681.

ON THE DERIVATIVES OF THREE BINARY QUANTICS.

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For a reason which will appear, instead of the ordinary factorial notation, I write $\{\alpha 012\}$ to denote the factorial $\alpha\,.\,\alpha+1\,.\,\alpha+2$, and so in other cases; and I consider the series of equations

$$\begin{aligned}
&(1) = X,\\
&(2) = (\{\alpha 0\},\ \{\beta 0\})\!(Y,\ -Y'),\\
&(3) = (\{\alpha 01\},\ 2\{\alpha 1\}\{\beta 1\},\ \{\beta 01\})\!(Z,\ -Z',\ Z''),\\
&(4) = (\{\alpha 012\},\ 3\{\alpha 12\}\{\beta 2\},\ 3\{\alpha 2\}\{\beta 12\},\ \{\beta 012\})\!(W,\ -W',\ -W'',\ -W'''),\\
&\&c.
\end{aligned}$$

where

$$\begin{aligned}
&X = Y + Y',\\
&Y = Z\ + Z'\ ,\ Y' = Z'\ + Z'',\\
&Z = W + W',\ Z' = W' + W'',\ Z'' = W'' + W''',\\
&\&c.
\end{aligned}$$

We have thus a series of linear equations serving to determine X; Y, Y'; Z, Z', Z''; W, W', W'', W'''; &c. We require in particular the values of X; Y, Y'; Z, Z''; W, W'''; &c., and I write down the results as follow:

$$X\ =\ (1),$$

$$\begin{array}{rl}
 & \quad (1)\quad\ (2) \\
\{\alpha+\beta 0\}\,Y & = \overline{\{\beta 0\},\ +1,} \\
\{\quad ,, \quad\}\,Y' & = \{\alpha 0\},\ -1;
\end{array}$$

$$\begin{array}{rclllll} & & \{\alpha+\beta 2\}(1), & \{\alpha+\beta 1\}(2), & \{\alpha+\beta 0\}(3), & & \\ \hline \{\alpha+\beta 012\}Z & = & \{\beta 01\}\ , & +2\{\beta 1\}\ , & +1\ , & & \\ \{\ \ ,,\ \ \}Z'' & = & \{\alpha 01\}\ , & -2\{\alpha 1\}\ , & +1\ ; & & \\ & & \{\alpha+\beta 34\}(1), & \{\alpha+\beta 14\}(2), & \{\alpha+\beta 03\}(3), & \{\alpha+\beta 01\}(4); & \\ \hline \{\alpha+\beta 01\ldots 4\}W & = & \{\beta 012\}\ , & +3\{\beta 12\}\ , & +3\{\beta 2\}\ , & +1\ , & \\ \{\ \ ,,\ \ \}W''' & = & \{\alpha 012\}\ , & -3\{\alpha 12\}\ , & +3\{\alpha 2\}\ , & -1\ ; & \\ & & \{\alpha+\beta 456\}(1), & \{\alpha+\beta 156\}(2), & \{\alpha+\beta 036\}(3), & \{\alpha+\beta 015\}(4), & \{\alpha+\beta 012\}(5); \\ \hline \{\alpha+\beta 01\ldots 6\}U & = & \{\beta 0123\}\ , & +4\{\beta 123\}\ , & +6\{\beta 23\}\ , & +4\{\beta 3\}\ , & +1\ , \\ \{\ \ ,,\ \ \}U'''' & = & \{\alpha 0123\}\ , & -4\{\alpha 123\}\ , & +6\{\alpha 23\}\ , & -4\{\alpha 3\}\ , & +1\ ; \end{array}$$

&c.

read

$$\alpha+\beta\,.\,Y=\beta\,(1)+(2),$$

$$,,\ \ .\,Y'=\alpha\,(1)-(2),$$

$$\alpha+\beta.\alpha+\beta+1.\alpha+\beta+2.Z\ =\beta.\beta+1.\alpha+\beta+2.(1)+2.\beta+1.\alpha+\beta+1.(2)+\alpha+\beta.(3),$$

$$,,\qquad ,,\qquad ,,\qquad .Z''=\alpha.\alpha+1.\alpha+\beta+2.(1)+2.\alpha+1.\alpha+\beta+1.(2)+\alpha+\beta.(3),$$

