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ADDITION TO THE MEMOIR ON THE TRANSFORMATION OF ELLIPTIC FUNCTIONS.

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I HAVE recently succeeded in completing a theory considered in my "Memoir on the Transformation of Elliptic Functions," *Phil. Trans.*, vol. CLXIV. (1874), pp. 397—456, [578],—that of the septic transformation, $n = 7$. We have here

$$\frac{1-y}{1+y} = \frac{1-x}{1+x}\left(\frac{\alpha - \beta x + \gamma x^2 - \delta x^3}{\alpha + \beta x + \gamma x^2 + \delta x^3}\right)^2,$$

a solution of

$$\frac{Mdy}{\sqrt{1-y^2 \,.\, 1 - v^8 y^2}} = \frac{dx}{\sqrt{1-x^2 \,.\, 1-u^8x^2}},$$

where $\dfrac{1}{M} = 1 + \dfrac{2\beta}{\alpha}$; and the ratios $\alpha : \beta : \gamma : \delta$, and the uv-modular equation are determined by the equations

$$u^{14}\alpha^2 = v^2\delta^2,$$

$$u^6(2\alpha\gamma + 2\alpha\beta + \beta^2) = v^2(\gamma^2 + 2\gamma\delta + 2\beta\delta),$$

$$\gamma^2 + 2\beta\gamma + 2\alpha\delta + 2\beta\delta = v^2u^2(2\alpha\gamma + 2\beta\gamma + 2\alpha\delta + \beta^2),$$

$$\delta^2 + 2\gamma\delta = v^2u^{10}(\alpha^2 + 2\alpha\beta);$$

or, what is the same thing, writing $\alpha = 1$, the first equation may be replaced by $\delta = \dfrac{u^7}{v}$, and then, α, δ having these values, the last three equations determine β, γ and the modular equation. If instead of β we introduce M, by means of the relation

$\frac{1}{M}=1+2\beta$, that is, $2\beta=\frac{1}{M}-1$, then the last equation gives $2\gamma=u^3v^3\left(\frac{1}{M}-\frac{u^4}{v^4}\right)$; and $\alpha, \beta, \gamma, \delta$ having these values, we have the residual two equations

$$u^6(2\alpha\gamma+2\alpha\beta+\beta^2)=\quad v^2(\gamma^2+2\gamma\delta+2\beta\delta),$$
$$\gamma^2+2\beta\gamma+2\alpha\delta+\beta\delta=v^2u^2(2\alpha\gamma+2\beta\gamma+2\alpha\delta+\beta^2),$$

viz. each of these is a quadric equation in $\frac{1}{M}$; hence eliminating $\frac{1}{M}$, we have the modular equation; and also (linearly) the value of $\frac{1}{M}$, and thence the values of $\alpha, \beta, \gamma, \delta$ in terms of u, v.

Before going further it is proper to remark that, writing as above $\alpha=1$, then if $\delta=\beta\gamma$, we have

$$1-\beta x+\gamma x^2-\delta x^3=(1-\beta x)(1+\gamma x^2),$$
$$1+\beta x+\gamma x^2+\delta x^3=(1+\beta x)(1+\gamma x^2),$$

and the equation of transformation becomes

$$\frac{1-y}{1+y}=\frac{1-x}{1+x}\left(\frac{1-\beta x}{1+\beta x}\right)^2,$$

viz. this belongs to the cubic transformation. The value of β in the cubic transformation was taken to be $\beta=\frac{u^3}{v}$, but for the present purpose it is necessary to pay attention to an omitted double sign, and write $\beta=\pm\frac{u^3}{v}$; this being so, $\delta=\beta\gamma$, and giving to γ the value $\mp u^4$, δ will have its foregoing value $=\frac{u^7}{v}$. And from the theory of the cubic equation, according as $\beta=\frac{u^3}{v}$ or $=-\frac{u^3}{v}$, the modular equation must be

