696.

CALCULATION OF THE MINIMUM N.G.F. OF THE BINARY SEVENTHIC.

[From the *American Journal of Mathematics*, t. II. (1879), pp. 71—84.]

FOR the binary seventhic $(a, \ldots \between x, y)^7$ the number of the asyzygetic covariants $(a, \ldots)^\theta (x, y)^\mu$, or say of the deg-order $(\theta . \mu)$, is given as the coefficient of $a^\theta x^\mu$ in the function

$$\frac{1-x^{-2}}{1-ax^7 . 1-ax^5 . 1-ax^3 . 1-ax . 1-ax^{-1} . 1-ax^{-3} . 1-ax^{-5} . 1-ax^{-7}}$$

developed in ascending powers of a. See my "Ninth Memoir on Quantics," *Phil. Trans.*, t. CLXI. (1871), pp. 17—50, [462].

This function is in fact

$$= A(x) - \frac{1}{x^2} A\left(\frac{1}{x}\right),$$

where, developing in ascending powers of a, the second term $-\frac{1}{x^2} A\left(\frac{1}{x}\right)$ contains only negative powers of x, and it may consequently be disregarded: the number of asyzygetic covariants of the deg-order $(\theta . \mu)$ is thus equal to the coefficient of $a^\theta x^\mu$ in the function $A(x)$, which function is for this reason called the Numerical Generating Function (N.G.F.) of the binary seventhic; and the function $A(x)$ expressed as a fraction in its least terms is said to be the minimum N.G.F.

According to a theorem of Professor Sylvester's (*Proc. Royal Soc.*, t. XXVIII. (1878), pp. 11—13), this minimum N.G.F. is of the form

$$\frac{Z_0 + aZ_1 + a^2Z_2 + \ldots + a^{36}Z_{36}}{1-ax . 1-ax^3 . 1-ax^5 . 1-ax^7 . 1-a^4 . 1-a^6 . 1-a^8 . 1-a^{10} . 1-a^{12}}$$

where $Z_0, Z_1, \ldots, Z_{36}$ are rational and integral functions of x of degrees not exceeding 14; and where, as will presently be seen, there is a symmetry in regard to the terms Z_0, Z_{36}; Z_1, Z_{35}; &c., equidistant from the middle term Z_{18}, such that the terms $Z_0, \ldots, Z_{18}$ being known, the remaining terms $Z_{19}, \ldots, Z_{36}$ can be at once written down.

Using only the foregoing properties, I obtained for the N.G.F. an expression which I communicated to Professor Sylvester, and which is published, *Comptes Rendus*, t. LXXXVII. (1878), p. 505, but with an erroneous value for the coefficient of a^7 and for that of the corresponding term a^{29}.* The correct value is

Numerator of Minimum N.G.F. is =

$$
\begin{aligned}
&1\\
&+a\ (-x-x^3-x^5)\\
&+a^2\ (x^2+x^4+2x^6+x^8+x^{10})\\
&+a^3\ (-x^7-x^9-x^{11}-x^{13})\\
&+a^4\ (2x^4+x^8+x^{14})\\
&+a^5\ (x+2x^3-x^9-x^{11})\\
&+a^6\ (-1+2x^2-x^4-x^8-x^{10}+x^{12})\\
&+a^7\ (4x+x^3+3x^5-x^9+x^{11})\\
&+a^8\ (2-x^2-3x^6-3x^8-x^{10}-x^{12})\\
&+a^9\ (x+3x^3+x^5-x^7+2x^9+2x^{13})\\
&+a^{10}(-1+4x^2-x^6-2x^8-2x^{10}-x^{14})\\
&+a^{11}(5x+3x^3+2x^5-x^7-2x^9-x^{11}+x^{13})\\
&+a^{12}(5+x^2-4x^6-6x^8-4x^{10}-x^{12}+2x^{14})\\
&+a^{13}(x-4x^5-4x^7-x^9+x^{11}+4x^{13})\\
&+a^{14}(2+5x^2+x^4+x^6-2x^8+3x^{12}-x^{14})\\
&+a^{15}(3x-x^3-x^5-7x^7-5x^9-x^{11}-x^{13})\\
&+a^{16}(6+3x^2+3x^4-4x^6-3x^8-x^{12}+5x^{14})\\
&+a^{17}(-x-2x^3-9x^5-8x^7-4x^9-3x^{11}+4x^{13})\\
&+a^{18}(2+6x^2+x^4+2x^6+2x^8+x^{10}+6x^{12}+2x^{14})\\
&+a^{19}(4x-3x^3-4x^5-8x^7-9x^9-2x^{11}-x^{13})\\
&+a^{20}(5-x^2-3x^6-4x^8+3x^{10}+3x^{12}+6x^{14})\\
&+a^{21}(-x-x^3-5x^5-7x^7-x^9-x^{11}+3x^{13})\\
&+a^{22}(-1+3x^2-2x^4+x^8+x^{10}+5x^{12}+2x^{14})\\
&+a^{23}(4x+x^3-x^5-4x^7-4x^9+x^{13})
\end{aligned}
$$

