ I &= 146\,.\,235, \\ J &= 156\,.\,234, \end{aligned}$$

then the fifteen results are

$$\begin{aligned} A + B - C - D &= 0, \\ A + B - E - J &= 0, \\ A - C + F - I &= 0, \\ A - D + G - H &= 0, \\ A - E + F + G &= 0, \\ A - H - I - J &= 0, \\ B - C - G + H &= 0, \\ B - D - F + I &= 0, \\ B - E + H + I &= 0, \\ B - F - G - J &= 0, \\ C + D - E - J &= 0, \\ C - E + G + I &= 0, \\ C - F - H - J &= 0, \\ D - E + F + H &= 0, \\ D - G - I - J &= 0, \end{aligned}$$

which are all satisfied if only

$$\begin{aligned} A &= \;.\quad\;\; .\; + H + I + J, \\ B &= F + G \quad .\quad\; . + J, \\ C &= F \quad\; . + H \quad . + J, \\ D &= \;.\quad G \quad\; . + I + J, \\ E &= F + G + H + I + J; \end{aligned}$$

and we thus have these five relations between the ten products of determinants A, B, C, D, E, F, G, H, I, J.