$$u^4-v^4+2uv(1-u^2v^2)=0, \text{ or } u^4-v^4-2uv(1-u^2v^2)=0.$$

We thus see *à priori*, and it is easy to verify that the equations of the septic transformation are satisfied by the values

$$\alpha=1,\ \beta=\ \frac{u^3}{v},\ \gamma=\ u^4,\ \delta=\frac{u^7}{v}, \text{ and } u^4-v^4+2uv(1-u^2v^2)=0;$$
$$\alpha=1,\ \beta=-\frac{u^3}{v},\ \gamma=-u^4,\ \delta=\frac{u^7}{v}, \text{ and } u^4-v^4-2uv(1-u^2v^2)=0;$$

and it hence follows that in obtaining the modular equation for the septic transformation, we shall meet with the factors $u^4-v^4\pm 2uv(1-u^2v^2)$. Writing for shortness $uv=\theta$, these factors are $u^4-v^4\pm 2\theta(1-\theta^2)$; the factor for the proper modular equation is $u^8+v^8-\Theta$, where

$$\Theta=8\theta-28\theta^2+56\theta^3-70\theta^4+56\theta^5-28\theta^6+8\theta^7,$$

viz. the equation $(1-u^8)(1-v^8)-(1-uv)^8=0$ is $u^8+v^8-\Theta=0$; and the modular equation, as obtained by the elimination from the two quadric equations, presents itself in the form

$$(u^4-v^4+2\theta-2\theta^3)^2(u^4-v^4-2\theta+2\theta^3)^2(u^8+v^8-\Theta)=0.$$

Proceeding to the investigation, we substitute the values

$$\alpha=1,\ \beta=\tfrac{1}{2}\left(\frac{1}{M}-1\right),\ \gamma=\tfrac{1}{2}u^3v^2\left(\frac{1}{M}-\frac{u^4}{v^4}\right),\ \delta=\frac{u^7}{v},$$

in the residual two equations, which thus become

$$\frac{1}{M^2}(1-v^8)+\frac{2}{M}(1-uv)^3(1+uv)+\left\{(1-u^8)-4(1-uv)\left(1+\frac{u^7}{v}\right)\right\}=0,$$

$$\frac{1}{M^2}\left\{-u^2v^2(1-uv)^3(1+uv)\right\}+\frac{2}{M}\left\{u^2v^2(1-u^8)+\frac{u^3}{v}(1+u^2v^2)(u^4-v^4)\right\}+\left\{\frac{u^{14}}{v^2}+6\frac{u^7}{v}(1-u^2v^2)-u^2v^2\right\}=0,$$

the first of which is given p. 432 of the "Memoir," [*Coll. Math. Papers*, vol. IX., p. 150]. Calling them

$$\left(\mathrm{a},\ \mathrm{b},\ \mathrm{c}\between\frac{1}{M},\ 1\right)^2=0,\quad \left(\mathrm{a}',\ \mathrm{b}',\ \mathrm{c}'\between\frac{1}{M},\ 1\right)^2=0,$$

we have

$$\frac{1}{M^2}:\frac{2}{M}:1=\mathrm{bc}'-\mathrm{b}'\mathrm{c}:\mathrm{ca}'-\mathrm{c}'\mathrm{a}:\mathrm{ab}'-\mathrm{a}'\mathrm{b},$$

and the result of the elimination therefore is

$$(\mathrm{ca}'-\mathrm{c}'\mathrm{a})^2-4(\mathrm{bc}'-\mathrm{b}'\mathrm{c})(\mathrm{ab}'-\mathrm{a}'\mathrm{b})=0.$$

