686.

ON A FUNCTIONAL EQUATION.

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I WAS led by a hydrodynamical problem to consider a certain functional equation; viz. writing for shortness $x_1 = \dfrac{ax+b}{cx+d}$, this is

$$\phi x - \phi x_1 = (x - x_1)\frac{Ax+B}{Cx+D}.$$

I find by a direct process, which I will afterwards explain, the solution

$$\phi x = \frac{A}{C}x + \frac{\sqrt{\{(a-d)^2+4bc\}}\,(AD-BC)}{C\,(dC-cD)}\int_0^\infty \frac{\sin\xi t\sin\eta t\,dt}{\sin\zeta t\sinh\pi t};$$

where ζ is a constant, but ξ, η are complicated logarithmic functions of x (ξ, η, ζ depend also on the quantities a, b, c, d, C, D); $\sinh \pi t$ denotes as usual the hyperbolic sine, $\tfrac{1}{2}(e^{\pi t} - e^{-\pi t})$.

The values of ξ, η, ζ are given by the formulæ

$$\lambda + \frac{1}{\lambda} = \frac{a^2+d^2+2bc}{ad-bc},$$

$$\mathrm{a} = ax+b,\quad \mathrm{b} = -\,dx+b,$$

$$\mathrm{c} = cx+d,\quad \mathrm{d} = \,cx-a,$$

$$W = C\mathrm{a} + D\mathrm{c},$$

$$Z = C\mathrm{b} + D\mathrm{d},$$

$$R = \lambda\mathrm{c} + \lambda\mathrm{d},$$

$$S = -\mathrm{c} - \mathrm{d},$$

$$R' = W + \frac{1}{\lambda}Z,$$

$$S' = -W - \lambda Z,$$

which determine λ, R, S, R', S' and then

$$\xi = \tfrac{1}{2}\log\frac{RS'}{R'S},\quad \eta = \tfrac{1}{2}\left(\log\lambda + \log\frac{RR'}{SS'}\right),\quad \zeta = \tfrac{1}{2}\log\lambda.$$

There is some difficulty as to the definite integral, on account of the denominator factor $\sin\zeta t$, which becomes $=0$ for the series of values $t = \dfrac{m\pi}{\zeta}$, but this is a point which I do not enter into.

I will in the first instance verify the result. Writing x_1 in place of x, and taking ξ_1, η_1 to denote the corresponding values of ξ, η, it will be shown that

$$\xi_1 = \xi,\quad \eta_1 = \eta + 2\zeta,$$ see *post*, (1).

Hence in the difference $\phi x - \phi x_1$ we have the integral

$$\int \frac{\sin\xi t\,\{\sin\eta t - \sin(\eta + 2\zeta)\,t\}\,dt}{\sin\zeta t\sinh\pi t},$$

(where and in all that follows the limits are ∞, 0 as before); here, since

$$\sin\eta t - \sin(\eta + 2\zeta)\,t = -2\sin\zeta t\cos(\eta+\zeta)\,t,$$

the factor $\sin\zeta t$ divides out, and the numerator is

$$= -2\sin\xi t\cos(\eta+\zeta)\,t,$$

which is

$$= \sin(\eta+\zeta-\xi)\,t - \sin(\eta+\zeta+\xi)\,t.$$

Hence the integral in question is

$$= \int\frac{\sin(\eta+\zeta-\xi)\,t\,dt}{\sinh\pi t} - \int\frac{\sin(\eta+\zeta+\xi)\,t\,dt}{\sinh\pi t}.$$

Now we have in general

$$\frac{1}{1+\exp.\alpha} = \tfrac{1}{2} - \int\frac{\sin\alpha t\,dt}{\sinh\pi t};$$