* The existence of these errors was pointed out to me by Professor Sylvester in a letter dated 13th November, 1878.

$$+ a^{24}(2 - x^2 - 4x^4 - 6x^6 - 4x^8 + x^{10} + 5x^{14})$$
$$+ a^{25}(x - x^3 - 2x^5 - x^7 + 2x^9 + 3x^{11} + 5x^{13})$$
$$+ a^{26}(-1 - 2x^4 - 2x^6 - x^8 + 4x^{10} - x^{14})$$
$$+ a^{27}(2x + 2x^5 - x^7 + x^9 + 3x^{11} + x^{13})$$
$$+ a^{28}(-x^2 - x^4 - 3x^6 - 3x^8 - x^{12} + 2x^{14})$$
$$+ a^{29}(x^3 - x^5 + 3x^9 + x^{11} + 4x^{13})$$
$$+ a^{30}(x^2 - x^4 - x^6 - x^{10} + 2x^{12} - x^{14})$$
$$+ a^{31}(-x^3 - x^5 + 2x^{11} + x^{13})$$
$$+ a^{32}(1 + x^6 + 2x^{10})$$
$$+ a^{33}(-x - x^3 - x^5 - x^7)$$
$$+ a^{34}(x^4 + x^6 + 2x^8 + x^{10} + x^{12})$$
$$+ a^{35}(-x^9 - x^{11} - x^{13})$$
$$+ a^{36} . x^{14}.$$

Denominator (as mentioned before) is

$$= 1 - ax . 1 - ax^3 . 1 - ax^5 . 1 - ax^7 . 1 - a^4 . 1 - a^6 . 1 - a^8 . 1 - a^{10} . 1 - a^{12}.$$

The method of calculation is as follows: write for a moment

$$\phi(a, x) = \frac{1 - x^{-2}}{1 - ax^7 . 1 - ax^5 . 1 - ax^3 . 1 - ax . 1 - ax^{-1} . 1 - ax^{-3} . 1 - ax^{-5} . 1 - ax^{-7}};$$

then $\phi(a, x)$, developed in ascending powers of a, and rejecting from the result all negative powers of x, is

$$= \frac{Z_0 + aZ_1 + \ldots + a^{36}Z_{36}}{1 - ax . 1 - ax^3 . 1 - ax^5 . 1 - ax^7 . 1 - a^4 . 1 - a^6 . 1 - a^8 . 1 - a^{10} . 1 - a^{12}},$$

developed in like manner in ascending powers of a; for the determination of the Z's up to Z_{18} we require only the development of $\phi(a, x)$ up to a^{18}; and, assuming that each Z is at most of the degree 14 in x, we require the coefficients of the different powers of a in $\phi(a, x)$ only up to x^{14}. Assuming then that $\phi(a, x)$ developed in ascending powers of a, up to a^{18}, rejecting all negative powers of x, and all positive powers greater than x^{14}, is

$$= X_0 + aX_1 + \ldots + a^{18}X_{18},$$

we have

$$X_0 + aX_1 + \ldots + a^{18}X_{18} = \frac{Z_0 + aZ_1 + \ldots + a^{18}Z_{18}}{1 - ax . 1 - ax^3 . 1 - ax^5 . 1 - ax^7 . 1 - a^4 . 1 - a^6 . 1 - a^8 . 1 - a^{10} . 1 - a^{12}},$$

or say

$$Z_0 + aZ_1 + \ldots + a^{18}Z_{18} = 1 - a^4 . 1 - a^6 . 1 - a^8 . 1 - a^{10} . 1 - a^{12} .$$
$$1 - ax . 1 - ax^3 . 1 - ax^5 . 1 - ax^7 . (X_0 + aX_1 + \ldots + a^{18}X_{18});$$

viz. developing here the right-hand side as far as a^{18}, but in each term rejecting the powers of x above x^{14}, the coefficients of the several powers $a^0, a^1, \ldots, a^{18}$ give the required values Z_0, $Z_1, \ldots, Z_{18}$. We require, therefore, only to know the values of these functions X_0, $X_1, \ldots, X_{18}$.