Write as before $uv=\theta$. In forming the expressions $\mathrm{ca}'-\mathrm{c}'\mathrm{a}$, &c., to avoid fractions we must in the first instance introduce the factor v^2: thus

$$\begin{aligned}v^2(\mathrm{ca}'-\mathrm{c}'\mathrm{a})&=v\{v(1-u^8)-4(1-\theta)(v+u^7)\}\{-\theta^2(1+\theta)(1-\theta)^3\}\\&\quad-\{u^{14}+6u^6\theta(1-\theta^2)-v^2\theta^2\}\{1-v^8\},\\&=-\theta^2(1+\theta)(1-\theta)^3\{v^2(-3+4\theta)+u^6(-4\theta+3\theta^2)\}\\&\quad-\{u^{14}+6u^6(\theta-\theta^3)-v^2\theta^2\}(1-v^8);\end{aligned}$$

but instead of θ^2v^2 writing u^2v^4, the expression on the right-hand side becomes divisible by u^2; and we find

$$\begin{aligned}\frac{v^2}{u^2}(\mathrm{ca}'-\mathrm{c}'\mathrm{a})&=-(1+\theta)(1-\theta)^3\{v^4(-3+4\theta)+u^4(-4\theta^3+3\theta^4)\}\\&\quad-\{u^{12}+6u^4(\theta-\theta^3)-v^4\}(1-v^8),\end{aligned}$$

and thence

$$-\frac{v^2}{u^2}(\text{ca}'-\text{c}'\text{a}) = u^{12} + u^4(6\theta - 10\theta^3 + 11\theta^4 - 6\theta^5 - 8\theta^6 + 10\theta^7 - 4\theta^8)$$
$$+ v^4(-4 + 10\theta - 8\theta^2 - 6\theta^3 + 11\theta^4 - 10\theta^5 + 6\theta^7) + v^{12}.$$

Similarly we have

$$\frac{v^2}{u^2}(\text{bc}'-\text{b}'\text{c}) = u^{12}(5 - 5\theta + 4\theta^2 - 5\theta^3 + 2\theta^4) + u^4(9\theta - 16\theta^2 + \theta^3 + 10\theta^4 + \theta^5 - 16\theta^6 + 9\theta^7)$$
$$+ v^4(2 - 5\theta + 4\theta^2 - 5\theta^3 + 5\theta^4),$$

$$\frac{v^2}{u^2}(\text{ab}'-\text{a}'\text{b}) = u^4(\theta + \theta^3 - \theta^4) + v^4(2 - 5\theta + 4\theta^2 + 3\theta^3 - 10\theta^4 + 3\theta^5 + 4\theta^6 - 5\theta^7 + 2\theta^8)$$
$$+ v^{12}(-1 + \theta + \theta^3);$$

say these values are

$$u^{12} + pu^4 + qv^4 + v^{12}, \quad \lambda u^{12} + \mu u^4 + \nu v^4, \quad \rho u^4 + \sigma v^4 + \tau v^{12}.$$

The required equation is thus

$$0 = (u^{12} + pu^4 + qv^4 + v^{12})^2 - 4(\lambda u^{12} + \mu u^4 + \nu v^4)(\rho u^4 + \sigma v^4 + \tau v^{12}),$$

viz. the function is

$$\begin{aligned} & u^{24} \\ & + u^{16}(2p - 4\lambda\rho) \\ & + u^8\ (2q\theta^4 + p^2 - 4\lambda\sigma\theta^4 - 4\mu\rho) \\ & + \quad (2\theta^{12} + 2pq\theta^4 - 4\lambda\tau\theta^{12} - 4\mu\sigma\theta^4 - 4\nu\rho\theta^4) \\ & + v^8\ (2p\theta^4 + q^2 - 4\mu\tau\theta^4 - 4\nu\sigma) \\ & + v^{16}(2q - 4\nu\tau) \\ & + v^{24}, \end{aligned}$$

or say it is

$$= (1,\ b,\ c,\ d,\ e,\ f,\ 1 \between u^{24},\ u^{16},\ u^8,\ 1,\ v^8,\ v^{16},\ v^{24}).$$