(this is, in fact, Poisson's formula

$$-\frac{1}{1+k\beta^{2n}} = \tfrac{1}{2} - 2\int\frac{\sin(2n\log\beta + \log k)\,t\,.\,dt}{e^{\pi t} - e^{-\pi t}},$$

in the second Memoir on the distribution of Electricity, &c., *Mém. de l'Inst.*, 1811, p. 223); and hence the value is

$$-\frac{1}{1+\exp.(\eta+\zeta-\xi)} + \frac{1}{1+\exp.(\eta+\zeta+\xi)},$$

or since

$$\eta+\zeta = \log\lambda + \tfrac{1}{2}\log\frac{RR'}{SS'},\quad \xi = \tfrac{1}{2}\log\frac{RS'}{R'S},$$

we have

$$\eta+\zeta-\xi = \log\lambda + \tfrac{1}{2}\log\frac{R'^2}{S'^2} = \log\lambda\frac{R'}{S'},$$

$$\eta+\zeta+\xi = \log\lambda + \tfrac{1}{2}\log\frac{R^2}{S^2} = \log\lambda\frac{R}{S},$$

and the value is thus

$$=-\frac{1}{1+\lambda\dfrac{R'}{S'}}+\frac{1}{1+\lambda\dfrac{R}{S}},\quad =-\frac{(RS'-R'S)\lambda}{(\lambda R'+S')(\lambda R+S)}.$$

Hence, from the assumed value of ϕx, we obtain

$$\phi x-\phi x_1=\frac{A}{C}(x-x_1)-\frac{\sqrt{\{(a-d)^2+4bc\}}\,(AD-BC)\,(RS'-R'S)\,\lambda}{C\,(dC-cD)\,(\lambda R'+S')\,(\lambda R+S)}.$$

We have

$$RS'-R'S=\frac{(\lambda-1)\,(a+d)^2}{ad-bc}\,(dC-cD)\,\{cx^2+(d-a)\,x-b\},$$

$$R\lambda+\quad S=(\lambda^2-1)\,(cx+d),\qquad\text{see } post,\ (2),$$

$$R'\lambda+\quad S'=(\lambda-1)\,(a+d)\,(Cx+D),$$

or since

$$\frac{cx^2+(d-a)\,x-b}{cx+d}=x-x_1,$$

this is

$$\phi x-\phi x_1=\frac{A}{C}(x-x_1)-\frac{\sqrt{\{(a-d)^2+4bc\}}\,(AD-BC)}{C}\,\frac{(a+d)\,\lambda}{(ad-bc)(\lambda^2-1)(Cx+D)}(x-x_1).$$

But from the value of λ,

$$\frac{\lambda}{\lambda^2-1}=\frac{ad-bc}{(a+d)\,\sqrt{\{(a-d)^2+4bc\}}},$$

and the equation thus is

$$\phi x-\phi x_1=(x-x_1)\left\{\frac{A}{C}-\frac{AD-BC}{C\,(Cx+D)}\right\},\quad =(x-x_1)\,\frac{Ax+B}{Cx+D},$$

as it should be.

(1) For the foregoing values of ξ_1, η_1, we require R_1, S_1, R_1', S_1', the values which R, S, R', S' assume on writing therein x_1 for x. We have

$$R_1=\quad\lambda\,(cx_1+d)+\quad(cx_1-a),$$

$$S_1=-\quad(cx_1+d)-\lambda\,(cx_1-a):$$

substituting for x_1 its value, we find

$$R_1\,(cx+d)=(a+d)\,\lambda\,(cx+d)-(ad-bc)\,(\lambda+1),$$

or writing herein

$$ad-bc=\frac{(a+d)^2\,\lambda}{(\lambda+1)^2},$$

this is

$$R_1\,(cx+d)=\frac{(a+d)\,\lambda}{\lambda+1}\,R;$$

and similarly

$$S_1\,(cx+d)=\frac{a+d}{\lambda+1}\,S.$$

We have in like manner

$$R_1' = \quad W_1 + \frac{1}{\lambda} Z_1, \text{ where } W_1 = C(\quad ax_1 + b) + D(cx_1 + d),$$