&c.,

the law being obvious, except as regards the numbers which in the top lines occur in connexion with $\alpha+\beta$ in the $\{\ \ \}$ symbols. As regards these, we form them by successive subtractions as shown by the diagrams

$$\begin{array}{r|l} \overline{34} & 34 \\ 2 & 14 \\ 11 & 03 \\ 2 & 01 \end{array}\qquad \begin{array}{r|l} \overline{456} & 456 \\ 3 & 156 \\ 12 & 036 \\ 21 & 015 \\ 3 & 012 \end{array}\qquad \begin{array}{r|l} \overline{5678} & 5678\ \ \&c.; \\ 4 & 1678 \\ 13 & 0378 \\ 22 & 0158 \\ 31 & 0127 \\ 4 & 0123 \end{array}$$

and the statement of the result is now complete.

In part verification, starting from the Y-formulæ (which are obtained at once), assume

$$\begin{array}{rclll} & & \{\alpha+\beta 2\}(1), & \{\alpha+\beta 1\}(2), & \{\alpha+\beta 0\}(3), \\ \hline \{\alpha+\beta 012\}\,Z & = & \lambda\ , & \mu\ , & \nu \\ \{\ \ ,,\ \ \}\,Z' & = & \lambda'\ , & \mu'\ , & \nu' \\ \{\ \ ,,\ \ \}\,Z'' & = & \lambda''\ , & \mu''\ , & \nu'' \end{array}$$

we must have

$$\{\alpha+\beta 012\}\,.\,Z\ +Z'=\{\alpha+\beta 012\}\ Y\,,=\{\alpha+\beta 12\}\ \overset{\ \ (1)\qquad (2)}{\overline{(\{\beta 0\},\ +1)}}$$

$$\{\ \ ,,\ \ \}\,.\,Z'+Z''=\{\ \ ,,\ \ \}\ Y',=\{\ \ ,,\ \ \}\ (\{\alpha 0\}\,,\ -1)$$

that is,

$$\{\alpha+\beta 2\}\,.\,\lambda\ +\lambda'\ =\{\alpha+\beta 12\}\ \{\beta 0\},$$

$$\{\ \ ,,\ \ \}\,.\,\lambda'+\lambda''=\{\ \ ,,\ \ \}\ \{\alpha 0\},$$

and further

$$\{\alpha+\beta 2\}\,(\{\alpha 01\},\ -2\,\{\alpha 1\}\,\{\beta 1\},\ \{\beta 01\}\between \lambda,\ \lambda',\ \lambda'')=0,$$

or, what is the same thing,

$$\lambda+\lambda'=\{\alpha+\beta 1\}\,\{\beta 0\},$$
$$\lambda'+\lambda''=\{\quad,,\quad\}\,\{\alpha 0\},$$
$$(\{\alpha 01\},\ -2\,\{\alpha 1\}\,\{\beta 1\},\ \{\beta 01\}\between \lambda,\ \lambda',\ \lambda'')=0.$$

And in like manner we have

$$\mu+\mu'=\{\alpha+\beta 2\}\,.\quad 1,$$
$$\mu'+\mu''=\{\quad,,\quad\}\,.-1,$$
$$(\{\alpha 01\},\ -2\,\{\alpha 1\}\,\{\beta 1\},\ \{\beta 01\}\between \mu,\ \mu',\ \mu'')=0;$$

and

$$\nu+\nu'=0,$$
$$\nu'+\nu''=0,$$
$$(\{\alpha 01\},\ -2\,\{\alpha 1\}\,\{\beta 1\},\ \{\beta 01\}\between \nu,\ \nu',\ \nu'')=0.$$

We hence find without difficulty

$$\lambda,\ \mu,\ \nu=\beta\,.\,\beta+1,\quad 2\,.\,\beta+1,\ +1,=\ \{\beta 01\},\ 2\,\{\beta 1\},\ +1,$$
$$\lambda',\ \mu',\ \nu'=\ \alpha\,.\,\beta\ ,\quad \alpha-\beta\ ,\ -1,=\{\alpha 0\}\,\{\beta 0\},\ \alpha-\beta\ ,\ -1,$$
$$\lambda'',\ \mu'',\ \nu''=\alpha\,.\,\alpha+1\ ,\ -2\,.\,\alpha+1,\ +1,=\ \{\alpha 01\},\ 2\,\{\alpha 1\},\ +1;$$