Supposing that this has a factor $u^8 - \Theta + v^8$, the form is

$$(u^{16} + Bu^8 + C + Dv^8 + v^{16})(u^8 - \Theta + v^8);$$

and comparing coefficients we have

$$\begin{aligned} B - \Theta &= b, \\ C - \Theta B + \theta^8 &= c, \\ D\theta^8 - \Theta C + B\theta^8 &= d, \\ \theta^8 - \Theta D + C &= e, \\ -\Theta + D &= f, \end{aligned}$$

where Θ has the before-mentioned value

$$= (8,\ -28,\ +56,\ -70,\ +56,\ -28,\ +8 \between \theta,\ \theta^2,\ \theta^3,\ \theta^4,\ \theta^5,\ \theta^6,\ \theta^7).$$

From the first, second, and fifth equations, $B = b + \Theta$, $C = c + \Theta B - \theta^8$, $D = f + \Theta$; and the third and fourth equations should then be verified identically. Writing down the coefficients of the different powers of θ, we find

$$2p = 0 + 12 \quad 0 - 20 + 22 - 12 - 16 + 20 - 8 \; (\theta^0, .., \theta^8)$$
$$-4\lambda\rho = 0 - 20 + 20 - 36 + 60 - 44 + 36 - 28 + 8 \quad ,,$$

$$b = 0 - 8 + 20 - 56 + 82 - 56 + 20 - 8 \quad 0 \quad ,,$$
$$\Theta = 0 + 8 - 28 + 56 - 70 + 56 - 28 + 8 \quad 0 \quad ,,$$

$$\therefore B = 0 \quad 0 - 8 \quad 0 + 12 \quad 0 - 8 \quad 0 \quad 0 \quad ,,$$

that is,

$$B = -8\theta^2 + 12\theta^4 - 8\theta^6;$$

and in precisely the same way the fifth equation gives

$$D = -8\theta^2 + 12\theta^4 - 8\theta^6.$$

We find similarly C from the second equation: writing down first the coefficients of p^2, $2q\theta^4$, $-4\lambda\sigma\theta^4$, and $-4\mu\rho$, the sum of these gives the coefficients of c; and then writing underneath these the coefficients of $B\Theta$ and of $-\theta^8$, the final sum gives the coefficients of C: the coefficients of each line belong to $(\theta^0, \theta^1, .., \theta^{16})$.

$$0 \quad 0 \quad 36 \quad 0 - 120 + 132 + 28 - 316 + 361 - 20 - 340 + 396 - 144 - 112 + 164 - 80 + 16$$
$$- 8 + 20 - 16 - 12 + 22 - 20 \quad 0 + 12$$
$$- 40 + 140 - 212 + 140 + 80 - 188 + 168 - 92 - 64 + 176 - 164 + 80 - 16$$
$$- 36 + 64 - 40 + 60 - 72 + 28 \quad 0 + 68 - 100 + 36$$

$$0 \quad 0 \quad 0 + 64 - 208 + 352 - 272 - 160 + 463 - 160 - 272 + 352 - 208 + 64 \quad 0 \quad 0 \quad 0$$
$$0 \quad 0 \quad 0 - 64 + 224 - 352 + 224 + 160 - 392 + 160 + 224 - 352 + 224 - 64 \quad 0 \quad 0 \quad 0$$
$$- 1$$

$$0 \quad 0 \quad 0 \quad 0 + 16 \quad 0 - 48 \quad 0 + 70 \quad 0 - 48 \quad 0 + 16 \quad 0 \quad 0 \quad 0 \quad 0,$$

that is,

$$C = 16\theta^4 - 48\theta^6 + 70\theta^8 - 48\theta^{10} + 16\theta^{12};$$

and in precisely the same way this value of C would be found from the fourth equation. There remains to be verified only the fourth equation $(D + B)\,\theta^8 - \Theta C = d$, that is,

$$2\theta^8(-8\theta^2 + 12\theta^4 - 8\theta^6) - \Theta C = (2 - 4\lambda\tau)\,\theta^{12} + (2pq - 4\mu\sigma - 4\nu\rho)\,\theta^4,$$

and this can be effected without difficulty.