$$S_1' = - W_1 - \lambda Z_1, \text{ where } Z_1 = C(-dx_1 + b) + D(cx_1 - a).$$

Substituting for x_1 its value, we find

$$W_1(cx+d) = C[(a+d)(ax+b) - (ad-bc)x] + D[(a+d)(cx+d) - (ad-bc)],$$

$$Z_1(cx+d) = C[\qquad\qquad - (ad-bc)x] + D[\qquad\qquad - (ad-bc)]:$$

hence, substituting for $ad - bc$ as before,

$$W_1(cx+d) = \frac{a+d}{(\lambda+1)^2}\{(\lambda+1)^2 W - (a+d)\lambda(Cx+D)\},$$

$$Z_1(cx+d) = \frac{a+d}{(\lambda+1)^2}\{\qquad\qquad - (a+d)\lambda(Cx+D)\},$$

whence without difficulty

$$R_1'(cx+d) = \frac{(a+d)\lambda}{\lambda+1} R',$$

$$S_1'(cx+d) = \quad \frac{a+d}{\lambda+1} S':$$

consequently

$$\frac{R_1 S_1'}{R_1' S_1} = \quad \frac{RS'}{R'S}, \text{ that is, } \xi_1 = \xi,$$

$$\frac{R_1 R_1'}{S_1 S_1'} = \lambda^2 \frac{RR'}{SS'}, \quad \text{,,} \quad , \eta_1 = \log\lambda + \eta, = 2\zeta + \eta,$$

which are the formulæ in question.

(2) For the value of $RS' - R'S$, we have

$$RS' - R'S = (\lambda \mathrm{c} + \mathrm{d})(-W - \lambda Z) - (-\lambda \mathrm{d} - \mathrm{c})\left(W + \frac{Z}{\lambda}\right)$$

$$= \left(-\lambda^2 + \frac{1}{\lambda}\right)\mathrm{c}Z + (\lambda+1)\{(\mathrm{d}-\mathrm{c})W - \mathrm{d}R\}$$

$$= -(\lambda-1)\left\{\left(1 + \lambda + \frac{1}{\lambda}\right)\mathrm{c}Z + (\mathrm{c}-\mathrm{d})W + \mathrm{d}Z\right\};$$

or substituting for $\lambda + \frac{1}{\lambda}$, Z and W their values, this is

$$= \frac{-(\lambda-1)}{ad-bc}\{(a^2+d^2+ad+bc)\,\mathrm{c}\,(\mathrm{b}C + \mathrm{d}D)$$

$$+ (ad-bc)[(\mathrm{c}-\mathrm{d})(\mathrm{a}C + \mathrm{c}D) + \mathrm{d}(\mathrm{b}C + \mathrm{d}D)]\}.$$

In the term in { }, the coefficient of C is

$$[(a^2+d^2+ad+bc)\,\mathrm{b} + (ad-bc)\,\mathrm{a}]\,\mathrm{c} - \mathrm{d}(\mathrm{a}-\mathrm{b})(ad-bc)$$

$$= (a+d)(d\mathrm{b} - b\mathrm{d})\,\mathrm{c} - (a+d)\,\mathrm{d}x\,(ad-bc),$$

and similarly the coefficient of D is

$$[(a^2+d^2+ad+bc)\,\mathrm{d}+(ad-bc)\,\mathrm{c}]\,\mathrm{c}-\mathrm{d}\,(\mathrm{c}-\mathrm{d})\,(ad-bc)$$
$$=(a+d)\,(a\mathrm{d}-c\mathrm{b})\,\mathrm{c}-(a+d)\,\mathrm{d}\,(ad-bc).$$