viz. for verification of the λ-equations we have

$$\beta\,.\,\beta+1\,.+\ \alpha\,.\,\beta\ ,\ \text{that is, }\alpha+\beta+1\,.\,\beta,\ =\{\alpha+\beta 1\}\,\{\beta 0\},$$
$$\alpha\,.\,\beta\,.\ +\alpha\,.\,\alpha+1,\quad ,,\quad \alpha+1+\beta\,.\,\alpha,\ =\{\quad,,\quad\}\,\{\alpha 0\},$$

and

$$(\alpha\,.\,\alpha+1,\ -2\,.\,\alpha+1\,.\,\beta+1,\ \beta\,.\,\beta+1\between \beta\,.\,\beta+1,\ \alpha\,.\,\beta,\ \alpha\,.\,\alpha+1)=0,$$

that is,

$$\alpha\,.\,\alpha+1\,.\,\beta\,.\,\beta+1\,.-2\,.\,\alpha+1\,.\,\beta+1\,.\,\alpha\,.\,\beta\,.+\beta\,.\,\beta+1\,.\,\alpha\,.\,\alpha+1=0;$$

and similarly the μ- and ν-equations may be verified.

We have thus for the Z's the equations

	$\{\alpha+\beta 2\}$ (1),	$\{\alpha+\beta 1\}$ (2),	$\{\alpha+\beta 0\}$ (3),
$\{\alpha+\beta 012\}\,Z\ =$	$\{\beta 01\}$,	$2\,\{\beta 1\}$,	$+1$,
$\{\quad,,\quad\}\,Z'\ =$	$\{\alpha 0\}\,\{\beta 0\}$,	$\alpha-\beta$,	-1 ,
$\{\quad,,\quad\}\,Z''=$	$\{\alpha 01\}$,	$-2\,\{\alpha 1\}$,	$+1$,

which include the foregoing expressions for Z and Z''.

We may then take the expressions for the W's to be

	$\{\alpha+\beta 34\}$ (1),	$\{\alpha+\beta 14\}$ (2),	$\{\alpha+\beta 03\}$ (3),	$\{\alpha+\beta 01\}$ (4),
$\{\alpha+\beta 0123\}\,W\ =$	λ ,	μ ,	ν ,	ρ ,
$\{\quad,,\quad\}\,W'\ =$	λ' ,	μ' ,	ν' ,	ρ' ,
$\{\quad,,\quad\}\,W''\ =$	λ'' ,	μ'' ,	ν'' ,	ρ'' ,
$\{\quad,,\quad\}\,W'''=$	λ''' ,	μ''' ,	ν''' ,	ρ''' ;

and we obtain in like manner the equations

$$\begin{aligned}\lambda\ +\lambda'\ &= \{\alpha+\beta 234\}\{\beta 01\},\\ \lambda'+\lambda'' &= \{\quad ,, \quad\}\{\alpha 0\}\{\beta 0\},\\ \lambda''+\lambda''' &= \{\quad ,, \quad\}\{\alpha 01\},\end{aligned}$$

$$(\{\alpha 012\},\ -3\{\alpha 12\}\{\beta 2\},\ +3\{\alpha 2\}\{\beta 12\},\ -\{\beta 012\}\between\lambda,\ \lambda',\ \lambda'',\ \lambda''')=0\,;$$

$$\begin{aligned}\mu\ +\mu'\ &= \{\alpha+\beta 134\}\,.\ \ 2\{\beta 1\},\\ \mu'+\mu'' &= \{\quad ,, \quad\}\,.\ \ \alpha-\beta,\\ \mu''+\mu''' &= \{\quad ,, \quad\}\,.-2\{\alpha 1\},\end{aligned}$$

$$(\{\alpha 012\},\ -3\{\alpha 12\}\{\beta 2\},\ +3\{\alpha 2\}\{\beta 12\},\ -\{\beta 012\}\between\mu,\ \mu',\ \mu'',\ \mu''')=0\,;$$

$$\begin{aligned}\nu\ +\nu'\ &= \{\alpha+\beta 034\}\,.\ \ 1,\\ \nu'+\nu'' &= \{\quad ,, \quad\}\,.-1,\\ \nu''+\nu''' &= \{\quad ,, \quad\}\,.\ \ 1,\end{aligned}$$