To make a break in the calculation, it is convenient to write

$$1 - ax\,.\,1 - ax^3\,.\,1 - ax^5\,.\,1 - ax^7\,(X_0 + aX_1 + \ldots + a^{18}X_{18}) = Y_0 + aY_1 + \ldots + a^{18}Y_{18};$$

putting then

$$1 - ax\,.\,1 - ax^3\,.\,1 - ax^5\,.\,1 - ax^7 = 1 - ap + a^2q - a^3r,$$

where (up to x^{14})

$$p = x + x^3 + x^5 + x^7,$$
$$q = x^4 + x^6 + 2x^8 + x^{10} + x^{12},$$
$$r = x^9 + x^{11} + x^{13},$$

we have

$$Y_0 + aY_1 + a^2Y_2 + \ldots + a^{18}Y_{18} = (1 - ap + a^2q - a^3r)(X_0 + aX_1 + a^2X_2 + \ldots + a^{18}X_{18}).$$

The values of Y_0, Y_1, …, Y_{18} then are

$$\begin{array}{rccccc}
 & Y_0 & Y_1 & Y_2 & Y_3\;.\;.\;.\;. & Y_{18} \\
\hline
= & X_0 & X_1 & X_2 & X_3 & X_{18} \\
 & & -pX_0 & -pX_1 & -pX_2 & -pX_{17} \\
 & & & +qX_0 & +qX_1 & +qX_{16} \\
 & & & & -rX_0 & -rX_{15}
\end{array}$$

the values being taken to x^{14} only; and we then have

$$Z_0 + aZ_1 + a^2Z_2 + \ldots + a^{18}Z_{18} = 1 - a^4\,.\,1 - a^6\,.\,1 - a^8\,.\,1 - a^{10}\,.\,1 - a^{12}\,(Y_0 + aY_1 + \ldots + a^{18}Y_{13});$$

viz. the values of Z_0, Z_1, …, Z_{18} are

$$\begin{array}{rcccccccccc}
 & Z_0 & Z_1 & Z_2 & Z_3 & Z_4 & Z_5 & Z_6 & Z_7 & Z_8 & Z_9 \\
\hline
= & Y_0 & Y_1 & Y_2 & Y_3 & Y_4 & Y_5 & Y_6 & Y_7 & Y_8 & Y_9 \\
 & & & & & -Y_0 & -Y_1 & -Y_2 & -Y_3 & -Y_4 & -Y_5 \\
 & & & & & & & -Y_0 & -Y_1 & -Y_2 & -Y_3 \\
 & & & & & & & & & -Y_0 & -Y_1
\end{array}$$

$$\begin{array}{rccccccccc}
 & Z_{10} & Z_{11} & Z_{12} & Z_{13} & Z_{14} & Z_{15} & Z_{16} & Z_{17} & Z_{18} \\
\hline
= & Y_{10} & Y_{11} & Y_{12} & Y_{13} & Y_{14} & Y_{15} & Y_{16} & Y_{17} & Y_{18} \\
 & -Y_6 & -Y_7 & -Y_8 & -Y_9 & -Y_{10} & -Y_{11} & -Y_{12} & -Y_{13} & -Y_{14} \\
 & -Y_4 & -Y_5 & -Y_6 & -Y_7 & -Y_8 & -Y_9 & -Y_{10} & -Y_{11} & -Y_{12} \\
 & -Y_2 & -Y_3 & -Y_4 & -Y_5 & -Y_6 & -Y_7 & -Y_8 & -Y_9 & -Y_{10} \\
 & & & & & +2Y_0 & +2Y_1 & +2Y_2 & +2Y_3 & +2Y_4 \\
 & & & & & & & +2Y_0 & +2Y_1 & +2Y_2 \\
 & & & & & & & & & +\;Y_0.
\end{array}$$

The rule of symmetry, before referred to, is that the coefficient Z_{36-p} of a^{36-p} is obtained from the coefficient Z_p of a^p by changing each power x^q into x^{14-q}, the coefficients being unaltered; in particular Z_{18}, the coefficient of a^{18}, must remain unaltered when each power x^q is changed into x^{14-q}; and the verification thus obtained of the value

$$Z_{18} = 2 + 6x^2 + x^4 + 2x^6 + 2x^8 + x^{10} + 6x^{12} + 2x^{14}$$

is in fact almost a complete verification of the whole work. Some other verifications, which present themselves in the course of the work, will be referred to further on.