Hence the whole term in { } is

$$=(a+d)\,\{[(d\mathrm{b}-b\mathrm{d})\,\mathrm{c}-\mathrm{d}\,(ad-bc)\,x]\,C+[(a\mathrm{d}-c\mathrm{b})\,\mathrm{c}-\mathrm{d}\,(ad-bc)]\,D\},$$

which is readily reduced to

$$(a+d)\,(\mathrm{ad}-\mathrm{bc})\,(-dC+cD);$$

also

$$\mathrm{ad}-\mathrm{bc}=(a+d)\,\{cx^2+(d-a)\,x-b\};$$

so that we have

$$RS'-R'S=\frac{(\lambda-1)\,(a+d)^2}{ad-bc}\,(dC-cD)\,[cx^2+(d-a)\,x-b],$$

which is the required value of $RS'-R'S$; and there is no difficulty in obtaining the other two formulæ,

$$R\lambda+S=(\lambda^2-1)\,(cx+d),$$
$$R'\lambda+S'=(\lambda-1)\,(a+d)\,(Cx+D);$$

the verification is thus completed.

To show how the formula was directly obtained, we have

$$\frac{Ax+B}{Cx+D}=\frac{A}{C}-\frac{AD-BC}{C}\,\frac{1}{Cx+D}$$
$$=\frac{A}{C}+\beta x \text{ suppose};$$

the equation then is

$$\phi x-\phi x_1=\frac{A}{C}\,(x-x_1)+(x-x_1)\,\beta x.$$

Hence, if x_1, x_2, x_3, ... denote the successive functions ϑx, $\vartheta^2 x$, $\vartheta^3 x$, &c., we have

$$\phi x_1-\phi x_2=\frac{A}{C}\,(x_1-x_2)+(x_1-x_2)\,\beta x_1,$$

$$\phi x_2-\phi x_3=\frac{A}{C}\,(x_2-x_3)+(x_2-x_3)\,\beta x_2,$$

whence adding, and neglecting ϕx_∞ and x_∞, we have

$$\phi x=\frac{A}{C}\,x+[(x-x_1)\,\beta x+(x_1-x_2)\,\beta x_1+(x_2-x_3)\,\beta x_2+\ldots],$$

where the term in [ ], regarding therein x_1, x_2, x_3, ... as given functions of x, is itself a given function of x; and it only remains to sum the series.

Starting from

$$x_1=\vartheta x=\frac{ax+b}{cx+d},$$

and writing

$$\lambda+\frac{1}{\lambda}=\frac{a^2+d^2+2bc}{ad-bc},$$

then the nth function is given by the formula

$$x_n = \vartheta_n x = \frac{(\lambda^{n+1}-1)(ax+b)+(\lambda^n-\lambda)(-dx+b)}{(\lambda^{n+1}-1)(cx+d)+(\lambda^n-\lambda)(cx-a)}$$

$$= \frac{(\lambda^{n+1}-1)\,\mathrm{a}+(\lambda^n-\lambda)\,\mathrm{b}}{(\lambda^{n+1}-1)\,\mathrm{c}+(\lambda^n-\lambda)\,\mathrm{d}}$$

$$= \frac{\lambda^n P+Q}{\lambda^n R+S},$$

if $P=\lambda\mathrm{a}+\mathrm{b}$, $Q=-\mathrm{a}-\lambda\mathrm{b}$, and as before $R=\lambda\mathrm{c}+\mathrm{d}$, $S=-\mathrm{c}-\lambda\mathrm{d}$.

I stop to remark that λ being real, then if $\lambda>1$ we have λ^n very large for n very large, and $x^n=\dfrac{P}{R}$ which is independent of n; the value in question is

$$x_n = \frac{\lambda(ax+b)+(-dx+b)}{\lambda(cx+d)+(\ \ cx-a)},$$

which, observing that the equation in λ may be written

$$\frac{\lambda a-d}{c(\lambda+1)} = \frac{b(\lambda+1)}{\lambda d-a},$$

is, in fact, independent of x, and is $=\dfrac{\lambda a-d}{c(\lambda+1)}$ or $\dfrac{b(\lambda+1)}{\lambda d-a}$; we have $x_{n-1}=x_n$, or calling each of these two equal values x, we have

$$x = \frac{ax+b}{cx+d},$$

which is the same equation as is obtainable by the elimination of λ from the equations

$$x = \frac{\lambda a-d}{c(\lambda+1)} = \frac{b(\lambda+1)}{\lambda d-a}.$$

The same result is obtained by taking $\lambda<1$ and consequently $x_n=\dfrac{Q}{S}$.

