# 691.

## NOTE ON MR MONRO'S PAPER "ON FLEXURE OF SPACES."

[From the *Proceedings of the London Mathematical Society*, vol. IX. (1878), pp. 171, 172. Read June 13, 1878.]

CONSIDER an element of surface, surrounding a point $P$; the flexure of the element may be interfered with by the continuity round $P$, and it is on this account proper to regard the element as cut or slit along a radius drawn from $P$ to the periphery of the element. This being understood, we have the well-known theorem that, considering in the neighbourhood of the origin elements of the surfaces

$$z=\tfrac{1}{2}(ax^2+2hxy+by^2),\quad \text{and}\quad z'=\tfrac{1}{2}(a'x'^2+2h'x'y'+b'y'^2),$$

these will be applicable the one on the other, provided only $ab-h^2=a'b'-h'^2$. But in connexion with Mr Monro's paper it is worth while to give the proof in detail.

It is to be shown that $z$, $z'$ denoting the above-mentioned functions of $(x, y)$ and $(x', y')$ respectively, it is possible to find (for small values) $x'$, $y'$ functions of $x$, $y$ such that identically

$$dx'^2+dy'^2+dz'^2=dx^2+dy^2+dz^2.$$

The solution is taken to be $x'=x+\xi$, $y'=y+\eta$, where $\xi$, $\eta$ denote cubic functions of $x$, $y$. We have then, attending only to the terms of an order not exceeding 3 in $x$, $y$,

$$dx^2+dy^2+2(dx\,d\xi+dy\,d\eta)+\{(a'x+h'y)\,dx+(h'x+b'y)\,dy\}^2$$
$$=dx^2+dy^2+\{(ax+hy)\,dx+(hx+by)\,dy\}^2,$$

so that the terms $dx^2+dy^2$ disappear; and then writing

$$d\xi=\frac{d\xi}{dx}dx+\frac{d\xi}{dy}dy,\quad d\eta=\frac{d\eta}{dx}dx+\frac{d\eta}{dy}dy,$$

the equation will be satisfied identically as regards $dx$, $dy$ if only

$$2\frac{d\xi}{dx} = (ax+hy)^2 - (a'x+h'y)^2,$$

$$\frac{d\xi}{dy} + \frac{d\eta}{dx} = (ax+hy)(hx+by) - (a'x+h'y)(h'x+b'y),$$

$$2\frac{d\eta}{dy} = (hx+by)^2 - (h'x+b'y)^2.$$

Calling the terms on the right-hand side $2\mathfrak{A}$, $\mathfrak{B}$, $2\mathfrak{C}$ respectively, we have

$$\frac{d^2\mathfrak{A}}{dy^2} - \frac{d^2\mathfrak{B}}{dx\,dy} + \frac{d^2\mathfrak{C}}{dx^2} = 0,$$

that is,

$$(h^2-h'^2) + (h^2-h'^2) - \{(ab+h^2)-(a'b'+h'^2)\} = 0,$$

or, what is the same thing,

$$a'b' - h'^2 = ab - h^2,$$

a relation which must exist between the constants $(a, b, h)$ and $(a', b', h')$.

It is easy to find the actual values of $\xi$, $\eta$; viz. these are

$$\xi = \tfrac{1}{6}(a^2-a'^2)\,x^3 + \tfrac{1}{2}(ah-a'h')\,x^2y + \tfrac{1}{2}(h^2-h'^2)\,x^2y + \tfrac{1}{6}(bh-b'h')\,y^3,$$
$$\eta = \tfrac{1}{6}(ah-a'h')\,x^3 + \tfrac{1}{2}(h^2-h'^2)\,x^2y + \tfrac{1}{2}(bh-b'h')\,x^2y + \tfrac{1}{6}(b^2-b'^2)\,y^3,$$

or, what is the same thing, we have

$$\xi = \tfrac{1}{24}\frac{d\Omega}{dx}, \quad \eta = \tfrac{1}{24}\frac{d\Omega}{dy},$$

where

$$\Omega = (a^2-a'^2)\,x^4 + 4(ah-a'h')\,x^3y + 6(h^2-h'^2)\,x^2y^2 + 4(bh-b'h')\,xy^3 + (b^2-b'^2)\,y^4,$$
$$= (ax^2+2hxy+by^2)^2 - (a'x^2+2h'xy+b'y^2)^2 = 4(z^2-z'^2),$$

in virtue of the relation $ab-h^2 = a'b'-h'^2$. The resulting values $x' = x+\xi$, $y' = y+\eta$ are obviously the first terms of two series which, if continued, would contain higher powers of $(x, y)$.