683.

ON THE FUNCTION arc sin $(x+iy)$.

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THE determination of the function in question, the arc to a given imaginary sine, is considered in Cauchy's *Exercises d'Analyse, &c.*, t. III. (1844), p. 382; but it appears, by two hydrodynamical papers by Mr Ferrers and Mr Lamb, *Quarterly Mathematical Journal*, t. XIII. (1874), p. 115, and t. XIV. (1875), p. 40, that the question is connected with the theory of confocal conics.

Taking $c=\sqrt{(a^2-b^2)}$ a positive real quantity which may ultimately be put $=1$, the question is to find the real quantities ξ, η, such that

$$\xi+i\eta=\text{arc sin}\,\frac{1}{c}(x+iy),$$

or say

$$x+iy=c\sin(\xi+i\eta),$$

so that

$$x=c\sin\xi\cos i\eta,\quad iy=c\cos\xi\sin i\eta.$$

It is convenient to remark that if a value of $\xi+i\eta$ be $\xi'+i\eta'$, then the general value is $2m\pi+\xi'+i\eta'$ or $(2m+1)\pi-(\xi'+i\eta')$; hence, η may be made positive or negative at pleasure; $\cos i\eta$ is in each case positive, but $\frac{1}{i}\sin i\eta$ has the same sign as η; hence $\cos\xi$ has the same sign as x, but $\sin\xi$ has the same sign as y or the reverse sign, according as η is positive or negative; for any given values of x and y, we obtain, as will appear, determinate positive values of $\sin^2\xi$ and $\cos^2\xi$; and the square roots of these must therefore be taken so as to give to $\sin\xi$, $\cos\xi$ their proper signs respectively.

Suppose that λ, μ are the elliptic coordinates of the point (x, y); viz. that we have

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1,$$

$$\frac{x^2}{a^2+\mu}+\frac{y^2}{b^2+\mu}=1,$$

where $a^2+\lambda$, $b^2+\lambda$, and $a^2+\mu$ are positive, but $b^2+\mu$ is negative. Calling ρ, σ the distances of the point x, y from the points $(c, 0)$ and $(-c, 0)$, that is, assuming

$$\rho=\surd\{(x-c)^2+y^2\},$$
$$\sigma=\surd\{(x+c)^2+y^2\},$$

then we have

$$\surd(a^2+\lambda)=\tfrac{1}{2}(\sigma+\rho), \text{ whence also } \surd(b^2+\lambda)=\tfrac{1}{2}\surd\{(\sigma+\rho)^2-4c^2\},$$
$$\surd(a^2+\mu)=\tfrac{1}{2}(\sigma\sim\rho), \quad\quad ,, \quad\quad \surd(b^2+\mu)=\tfrac{1}{2}\surd\{(\sigma\sim\rho)^2-4c^2\},$$

which equations determine λ, μ as functions of x, y.

Now we have

$$\rho\sigma=\surd\{(x^2+y^2-c^2)^2-4c^2x^2\}=\surd\{(x^2-y^2-c^2)^2+4x^2y^2\},$$
$$\rho^2+\sigma^2=2(x^2+y^2+c^2);$$

substituting herein for x, y their values

$$c\sin\xi\cos i\eta,\ -ci\cos\xi\sin i\eta,$$

we find

$$x^2-y^2-c^2=c^2\{\sin^2\xi\cos^2 i\eta+\cos^2\xi\sin^2 i\eta-(\sin^2\xi+\cos^2\xi)(\sin^2 i\eta+\cos^2 i\eta)\}$$
$$=-c^2(\sin^2\xi\sin^2 i\eta+\cos^2\xi\cos^2 i\eta),$$

whence

$$\begin{aligned}(x^2-y^2-c^2)^2 &= c^4(\cos^2\xi\cos^2 i\eta+\sin^2\xi\sin^2 i\eta)^2\\ +4x^2y^2 &\quad -4c^4\sin^2\xi\cos^2\xi\sin^2 i\eta\cos^2 i\eta\\ &= c^4(\cos^2\xi\cos^2 i\eta-\sin^2\xi\sin^2 i\eta)^2.\end{aligned}$$

Hence

$$2\rho\sigma=2c^2(\cos^2\xi\cos^2 i\eta-\sin^2\xi\sin^2 i\eta),$$

and

$$\rho^2+\sigma^2=2c^2(\sin^2\xi\cos^2 i\eta-\cos^2\xi\sin^2 i\eta+1);$$

hence

$$(\rho+\sigma)^2=2c^2(\cos^2 i\eta-\sin^2 i\eta+1),\ =4c^2\cos^2 i\eta,$$
$$(\rho-\sigma)^2=2c^2(\sin^2\xi-\cos^2\xi+1),\ =4c^2\sin^2\xi.$$

Consequently

$$a^2+\lambda=c^2\cos^2 i\eta, \text{ and thence } b^2+\lambda=-c^2\sin^2 i\eta,$$
$$a^2+\mu=c^2\sin^2\xi, \quad\quad ,, \quad\quad b^2+\mu=-c^2\cos^2\xi,$$

values which verify as they should do the equations

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1,$$

$$\frac{x^2}{a^2+\mu}+\frac{y^2}{b^2+\mu}=1,$$

viz. these become

$$\frac{x^2}{c^2\cos^2 i\eta}+\frac{y^2}{-c^2\sin^2 i\eta}=\sin^2\xi+\cos^2\xi,\ =1,$$

$$\frac{x^2}{c^2\sin^2\xi}+\frac{y^2}{-c^2\cos^2\xi}=\cos^2 i\eta+\sin^2 i\eta,\ =1.$$

The same equations, or as we may also write them,

$$\lambda=-a^2\sin^2 i\eta-b^2\cos^2 i\eta,$$

$$\mu=-a^2\cos^2\xi-b^2\sin^2\xi,$$

determine η as a function of λ, and ξ as a function of μ; λ, μ being by what precedes, given functions of x, y.

Or more simply, starting from the last-mentioned values of λ, μ, and substituting these in the expressions

$$x^2=\frac{a^2+\lambda\,.\,a^2+\mu}{a^2-b^2},\quad y^2=\frac{b^2+\lambda\,.\,b^2+\mu}{b^2-a^2},$$

we find

$$x^2=c^2\sin^2\xi\cos^2 i\eta,\quad y^2=-c^2\cos^2\xi\sin^2 i\eta,$$

or say

$$x=c\sin\xi\cos i\eta,\quad iy=c\cos\xi\sin i\eta,$$

whence

$$x+iy=c\sin(\xi+i\eta),$$

the original relation between x, y and ξ, η.