We find

$$x_{n-1}-x_n = \frac{\lambda^{n-1}P+Q}{\lambda^{n-1}R+S} - \frac{\lambda^n P+Q}{\lambda^n R+S},$$

$$= \frac{-\lambda^{n-1}(\lambda-1)(PS-QR)}{(\lambda^{n-1}R+S)(\lambda^n R+S)},$$

where

$$PS-QR = -(\lambda^2-1)(\mathrm{a}d-\mathrm{bc}) = -(\lambda^2-1)(a+d)\{cx^2+(d-a)x-b\};$$

and therefore

$$x_{n-1}-x_n = \frac{(\lambda-1)(\lambda^2-1)(a+d)\{cx^2+(d-a)x-b\}\lambda^n}{\lambda(\lambda^{n-1}R+S)(\lambda^n R+S)}.$$

Also

$$\beta x_{n-1} = -\frac{AD-BC}{C}\,\frac{1}{Cx_{n-1}+D},$$

where

$$Cx_{n-1}+D=\frac{C(\lambda^{n-1}P+Q)+D(\lambda^{n-1}R+S)}{\lambda^{n-1}R+S}=\frac{R'\lambda^n+S'}{R\lambda^{n-1}+S},$$

where

$$R'=\frac{CP}{\lambda}+\frac{DR}{\lambda},\ =C\left(\ \mathrm{a}+\frac{\mathrm{b}}{\lambda}\right)+D\left(\ \mathrm{c}+\frac{\mathrm{d}}{\lambda}\right),$$

$$S'=CQ+DS,\ =C(-\mathrm{a}-\mathrm{b}\lambda)+D(-\mathrm{c}-\mathrm{d}\lambda);$$

viz.

$$R'=W+\frac{1}{\lambda}Z,\ \ S'=-W-\lambda Z,$$

where Z and W denote $\mathrm{a}C+\mathrm{c}D$ and $\mathrm{b}C+\mathrm{d}D$ as before.

We hence obtain

$$(x_{n-1}-x_n)\,\beta x_n=\frac{-(AD-BC)}{C}\times\frac{(\lambda-1)(\lambda^2-1)(a+d)\{cx^2+(d-a)x-b\}}{\lambda}\,\frac{\lambda^n}{(R\lambda^n+S)(R'\lambda^n+S')}$$

$$=\frac{-(AD-BC)}{C}\times\frac{(\lambda-1)(\lambda^2-1)(a+d)\{cx^2+(d-a)x-b\}}{\lambda(RS'-R'S)}\,\frac{(RS'-R'S)\lambda^n}{(R\lambda^n+S)(R'\lambda^n+S')},$$

or, substituting for $RS'-R'S$ its value in the denominator, this is

$$(x_{n-1}-x_n)\,\beta x_n=-\frac{AD-BC}{C}\,\frac{(ad-bc)(\lambda^2-1)}{(a+d)\lambda(cD-dC)}\,\frac{(RS'-R'S)\lambda^n}{(R\lambda^n+S)(R'\lambda^n+S')}$$

$$=-\frac{\sqrt{\{(a-d)^2+4bc\}}\,(AD-BC)}{C(cD-dC)}\,\frac{(RS'-R'S)\lambda^n}{(R\lambda^n+S)(R'\lambda^n+S')},$$

and thence

$$\phi x=\frac{A}{C}x-\frac{\sqrt{\{(a-d)^2+4bc\}}\,(AD-BC)}{C(cD-dC)}\,\Sigma\,\frac{(RS'-R'S)\lambda^n}{(R\lambda^n+S)(R'\lambda^n+S')},$$

the summation extending from 1 to ∞.