$$(\{\alpha 012\},\ -3\{\alpha 12\}\{\beta 2\},\ +3\{\alpha 2\}\{\beta 12\},\ -\{\beta 012\}\between\nu,\ \nu',\ \nu'',\ \nu''')=0\,;$$

$$\begin{aligned}\rho\ +\rho'\ &= 0,\\ \rho'+\rho'' &= 0,\\ \rho''+\rho''' &= 0,\end{aligned}$$

$$(\{\alpha 012\},\ -3\{\alpha 12\}\{\beta 2\},\ +3\{\alpha 2\}\{\beta 12\},\ -\{\beta 012\}\between\rho,\ \rho',\ \rho'',\ \rho''')=\{\alpha+\beta 01234\}.$$

These give for the $\lambda\rho'''$ square the values

$$\begin{array}{llll}\{\beta 012\}, & 3\{\beta 12\}, & 3\{\beta 2\}, & +1,\\ \{\alpha 0\}\{\beta 01\}, & 2\alpha-\beta\,.\{\beta 1\}, & \alpha-2\beta-2, & -1,\\ \{\alpha 01\}\{\beta 0\}, & \alpha-2\beta\,.\{\alpha 1\}, & -2\alpha+\beta-2, & +1,\\ \{\alpha 012\}, & -3\{\alpha 12\}, & +3\{\alpha 2\}, & -1,\end{array}$$

and so on; the law however of the terms in the intermediate lines is not by any means obvious.

Consider now the binary quantics P, Q, R, of the forms $(*\between x,\ y)^p$, $(*\between x,\ y)^q$, $(*\between x,\ y)^r$; we have for any, for instance for the fourth, order, the derivates

$$P(Q,\ R)^4,\ \ (P,\ (Q,\ R)^3)^1,\ \ (P,\ (Q,\ R)^2)^2,\ \ (P,\ (Q,\ R)^1)^3,\ \ (P,\ QR)^4\,;$$

and it is required to express

$$Q(P,\ R)^4 \text{ and } R(P,\ Q)^4,$$

each of them as a linear function of these.

I recall that we have $(P,\ Q)^0 = PQ$, so that the first and the last terms of the series might have been written $(P,\ (Q,\ R)^4)^0$ and $(P,\ (Q,\ R)^0)^4$ respectively; and, further, that $(P,\ Q)^1$ denotes $d_xP\,.\,d_yQ - d_yP\,.\,d_xQ$; $(P,\ Q)^2$ denotes

$$d_x^2P\,.\,d_y^2Q - 2d_xd_yP\,.\,d_xd_yQ + d_y^2P\,.\,d_x^2Q\,;$$

and so on.

I write $(a,\ b,\ c,\ d,\ e)$ for the fourth derived functions of any quantic $U, = (*\between x,\ y)^m$; we have, in a notation which will be at once understood,

$$\begin{aligned}
U &= (a,\ b,\ c,\ d,\ e\between x,\ y)^4 \div [m]^4\ ,\\
(d_x,\ d_y)\ U &= (a,\ b,\ c,\ d),\ (b,\ c,\ d,\ e)\,(x,\ y)^3 \div [m-1]^3,\\
(d_x,\ d_y)^2\ U &= (a,\ b,\ c),\ (b,\ c,\ d),\ (c,\ d,\ e)\,(x,\ y)^2 \div [m-2]^2,\\
(d_x,\ d_y)^3\ U &= (a,\ b),\ (b,\ c),\ (c,\ d),\ (d,\ e)\,(x,\ y)^1 \div [m-3]^1,\\
(d_x,\ d_y)^4\ U &= (a,\ b,\ c,\ d,\ e)\,;
\end{aligned}$$

and then, taking

$$(a_1,\ b_1,\ c_1,\ d_1,\ e_1),\quad (a_2,\ b_2,\ c_2,\ d_2,\ e_2),\quad (a_3,\ b_3,\ c_3,\ d_3,\ e_3),$$

to belong to P, Q, R, respectively, we must, instead of m, write p, q, r for the three functions respectively.