We have, therefore, to calculate the coefficients $X_0, X_1, \ldots, X_{18}$; the function $\phi(a, x)$ regarded as a function of a is at once decomposed into simple fractions; viz. we have

$$\begin{aligned}
\phi(a, x) &= \frac{1 - x^{-2}}{1 - ax^7 . 1 - ax^5 . 1 - ax^3 . 1 - ax . 1 - ax^{-1} . 1 - ax^{-3} . 1 - ax^{-5} . 1 - ax^{-7}} \\
&= \frac{x^{54}}{1 - x^4 . 1 - x^6 . 1 - x^8 . 1 - x^{10} . 1 - x^{12} . 1 - x^{14}} \, \frac{1}{1 - ax^7} \\
&- \frac{x^{40}}{1 - x^2 . 1 - x^4 . 1 - x^6 . 1 - x^8 . 1 - x^{10} . 1 - x^{12}} \, \frac{1}{1 - ax^5} \\
&+ \frac{x^{28}}{1 - x^2 . (1 - x^4)^2 . 1 - x^6 . 1 - x^8 . 1 - x^{10}} \, \frac{1}{1 - ax^3} \\
&- \frac{x^{18}}{1 - x^2 . (1 - x^4)^2 . (1 - x^6)^2 . 1 - x^8} \, \frac{1}{1 - ax} \\
&+ \frac{x^{10}}{1 - x^2 . (1 - x^4)^2 . (1 - x^6)^2 . 1 - x^8} \, \frac{1}{1 - ax^{-1}} \\
&- \frac{x^4}{1 - x^2 . (1 - x^4)^2 . 1 - x^6 . 1 - x^8 . 1 - x^{10}} \, \frac{1}{1 - ax^{-3}} \\
&+ \frac{1}{1 - x^2 . 1 - x^4 . 1 - x^6 . 1 - x^8 . 1 - x^{10} . 1 - x^{12}} \, \frac{1}{1 - ax^{-5}} \\
&- \frac{x^{-2}}{1 - x^4 . 1 - x^6 . 1 - x^8 . 1 - x^{10} . 1 - x^{12} . 1 - x^{14}} \, \frac{1}{1 - ax^{-7}} .
\end{aligned}$$

In order to obtain the expansion of $\phi(a, x)$ in the assumed form of an expansion in ascending powers of a, we must of course expand the simple fractions $\frac{1}{1 - ax^7}$, &c., in ascending powers of a, but it requires a little consideration to see that we must also expand the x-coefficients of these simple fractions *in ascending powers of x.* For instance, as regards the term independent of a, here developing the several coefficients as far as x^{18}, the last five terms give (see *post*)

$$\begin{array}{rrrrrrrrrrr}
 & & & & & & & & & & - \; x^{18} \\
 & & & & & & + \; x^{10} + & x^{12} + & 3x^{14} + & 5x^{16} + & 9x^{18} \\
 & & & - \; x^4 - & x^6 - & 3x^8 - & 4x^{10} - & 8x^{12} - & 11x^{14} - & 18x^{16} - & 24x^{18} \\
 & 1 + & x^2 + & 2x^4 + & 3x^6 + & 5x^8 + & 7x^{10} + & 11x^{12} + & 14x^{14} + & 20x^{16} + & 26x^{18} \\
-x^{-2} & . - & x^2 - & x^4 - & 2x^6 - & 2x^8 - & 4x^{10} - & 4x^{12} - & 6x^{14} - & 7x^{16} - & 10x^{18} \\
\hline
= -x^{-2} & +1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}$$

viz. the sum is $= 1 - x^{-2}$ as it should be*.

* To give the last degree of perfection to the beautiful method of Professor Cayley it would seem desirable that a proof should be given of the principle illustrated by the example in the text, and the nature of the mischief resulting from its neglect clearly pointed out.—Eds. of the A. J. M.

