680.

ON THE HESSIAN OF A QUARTIC SURFACE.

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THE surface considered is

$$U = k^2w^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right) - (x^2+y^2+z^2)^2 = 0,$$

or say

$$U = k^2w^2P - Q^2 = 0,$$

viz. this may be considered as the central inverse of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} - 1 = 0.$$

The values of the second derived functions, or terms of the Hessian determinant

$$\begin{vmatrix} a, & h, & g, & l \\ h, & b, & f, & m \\ g, & f, & c, & n \\ l, & m, & n, & d \end{vmatrix},$$

are

$$\begin{vmatrix} \frac{k^2}{a^2}w^2 - 2Q - 4x^2, & -4xy, & -4xz, & \frac{2k^2}{a^2}wx \\ -4xy, & \frac{k^2}{b^2}w^2 - 2Q - 4y^2, & -4yz, & \frac{2k^2}{b^2}wy \\ -4xz, & -4yz, & \frac{k^2}{c^2}w^2 - 2Q - 4z^2, & \frac{2k^2}{c^2}wz \\ \frac{2k^2}{a^2}wx, & \frac{2k^2}{b^2}wy, & \frac{2k^2}{c^2}wz, & k^2P \end{vmatrix}$$

and we thence have

$$bc - f^2 = \frac{k^4}{b^2c^2} - \frac{k^2w^2}{b^2}(2Q + 4z^2) - \frac{k^2w^2}{c^2}(2Q + 4y^2) + 4Q^2 + 8Q(y^2 + z^2),$$

$$gh - af = 4yz\left(\frac{k^2w^2}{a^2} - 2Q\right),$$

whence, forming the analogous quantities $ca - g^2$, &c., it is easy to obtain

$$\begin{aligned} abc - af^2 - bg^2 - ch^2 + 2fgh \\ = \;& \frac{k^6w^6}{a^2b^2c^2} \\ & - k^4w^4\left\{2Q\left(\frac{1}{b^2c^2} + \frac{1}{c^2a^2} + \frac{1}{a^2b^2}\right) + 4\left(\frac{x^2}{b^2c^2} + \frac{y^2}{c^2a^2} + \frac{z^2}{a^2b^2}\right)\right\} \\ & + k^2w^2\left\{12Q^2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) - 8QP\right\} \\ & - 24Q^3, \end{aligned}$$

which is to be multiplied by $d, = k^2P$. And

$$\begin{aligned} & -[l^2(bc - f^2) + m^2(ca - g^2) + n^2(ab - h^2) \\ & + 2mn(gh - af) + 2nl(hf - bg) + 2lm(fg - ch)] \\ & = -\frac{4k^8w^6P}{a^2b^2c^2} \\ & \quad - 4k^6w^4\left[2Q\left\{\frac{x^2}{a^4}\left(\frac{1}{b^2} + \frac{1}{c^2}\right) + \frac{y^2}{b^4}\left(\frac{1}{c^2} + \frac{1}{a^2}\right) + \frac{z^2}{c^4}\left(\frac{1}{a^2} + \frac{1}{b^2}\right)\right\}\right. \\ & \qquad \left. + \frac{4y^2z^2}{a^2}\left(\frac{1}{b^2} - \frac{1}{c^2}\right)^2 + \frac{4z^2x^2}{b^2}\left(\frac{1}{c^2} - \frac{1}{a^2}\right)^2 + \frac{4x^2y^2}{c^2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right)^2\right] \\ & \quad + 4k^4w^2\left[4Q^2\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)\right. \\ & \qquad \left. + 8Q\left\{y^2z^2\left(\frac{1}{b^2} - \frac{1}{c^2}\right)^2 + z^2x^2\left(\frac{1}{c^2} - \frac{1}{a^2}\right)^2 + x^2y^2\left(\frac{1}{a^2} - \frac{1}{b^2}\right)^2\right\}\right], \end{aligned}$$