If we attend only to the highest terms in x, we have

$$\begin{aligned}
U &= ax^4 \div [m]^4\ ,\\
(d_x,\ d_y)\ U &= (a,\ b)\,x^3 \div [m-1]^3,\\
(d_x,\ d_y)^2\ U &= (a,\ b,\ c)\,x^2 \div [m-2]^2,\\
(d_x,\ d_y)^3\ U &= (a,\ b,\ c,\ d)\,x \div [m-3]^1,\\
(d_x,\ d_y)^4\ U &= (a,\ b,\ c,\ d,\ e).
\end{aligned}$$

Consider now $P(Q,\ R)^4$, $(P,\ (Q,\ R)^3)^1$, &c.; in each case attending only to the term in a_1, and in this term to the highest term in x, we have

$$\begin{aligned}
&(1)\quad [p]^4\,P\,(Q,\ R)^4 = a_2e_3 - 4b_2d_3 + 6c_2c_3 - 4d_2b_3 + e_2a_3 \quad (X),\\
&(2)\quad [p-1]^3\,[q-3]^1\,[r-3]^1\,(P,\ (Q,\ R)^3)^1 = [q-3]^1\,.\,b_2d_3 - 3c_2c_3 + 3d_2b_3 - e_2a_3\ (-Y'),\\
&\qquad + [r-3]^1\,.\,a_2e_3 - 3b_2d_3 + 3c_2c_3 - d_2b_3\ (Y),\\
&(3)\quad [p-2]^2\,[q-2]^2\,[r-2]^2\,(P,\ (Q,\ R)^2)^2 = [q-2]^2\,.\,c_2c_3 - 2d_2b_3 + e_2a_3\ (Z''),\\
&\qquad + 2[q-2]^1\,[r-2]^1\,.\,b_2d_3 - 2c_2c_3 + d_2b_3\ (-Z'),\\
&\qquad + [r-2]^2\,.\,a_2e_3 - 2b_2d_3 + c_2c_3\ (Z),\\
&(4)\quad [p-3]^1\,[q-1]^3\,[r-1]^3\,(P,\ (Q,\ R)^1)^3 = [q-1]^3\,.\,d_2b_3 - e_2a_3 \quad (-W'''),\\
&\qquad + 3[q-1]^2\,[r-1]^1\,.\,c_2c_3 - d_2b_3 \quad (W''),\\
&\qquad + 3[q-1]^1\,[r-1]^2\,.\,b_2d_3 - c_2c_3 \quad (-W'),\\
&\qquad + [r-1]^3\,.\,a_2e_3 - b_2d_3 \quad (W),
\end{aligned}$$

$$(5)\qquad [p-4]^0\,[q]^4\,[r]^4\,(P,\ QR)^4 = \begin{array}{lr} \qquad [q]^4\ .\ e_2a_3 & (U''''), \\ +\,4\,[q]^3\,[r]^1\,.\ d_2b_3 & (-\,U'''), \\ +\,6\,[q]^2\,[r]^2\,.\ c_2c_3 & (U''), \\ +\,4\,[q]^1\,[r]^3\,.\ b_2d_3 & (-\,U'), \\ +\qquad [r]^4\,.\ a_2e_3 & (U). \end{array}$$

Thus, for the second of these equations,

$$(P,\ (Q,\ R)^3)^1 = d_xP\,.\,d_y\,(Q,\ R)^3 - \&c.\,;$$

the term in a_1 is $d_y\,(Q,\ R)^3, = (d_xQ,\ R)^3 + (Q,\ d_yR)^3$, the whole being divided by $[p-1]^3$; where attending only to the highest terms in x, the two terms are respectively

$$(b_2d_3 - 3c_2c_3 + 3d_2b_3 - e_2a_3) \div [r-3]^1,$$

and

$$(a_2e_3 - 3b_2d_3 + 3c_2c_3 - d_2b_3) \div [q-3]^1,$$

which are each divided by $[p-1]^3$ as above; whence, multiplying by

$$[p-1]^3\,[q-1]^2\,[r-1]^1,$$

we have the formula in question; and so for the other cases.