The expansion is required only as far as x^{14}: the first four terms are therefore to be disregarded, and, writing for shortness

$$E = \frac{1}{1-x^2.(1-x^4)^2(1-x^6)^2.1-x^8},$$

$$F = \frac{1}{1-x^2.(1-x^4)^2.1-x^6.1-x^8.1-x^{10}},$$

$$G = \frac{1}{1-x^2.1-x^4.1-x^6.1-x^8.1-x^{10}.1-x^{12}},$$

$$H = \frac{1}{1-x^4.1-x^6.1-x^8.1-x^{10}.1-x^{12}.1-x^{14}},$$

we have

$$\phi(a, x) = \frac{x^{10}E}{1-ax^{-1}} - \frac{x^4F}{1-ax^{-3}} + \frac{G}{1-ax^{-5}} - \frac{x^{-2}H}{1-ax^{-7}},$$

which is

$$\begin{aligned} = \quad & x^{10}E(1+ax^{-1}+a^2x^{-2}+\ldots) \\ & - x^4F(1+ax^{-3}+a^2x^{-6}+\ldots) \\ & + \quad G(1+ax^{-5}+a^2x^{-10}+\ldots) \\ & - x^{-2}H(1+ax^{-7}+a^2x^{-14}+\ldots), \end{aligned}$$

where the several series are to be continued up to a^{18}, and, after substituting for E, F, G, H their expansions in ascending powers of x, we are to reject negative powers of x, and also powers higher than x^{14}. The functions E, F, G, H contain each of them only even powers of x, and it is easy to see that we require the expansions up to x^{22}, x^{64}, x^{104} and x^{142} respectively. For the sake of a verification, I in fact calculated E, F up to x^{64} and G, H up to x^{142}: viz. we have

$$(1-x^6)E = (1-x^{10})F,$$

from the coefficients of E we have those of $(1-x^6)E$, and in the process of calculating F we have at the last step but one the coefficients of $(1-x^{10})F$, the agreement of the two sets being the verification; similarly,

$$(1-x^2)G = (1-x^{14})H$$

gives a verification. The process for the calculation of E,

$$= \frac{1}{1-x^2.(1-x^4)^2(1-x^6)^2.1-x^8},$$

is as follows:

Ind. x	0	2	4	6	8	10	12	14	16	18	20	22
$(1-x^2)^{-1}$	1	1	1	1	1	1	1	1	1	1	1	1
			1	1	2	2	3	3	4	4	5	5
$(1-x^4)^{-1}$	1	1	2	2	3	3	4	4	5	5	6	6
			1	1	3	3	6	6	10	10	15	15
$(1-x^4)^{-1}$	1	1	3	3	6	6	10	10	15	15	21	21
				1	1	3	4	7	9	14	17	24
$(1-x^6)^{-1}$	1	1	3	4	7	9	14	17	24	29	38	45
				1	1	3	5	8	12	19	25	36
$(1-x^6)^{-1}$	1	1	3	5	8	12	19	25	36	48	63	81
					1	1	3	5	9	13	22	30
$E=(1-x^8)^{-1}$	1	1	3	5	9	13	22	30	45	61	85	111

the alternate lines giving the developments of the functions

$$(1-x^2)^{-1},\ (1-x^2)^{-1}(1-x^4)^{-1},\ (1-x^2)^{-1}(1-x^4)^{-2},\ \ldots,$$

which are the products of the x-functions down to any particular line. And in like manner we have the expansions of the other functions F, G, H respectively. I give first the expansions of E, F, G, H; next the calculation of the X's; then the calculation of the Y's: and from these the Z's up to Z_{18}, or coefficients of the powers $a^0, a^1, \ldots, a^{18}$ in the numerator of the N.G.F. are at once found; and the coefficients of the remaining powers $a^{19}, \ldots, a^{36}$ are then deduced, as already mentioned.

Writing in the formula $x=0$, we have, for the numerator of the N.G.F. of the *invariants*, the expression

$$1-a^6+2a^8-a^{10}+5a^{12}+2a^{14}+6a^{16}+2a^{18}+5a^{20}-a^{22}+2a^{24}-a^{26}+a^{32},$$

agreeing with a result in my "Second Memoir on Quantics," *Phil. Trans.*, t. CXLVI. (1856), [Number 141, vol. II. in this Collection, p. 266]; this, then, was a known result, and it affords a verification, not only of the terms in x^0, but also of those in x^{14}. Thus, in calculating the foregoing expression of the numerator, we obtain $Z_4=(2x^4+x^8+x^{14})$, viz. the term is

$$a^4(2x^4+x^8+x^{14}),$$

and we thence have the corresponding term $a^{32}(1+x^6+2x^{10})$, which, when $x=0$, becomes $=a^{32}$, a term of the numerator for the invariants: and the term $1x^{14}$ of Z_4

is thus verified, viz. so soon as, in the calculation, we arrive at this term, we know that it is right, and the calculation up to this point is, to a considerable extent, verified. And similarly, in continuing the calculation, we arrive at other terms which are verified in the like manner.