Now the before-mentioned integral formula gives

$$\frac{1}{1+k\lambda^n}=\tfrac{1}{2}-\int\frac{\sin(n\log\lambda+\log k)\,t\,dt}{\sinh\pi t},$$

$$\frac{1}{1+k'\lambda^n}=\tfrac{1}{2}-\int\frac{\sin(n\log\lambda+\log k')\,t\,dt}{\sinh\pi t}.$$

Taking the difference, and then writing $k=\dfrac{R}{S}$, $k'=\dfrac{R'}{S'}$, we have under the integral sign

$$\sin\left(n\log\lambda+\log\frac{R}{S}\right)t-\sin\left(n\log\lambda+\log\frac{R'}{S'}\right)t,$$

which is

$$= 2 \sin \tfrac{1}{2} \left(\log \frac{RS'}{R'S} \right) t \cos \left(n \log \lambda + \tfrac{1}{2} \log \frac{RR'}{SS'} \right) t,$$

which attending to the before-mentioned values of ξ, η, ζ is

$$= 2 \sin \xi t \cos (2n\zeta - \zeta + \eta) t,$$

and the formula thus is

$$\frac{S}{R\lambda^n + S} - \frac{S'}{R'\lambda^n + S'}, = - \frac{(RS' - R'S)\lambda^n}{(R\lambda^n + S)(R'\lambda^n + S')} = - \int \frac{2 \sin \xi t \cos (2n\zeta - \zeta + \eta) t \, dt}{\sinh \pi t}.$$

We have here

$$\cos (2n\zeta - \zeta + \eta) t = \cos 2n\zeta t \cos (\eta - \zeta) t - \sin 2n\zeta t \sin (\eta - \zeta) t,$$

whence summing from 1 to ∞ by means of the formulæ

$$\cos 2\zeta t + \cos 4\zeta t + \ldots = - \tfrac{1}{2},$$
$$\sin 2\zeta t + \sin 4\zeta t + \ldots = \tfrac{1}{2} \cot \zeta t,$$

(which series however are not convergent), the numerator under the integral sign becomes

$$\sin \xi t \{ - \cos (\eta - \zeta) t - \cot \zeta t \sin (\eta - \zeta) t \},$$

which is

$$= - \frac{\sin \xi t \sin \eta t}{\sin \zeta t},$$

and the formula thus is

$$\Sigma \frac{(RS' - R'S)\lambda^n}{(R\lambda^n + S)(R'\lambda^n + S')} = - \int \frac{\sin \xi t \sin \eta t \, dt}{\sin \zeta t \sinh \pi t};$$

and we therefore find

$$\phi x = \frac{A}{C} x + \frac{\sqrt{\{(a-d)^2 + 4bc\}}\,(AD - BC)}{C(cD - dC)} \int \frac{\sin \xi t \sin \eta t \, dt}{\sin \zeta t \sinh \pi t},$$

which is the result in question.

The solution is a particular one; calling it for a moment (ϕx), then, if the general solution be $\phi x = \Phi x + (\phi x)$, it at once appears that we must have $\Phi x - \Phi x_1 = 0$; and as it has been shown that $\frac{RS'}{R'S}$ is a function of x which remains unaltered by the change of x into x_1, this is satisfied by assuming $\Phi x = f\left(\frac{RS'}{R'S}\right)$, an arbitrary function of $\frac{RS'}{R'S}$. Hence we may to the foregoing expression of ϕx add this term $f\left(\frac{RS'}{R'S}\right)$.