Writing now (1), (2), (3), (4), (5) for the left-hand sides of the five equations respectively; and

$$\begin{array}{r} X: \\ -\,Y',\ Y: \\ Z'',\quad Z',\ Z: \\ -\,W''',\ W'',\ -\,W',\ W: \\ U'''',\ -\,U''',\ \ U'',\ -\,U',\ U: \end{array}$$

for the literal parts on the right-hand sides of the same equations respectively; then we have

$$X = Y + Y',$$
$$Y = Z + Z',\quad Y' = Z' + Z'',$$
&c.,

and the equations become

$$\begin{array}{l} (1) = \qquad X \qquad\qquad\qquad\qquad , \\ (2) = [r-1]^1\,Y - 1 \qquad [q-3]^1\ \ Y' \qquad\qquad , \\ (3) = [r-2]^2\,Z\ - 2\,[r-2]^1\,[q-2]^1\ \ Z' + 1 \qquad [q-2]^2\ \ Z'' \qquad , \\ (4) = [r-1]^3\,W - 3\,[r-1]^2\,[q-1]^1\,W' + 3\,[r-1]^1\,[q-1]^2\,W'' - 1\,[q-1]^3\,W''' \quad , \\ (5) = \quad [r]^4\,U - 4\quad [r]^3\quad [q]^1\quad U' + 6\quad [r]^2\quad [q]^2\quad U'' - 4\,[r]^1\,[q]^3\ \ U''' + [q]^4\,U'''', \end{array}$$

which are, in fact, the equations considered at the beginning of the present paper, putting therein $\alpha = r-3$ and $\beta = q-3$, they consequently give

$$\begin{array}{ll} & \{q+r-6,\,456\}(1),\ \{q+r-6,\,156\}(2),\ \{q+r-6,\,036\}(3),\ \{q+r-6,\,015\}(4),\ \{q+r-6, \\ \hline \{q+r-6,\,01\ldots6\}U\ = & \{q-3,\,0123\}\ \ ,\ +4\,\{q-3,\,123\}\ \ ,\ \ +6\,\{q-3,\,23\}\ \ ,\ \ +4\,\{q-3,\,3\}\ \ ,\qquad + \\ \{\qquad ,,\qquad \}U'''' = & \{r-3,\,0123\}\ \ ,\ -4\,\{r-3,\,123\}\ \ ,\ \ +6\,\{r-3,\,23\}\ \ ,\ \ -4\,\{r-3,\,3\}\ \ ,\qquad + \end{array}$$

Also, attending as before only to the terms in a, and therein to the highest power of x, we have

$$Q(R,\ P)^4 = a_2e_3 \div [q]^4,$$
$$R(P,\ Q)^4 = a_3e_2 \div [r]^4;$$

that is,

$$[q]^4\, Q(R,\ P)^4 = U, \quad [r]^4\, R(P,\ Q)^4 = U'''';$$

and, observing that $\{q+r-6,\ 01\ldots6\}$ is $=[q+r]^7$, and that $\{q+r-6,\ 456\}$, &c., may be written $\{q-r,\ \bar{2}\bar{1}0\}$, &c., where the superscript bars are the signs $-$, the formulæ become

$$\begin{array}{l|ccccc}
 & \{q+r,\ \bar{2}\bar{1}0\}(1), & \{q+r,\ \bar{5}\bar{1}0(2), & \{q+r,\ \bar{6}\bar{3}0\}(3), & \{q+r,\ \bar{6}\bar{5}\bar{1}\}(4), & \{q+r,\ \bar{6}\bar{5}\bar{4}\}(5), \\
\hline
[q+r]^7[q]^4Q(P,R)^4 = & [q]^4\ , & +4[q]^3\ , & +6[q]^2\ , & +4[q]^1\ , & +1\ , \\
[q+r]^7[r]^4R(P,Q)^4 = & [r]^4\ , & -4[r]^3\ , & +6[r]^2\ , & -4[r]^1\ , & +1\ .
\end{array}$$

Written at full length, the first of these equations (which, as being the fourth in a series, I mark 4th equation) is

$$\begin{aligned}
[q+r]^7[q]^4Q(P,R)^4 = {} & 1.q+r\ .q+r-1.q+r-2.\ [p]^4\ [q]^4 \qquad .P,(Q,R)^4 \quad \text{(4th equat.)} \\
& +4.q+r\ .q+r-1.q+r-5.[p-1]^3[q]^3[q-3]^1[r-1]^1.(P,(Q,R)^3)^1 \\
& +6.q+r\ .q+r-3.q+r-6.[p-2]^2[q]^2[q-2]^2[r-2]^2.(P,(Q,R)^2)^2 \\
& +4.q+r-1.q+r-5.q+r-6.[p-3]^1[q]^1[q-1]^3[r-1]^3.(P,(Q,R)^1)^3 \\
& +1.q+r-1.q+r-5.q+r-6.\qquad [q]^4\quad [r]^4\ .P,(Q,R)^4\ ,
\end{aligned}$$

and the other is, in fact, the same equation with q, Q, r, R interchanged with r, R, q, Q; the alternate $+$ and $-$ signs arise evidently from the terms

$$(R,\ Q)^4,\ = (Q,\ R)^4;\ (R,\ Q)^3,\ = -(Q,\ R)^3;\ \&c.,$$

which present themselves on the right-hand side.