Expansions of the Functions E, F, G, H.

Ind. x	E	F	G	H	Ind. x	E	F	G	H
0	1	1	1	1	16	45	36	20	6
2	1	1	1	0	18	61	47	26	7
4	3	3	2	1	20	85	66	35	10
6	5	4	3	1	22	111	84	44	11
8	9	8	5	2	24		113	58	16
10	13	11	7	2	26		141	71	17
12	22	18	11	4	28		183	90	23
14	30	24	14	4	30		225	110	26

Ind. x	F	G	H	Ind. x	G	H	Ind. x	H
32	284	136	33	70	2172	419	108	2265
34	344	163	37	72	2432	472	110	2426
36	425	199	47	74	2702	515	112	2623
38	508	235	52	76	3009	576	114	2807
40	617	282	64	78	3331	629	116	3026
42	729	331	72	80	3692	699	118	3232
44	872	391	86	82	4070	760	120	3479
46	1020	454	96	84	4494	843	122	3708
48	1205	532	115	86	4935	913	124	3981
50	1397	612	127	88	5427	1007	126	4240
52	1632	709	149	90	5942	1091	128	4541
54	1877	811	166	92	6510	1197	130	4828
56	2172	931	192	94	7104	1293	132	5164
58	2480	1057	212	96	7760	1416	134	5481
60	2846	1206	245	98	8442	1525	136	5850
62	3228	1360	269	100	9192	1663	138	6204
64	3677	1540	307	102	9975	1790	140	6609
66		1729	338	104	10829	1945	142	6998
68		1945	382	106		2088		

Calculation of the X's.

Ind. x even or odd according as suffix X is even or odd.

	0_1	2_3	4_5	6_7	8_9	10_{11}	12_{13}	14
						1	1	3
			− 1	− 1	− 3	− 4	− 8	− 11
	1	1	2	3	5	7	11	14
		− 1	− 1	− 2	− 2	− 4	− 4	− 6
$X_0=$	1	0	0	0	0	0	0	0
					1	1	3	
	− 1	− 1	− 3	− 4	− 8	− 11	− 18	
	3	5	7	11	14	20	26	
	− 2	− 4	− 4	− 6	− 7	− 10	− 11	
$X_1=$	0	0	0	+ 1	0	0	0	
					1	1	3	5
	− 1	− 3	− 4	− 8	− 11	− 18	− 24	− 36
	7	11	14	20	26	35	44	58
	− 6	− 7	− 10	− 11	− 16	− 17	− 23	− 26
$X_2=$	0	+ 1	0	+ 1	0	+ 1	0	+ 1
				1	1	3	5	
	− 4	− 8	− 11	− 18	− 24	− 36	− 47	
	20	26	35	44	58	71	90	
	− 16	− 17	− 23	− 26	− 33	− 37	− 47	
$X_3=$	0	+ 1	+ 1	+ 1	+ 2	+ 1	+ 1	
				1	1	3	5	9
	− 8	− 11	− 18	− 24	− 36	− 47	− 66	− 84
	35	44	58	71	90	110	136	163
	− 26	− 33	− 37	− 47	− 52	− 64	− 72	− 86
$X_4=$	1	0	+ 3	+ 1	+ 3	+ 2	+ 3	+ 2
			1	1	3	5	9	
	− 18	− 24	− 36	− 47	− 66	− 84	− 113	
	71	90	110	136	163	199	235	
	− 52	− 64	− 72	− 86	− 96	− 115	− 127	
$X_5=$	1	+ 2	+ 3	+ 4	+ 4	+ 5	+ 4	