The factor of the modular equation thus is

$$u^{16} + v^{16} + (-8\theta^2 + 12\theta^4 - 8\theta^6)(u^8 + v^8) + 16\theta^4 - 48\theta^6 + 70\theta^8 - 48\theta^{10} + 16\theta^{12},$$

viz. this is

$$(u^8+v^8)^2+(-4\theta^2+6\theta^4-4\theta^6)\,2\,(u^8+v^8)+16\theta^4-48\theta^6+68\theta^8-48\theta^{10}+16\theta^{12}$$
$$=(u^8+v^8-4\theta^2+6\theta^4-4\theta^6)^2$$
$$=\{(u^4-v^4)^2-4\theta^2(1-\theta^2)^2\}^2,$$

that is,

$$\{u^4-v^4-2\theta(1-\theta^2)\}^2\{u^4-v^4+2\theta(1-\theta^2)\}^2;$$

or the modular equation is

$$\{u^4-v^4-2\theta(1-\theta^2)\}^2\{u^4-v^4+2\theta(1-\theta^2)\}^2(u^8+v^8-\Theta)=0;$$

viz. the first and second factors belong to the cubic transformation; and we have for the proper modular equation in the septic transformation $u^8+v^8-\Theta=0$, or what is the same thing $(1-u^8)(1-v^8)-(1-\theta)^8=0$, that is, $(1-u^8)(1-v^8)-(1-uv)^8=0$, the known result; or, as it may also be written,

$$(\theta-u^8)(\theta-v^8)+7\theta^2(1-\theta)^2(1-\theta+\theta^2)^2=0.$$

The value of M is given by the foregoing relations

$$\frac{1}{M^2}:\frac{2}{M}:1=\lambda u^{12}+\mu u^4+\nu v^4:-(u^{12}+pu^4+qv^4+v^{12}):\rho u^4+\sigma v^4+\tau v^{12};$$

but these can be, by virtue of the proper modular equation $u^8+v^8-\Theta=0$, reduced into the form

$$\frac{1}{M^2}:\frac{2}{M}:1=7(\theta-u^8):14(\theta-2\theta^2+2\theta^3-\theta^4):-\theta+v^8,$$

viz. the equality of these two sets of ratios depends upon the following identities,

$$(-\theta+v^8)(u^{12}+pu^4+qv^4+v^{12})+14(\theta-2\theta^2+2\theta^3-\theta^4)(\rho u^4+\sigma v^4+\tau v^{12})$$
$$=\{-\theta u^4+(1-\theta)(-4-\theta+5\theta^2-\theta^3-4\theta^4)v^4+v^{12}\}(u^8-\Theta+v^8),$$
$$-7(\theta-u^8)(\rho u^4+\sigma v^4+\tau v^{12})-(\theta-v^8)(\lambda u^{12}+\mu u^4+\nu v^4)$$
$$=\{(2\theta+5\theta^2+3\theta^3-2\theta^4-2\theta^5)u^4+(2+2\theta-3\theta^2-5\theta^3-2\theta^4)v^4\}(u^8-\Theta+v^8),$$
$$-2(\theta-2\theta^2+2\theta^3-\theta^4)(\lambda u^{12}+\mu u^4+\nu v^4)+(u^8-\theta)(u^{12}+pu^4+qv^4+v^{12})$$
$$=\{u^{12}+\theta(1-\theta)(3+5\theta+3\theta^2)u^4-\theta v^4\}(u^8-\Theta+v^8),$$

which can be verified without difficulty: from the last-mentioned system of values, replacing θ by its value uv, we then have

$$\frac{1}{M^2}:\frac{2}{M}:1=7u(v-u^7):14uv(1-uv)(1-uv+u^2v^2):-v(u-v^7),$$

which agree with the values given p. 482 of the "Memoir"; and the analytical theory is thus completed.