It will be observed that the identity has been derived from the comparison of the terms in a, which are the highest terms in x, the other terms not having been written down or considered; but it is easy to see that an identity of the form in question exists, and, this being admitted, the process is a legitimate one.

The preceding equations of the series are

$$\begin{aligned}
[q+r]^1[q]^1Q(P,R)^1 = {} & 1.\ [p]^1\ [q]^1 \qquad P(Q,\ R)^1 \quad \text{(1st equation)} \\
& +1.\qquad [q]^1\quad [r]^1\quad (P,\ QR)^1;
\end{aligned}$$

$$\begin{aligned}
[q+r]^3[q]^2Q(P,R)^2 = {} & 1.q+r\ .\ [p]^2\ [q]^2 \qquad P,(Q,\ R)^2 \quad \text{(2nd equation)} \\
& +2.q+r-1.[p-1]^1[q]^1[q-1]^1[r-1]^1\quad (P,(Q,\ R)^1)^1 \\
& +1.q+r-2.\qquad [q]^2\quad [r]^2\quad (P,\ QR)^2;
\end{aligned}$$

$$\begin{aligned}
[q+r]^5[q]^3Q(P,R)^2 = {} & 1.q+r\ .q+r-1.\ [p]^3\ [q]^3 \qquad P,(Q,\ R)^3 \quad \text{(3rd equation)} \\
& +3.q+r\ .q+r-3.[p-1]^2[q]^2[q-2]^1[r-2]^1\,(P,(Q,\ R)^2)^1 \\
& +3.q+r-1.q+r-4.[p-2]^1[q]^1[q-1]^2[r-1]^2(P,\ QR)^3 \\
& +1.q+r-3.q+r-4.\qquad [q]^3\quad [r]^3\quad (P,\ QR)^3.
\end{aligned}$$

From these four equations the law is evident, except as to the numbers subtracted from $q+r$. These are obtained, as explained above, in regard to the numbers added to $\alpha+\beta$ in the { } symbols; transforming the diagrams so as to be directly applicable to the case now in question, they become

0	0	01	01	012	012	0123	0123
1	1	2	03	3	015	4	0127
1	2	11	14	21	036	31	0158
		2	34	12	156	22	0378
				3	456	13	1678
						4	5678,

showing how the numbers are obtained for the equations 2, 3, 4, 5 respectively. The first equation is

$$(q^2+qr)\,Q\,(P, R) = pq\,P\,(Q, R) + qr\,[Q\,(P, R) + R\,(P, Q)],$$

viz. this is

$$0 = pq\,P\,(Q, R) - qr\,Q\,(RP) + qr\,R\,(P, Q) + (q^2+qr)\,Q\,(R, P);$$

or, dividing by q, this is

$$0 = pP\,(Q, R) + qQ\,(R, P) + rR\,(P, Q),$$

which is a well-known identity.

We may verify any of the equations, though the process is rather laborious, for the particular values

$$P = x^{\frac{1}{2}(p+\alpha)}\,y^{\frac{1}{2}(p-\alpha)},\quad Q = x^{\frac{1}{2}(q+\beta)}\,y^{\frac{1}{2}(q-\beta)},\quad R = x^{\frac{1}{2}(r+\gamma)}\,y^{\frac{1}{2}(r-\gamma)};$$

thus, taking the second equation, we have, omitting common factors,

$$\begin{aligned}(Q, R)^2 = {} & q+\beta\,.\,q+\beta-2\,.\,r-\gamma\,.\,r-\gamma-2\\ & -2\quad .\,q+\beta\quad .\,q-\beta\,.\,r+\gamma\,.\,r-\gamma\\ & +\,.\,q-\beta\,.\,q-\beta-2\,.\,r+\gamma\,.\,r+\gamma-2\\ = {} & \beta^2(r^2-r)+\gamma^2(q^2-q)-2\beta\gamma\,(q-1)(r-1)-qr\,(q+r-2),\end{aligned}$$