which is

$$\begin{aligned} & = -\frac{4k^8w^6P}{a^2b^2c^2} \\ & \quad + k^6w^4\left\{8\left(\frac{1}{b^2c^2} + \frac{1}{c^2a^2} + \frac{1}{a^2b^2}\right)PQ - \frac{24}{a^2b^2c^2}Q^2 + 16\left(\frac{x^2}{b^2c^2} + \frac{y^2}{c^2a^2} + \frac{z^2}{a^2b^2}\right)P\right\} \\ & \quad + k^4w^2\left\{-48\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)Q^2 + 32P^2Q\right\}. \end{aligned}$$

Hence, uniting the two parts, we have

$$H = k^8w^6\left(-\frac{3}{a^2b^2c^2}P\right)$$

$$+k^6w^4\left\{\begin{array}{l} 6\left(\frac{1}{b^2c^2}+\frac{1}{c^2a^2}+\frac{1}{a^2b^2}\right)PQ \\ -\frac{24}{a^2b^2c^2}Q^2 \\ +12\left(\frac{x^2}{b^2c^2}+\frac{y^2}{c^2a^2}+\frac{z^2}{a^2b^2}\right)P \end{array}\right\}$$

$$+k^4w^2\left\{\begin{array}{l} 12\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)PQ^2 \\ -48\left(\frac{x^2}{b^2c^2}+\frac{y^2}{c^2a^2}+\frac{z^2}{a^2b^2}\right)Q^2 \\ +24P^2Q \end{array}\right\}$$

$$+k^2\quad\{-24PQ^3\}.$$

Writing herein $Q^2 = k^2w^2P - U$, and transposing all the terms which contain U, we have

$$H + k^2U\left\{-\frac{24k^4w^4}{a^2b^2c^2}+12k^2w^2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)P-48k^2w^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)-24PQ\right\}$$

$$= k^6w^4P\left\{\begin{array}{l} -\frac{27k^2}{a^2b^2c^2}w^2 \\ +16\left(\frac{1}{b^2c^2}+\frac{1}{c^2a^2}+\frac{1}{a^2b^2}\right)Q \\ +12\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)P \\ +12\left(\frac{x^2}{b^2c^2}+\frac{y^2}{c^2a^2}+\frac{z^2}{a^2b^2}\right) \\ -48\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right) \end{array}\right\}$$

where, in the term in { }, the last four lines are

$$= 18\left(\frac{1}{b^2c^2}+\frac{1}{c^2a^2}+\frac{1}{a^2b^2}\right)Q$$

$$-36\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right).$$

Hence, writing for shortness

$$\Theta = -\frac{2k^4}{a^2b^2c^2}w^4 + k^2w^2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)P - 4k^2w^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right) - 2PQ,$$

we have

$$H + 12k^2\Theta U = 9k^6w^4P\left\{-\frac{3k^2}{a^2b^2c^2}w^2 + 2\left(\frac{1}{b^2c^2}+\frac{1}{c^2a^2}+\frac{1}{a^2b^2}\right)Q - 4\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right)\right\}.$$

Hence, recollecting that $U = k^2w^2P - Q^2$, the Hessian curve of the order 32 breaks up into

$U = 0$, $w^4 = 0$, that is, $Q^2 = 0$, $w^4 = 0$, or the nodal conic,

$w = 0$, $Q = 0$, 8 times (order 16),

$U = 0$, $P = 0$, that is, $Q^2 = 0$, $P = 0$, or the quadriquadric,

$P = 0$, $Q = 0$, 2 times (order 8),

and into a curve (order 8) which is

$$k^2w^2P - Q^2 = 0,$$

$$-\frac{3k^2}{a^2b^2c^2}w^2 + 2\left(\frac{1}{b^2c^2}+\frac{1}{c^2a^2}+\frac{1}{a^2b^2}\right)Q - 4\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}\right) = 0,$$

viz. this, the intersection of the surface with a quadric surface, is the proper Hessian curve.