Postscript. The new formula

$$\vartheta^n x = \frac{(\lambda^{n+1} - 1)(ax + b) + (\lambda^n - \lambda)(-dx + b)}{(\lambda^{n+1} - 1)(cx + d) + (\lambda^n - \lambda)(\ \ cx - a)},$$

where

$$\lambda + \frac{1}{\lambda} = \frac{a^2 + d^2 + 2bc}{ad - bc},$$

for the nth repetition of ϑx, $=\dfrac{ax+b}{cx+d}$, is a very interesting one. It is to be remembered that, when n is even the numerator and denominator each divide by $\lambda-1$, but when n is odd they each divide by λ^2-1; after such division, then further dividing by a power of λ, they each consist of terms of the form $\lambda^\alpha+\dfrac{1}{\lambda^\alpha}$, that is, they are each of them a rational function of $\lambda+\dfrac{1}{\lambda}$. Substituting and multiplying by the proper power of $ad-bc$, the numerator and denominator become each of them a rational and integral function of a, b, c, d of the order $n+1$ when n is even, but of the order n when n is odd; in the former case, however, the numerator and denominator each divide by $a+d$, so that ultimately, whether n be even or odd, the order is $=n$ as it should be.

For example, when $n=2$, the value is

$$\frac{(\lambda^3-1)\,\mathrm{a}+(\lambda^2-\lambda)\,\mathrm{b}}{(\lambda^3-1)\,\mathrm{c}+(\lambda^2-\lambda)\,\mathrm{d}},\ =\frac{(\lambda^2+\lambda+1)\,\mathrm{a}+\lambda\mathrm{b}}{(\lambda^2+\lambda+1)\,\mathrm{c}+\lambda\mathrm{d}},\ =\frac{\left(\lambda+\frac{1}{\lambda}+1\right)\mathrm{a}+\mathrm{b}}{\left(\lambda+\frac{1}{\lambda}+1\right)\mathrm{c}+\mathrm{d}},$$

or, as this may be written,

$$=\frac{\left(\lambda+\frac{1}{\lambda}+2\right)\mathrm{a}-\mathrm{a}+\mathrm{b}}{\left(\lambda+\frac{1}{\lambda}+2\right)\mathrm{c}-\mathrm{c}+\mathrm{d}},$$

where, observing that

$$\lambda+\frac{1}{\lambda}+2=\frac{(a+d)^2}{ad-bc},\quad -\mathrm{a}+\mathrm{b}=-(\mathrm{a}+\mathrm{d})\,x,\quad -\mathrm{c}+\mathrm{d}=-(a+d),$$

the numerator and denominator each divide by $a+d$, and the final value is

$$=\frac{(a+d)(ax+b)-(ad-bc)\,x}{(a+d)(cx+d)-(ad-bc)},\ =\frac{(a^2+bc)\,x+b\,(a+d)}{c\,(a+d)\,x+bc+d^2},$$

which is the proper value of ϑ^2x. But, when $n=3$, the value is

$$\frac{(\lambda^4-1)\,\mathrm{a}+(\lambda^3-\lambda)\,\mathrm{b}}{(\lambda^4-1)\,\mathrm{c}+(\lambda^3-\lambda)\,\mathrm{d}},\ =\frac{(\lambda^2+1)\,\mathrm{a}+\lambda\mathrm{b}}{(\lambda^2+1)\,\mathrm{c}+\lambda\mathrm{d}},\ =\frac{\left(\lambda+\frac{1}{\lambda}\right)\mathrm{a}+\mathrm{b}}{\left(\lambda+\frac{1}{\lambda}\right)\mathrm{c}+\mathrm{d}};$$

and this is

$$=\frac{(a^2+d^2+2bc)(ax+b)+(ad-bc)(-dx+b)}{(a^2+d^2+2bc)(cx+d)+(ad-bc)(\ \ cx-a)},$$

or finally

$$=\frac{(a^3+2abc+bcd)\,x+b\,(a^3+ad+bc+d^3)}{c\,(a^2+ad+bc+d^2)\,x+(abc+2bcd+d^3)},$$

which is the proper value of ϑ^3x.