$$\begin{aligned}(P, (Q, R)^1)^1 &= (q+\beta\,.\,r-\gamma\,.-.\,q-\beta\,.\,r+\gamma)(p+\alpha\,.\,q+r-\beta-\gamma-2\,.-.\,p-\alpha\,.\,q+r+\beta+\gamma-2)\\ &= (\beta r-q\gamma)(\alpha\,.\,q+r-2\,.-p\,.\,\beta+\gamma)\\ &= \alpha\beta r\,(r+q-2)-\alpha\gamma q\,(q+r-2)-pr\beta^2+p\,(q-r)\,\beta\gamma+pq\gamma^2,\end{aligned}$$

and from the first of these the expressions of $Q\,(P, R)^2$ and $(P, QR)^2$ are at once obtained. The identity to be verified then becomes

$$\begin{aligned}&[q+r]^3\,[q]^2\,\{\alpha^2(r^2-r)+\gamma^2(p^2-p)-2\alpha\gamma\,(p-1)(r-1)-pr\,(p+r-2)\}\\ &= (q+r)\,[q]^2\,[p]^2\,\{\beta^2(r^2-r)+\gamma^2(q^2-q)-2\beta\gamma\,(q-1)(r-1)-qr\,(q+r-2)\}\\ &\quad +2\,(q+r-1)\,[q]^2\,(p-1)(r-1)\,\{\alpha\beta r\,(q+r-2)-\alpha\gamma q\,(q+r-2)\\ &\qquad\qquad -pr\beta^2+p\,(q-r)\,\beta\gamma+pq\gamma^2\}\\ &\quad +(q+r-2)\,[q]^2\,[r]^2\,\{\alpha^2\,(q+r)(q+r-1)+(\beta+\gamma)^2\,(p^2-p)\\ &\qquad\qquad -2\alpha\,(\beta+\gamma)(p-1)(q+r-1)-p\,(q+r)(p+q+r-2)\},\end{aligned}$$

which is easily verified, term by term; for instance, the terms with α, β, or γ, give

$$[q+r]^3[q]^2 pr(p+r-2) = (q+r)[q]^2[p]^2 qr(q+r-2) + (q+r-2)[q]^2[r]^2 p(q+r)(p+q+r-2),$$

which, omitting the factor $(q+r)(q+r-2)[q]^2 pr$, is

$$(q+r-1)(p+r-2) = (p-1)q + (r-1)(p+q+r+2);$$

viz. the right-hand side is

$$(p-1)q + (r-1)q + (r-1)(p+r-2), \quad = (q+r-1)(p+r-2),$$

as it should be.

The equations are useful for the demonstration of a subsidiary theorem employed in Gordan's demonstration of the finite number of the covariants of any binary form U. Suppose that a system of covariants (including the quantic itself) is

$$P, Q, R, S, \ldots;$$

this may be the complete system of covariants; and if it is so, then, T and V being any functions of the form $P^\alpha Q^\beta R^\gamma\ldots$, every derivative $(T, V)^\theta$ must be a term or sum of terms of the like form $P^\alpha Q^\beta R^\gamma\ldots$; the subsidiary theorem is that in order to prove that the case is so, it is sufficient to prove that every derivative $(P, Q)^\theta$, where P and Q are any two terms of the proposed system, is a term or sum of terms of the form in question $P^\alpha Q^\beta R^\gamma\ldots$.

In fact, supposing it shown that every derivative $(T, V)^\theta$ up to a given value θ_0 of θ is of the form $P^\alpha Q^\beta R^\gamma\ldots$, we can by successive application of the equation for $Q(P, R)^{\theta+1}$, regarded as an equation for the reduction of the last term on the right-hand side $(P, QR)^{\theta+1}$, bring first $(P, QR)^{\theta+1}$, and then $(P, QRS)^{\theta+1}, \ldots$, and so ultimately any function $(P, V)^{\theta+1}$, and then again any functions $(PQ, V)^{\theta+1}$, $(PQR, V)^{\theta+1}, \ldots$, and so ultimately any function $(T, V)^{\theta+1}$, into the required form $P^\alpha Q^\beta R^\gamma\ldots$: or the theorem, being true for θ, will be true for $\theta+1$; whence it is true generally.