	0_1	2_3	4_5	6_7	8_9	10_{11}	12_{13}	14
			1	1	3	5	9	13
	− 24	− 36	− 47	− 66	− 84	− 113	− 141	− 183
	110	136	163	199	235	282	331	391
	− 86	− 96	− 115	− 127	− 149	− 166	− 191	− 212
$X_6 =$	0	+ 4	+ 2	+ 7	+ 5	+ 8	+ 8	+ 9
		1	1	3	5	9	13	
	− 47	− 66	− 84	− 113	− 141	− 183	− 225	
	199	235	282	331	391	454	532	
	− 149	− 166	− 192	− 212	− 245	− 269	− 307	
$X_7 =$	3	+ 4	+ 7	+ 9	+ 10	+ 11	+ 13	
		1	1	3	5	9	13	22
	− 66	− 84	− 113	− 141	− 183	− 225	− 284	− 344
	282	331	391	454	532	612	709	811
	− 212	− 245	− 269	− 307	− 338	− 382	− 419	− 472
$X_8 =$	4	+ 3	+ 10	+ 9	+ 16	+ 14	+ 19	+ 17
	1	1	3	5	9	13	22	
	− 113	− 141	− 183	− 225	− 284	− 344	− 425	
	454	532	612	709	811	931	1057	
	− 338	− 382	− 419	− 472	− 515	− 576	− 629	
$X_9 =$	4	+ 10	+ 13	+ 17	+ 21	+ 24	+ 25	
	1	1	3	5	9	13	22	30
	− 141	− 183	− 225	− 284	− 344	− 425	− 508	− 617
	612	709	811	931	1057	1206	1360	1540
	− 472	− 515	− 576	− 629	− 699	− 760	− 843	− 913
$X_{10} =$	0	+ 12	+ 13	+ 23	+ 23	+ 34	+ 31	+ 40
	1	3	5	9	13	22	30	
	− 225	− 284	− 344	− 425	− 508	− 617	− 729	
	931	1057	1206	1360	1540	1729	1945	
	− 699	− 760	− 843	− 913	− 1007	− 1091	− 1197	
$X_{11} =$	8	+ 16	+ 24	+ 31	+ 38	+ 43	+ 49	
	1	3	5	9	13	22	30	45
	− 284	− 344	− 425	− 508	− 617	− 729	− 872	− 1020
	1206	1360	1540	1729	1945	2172	2432	2702
	− 913	− 1007	− 1091	− 1197	− 1293	− 1416	− 1525	− 1663
$X_{12} =$	10	+ 12	+ 29	+ 33	+ 48	+ 49	+ 65	+ 64

	0_1	2_3	4_5	6_7	8_9	10_{11}	12_{13}	14
	3	5	9	13	22	30	45	
	− 425	− 508	− 617	− 729	− 872	− 1020	− 1205	
	1729	1945	2172	2432	2702	3009	3331	
	− 1293	− 1416	− 1525	− 1663	− 1790	− 1945	− 2088	
$X_{13}=$	14	+ 26	+ 39	+ 53	+ 62	+ 74	+ 83	
	3	5	9	13	22	30	45	61
	− 508	− 617	− 729	− 872	− 1020	− 1205	− 1397	− 1632
	2172	2432	2702	3009	3331	3692	4070	4494
	− 1663	− 1790	− 1945	− 2088	− 2265	− 2426	− 2623	− 2807
$X_{14}=$	4	+ 30	+ 37	+ 62	+ 68	+ 91	+ 95	+ 116
	5	9	13	22	30	45	61	
	− 729	− 872	− 1020	− 1205	− 1397	− 1632	− 1877	
	3009	3331	3692	4070	4494	4935	5427	
	− 2265	− 2426	− 2623	− 2807	− 3026	− 3232	− 3479	
$X_{15}=$	20	+ 42	+ 62	+ 80	+ 101	+ 116	+ 132	
	5	9	13	22	30	45	61	85
	− 872	− 1020	− 1205	− 1397	− 1632	− 1877	− 2172	− 2480
	3692	4070	4494	4935	5427	5942	6510	7104
	− 2807	− 3026	− 3232	− 3479	− 3708	− 3981	− 4240	− 4541
$X_{16}=$	18	+ 33	+ 70	+ 81	+ 117	+ 129	+ 159	+ 168
	9	13	22	30	45	61	85	
	− 1205	− 1397	− 1632	− 1877	− 2172	− 2480	− 2846	
	4935	5427	5942	6510	7104	7760	8442	
	− 3708	− 3981	− 4240	− 4541	− 4828	− 5164	− 5481	
$X_{17}=$	31	+ 62	+ 92	+ 122	+ 149	+ 177	+ 200	
	9	13	22	30	45	61	85	111
	− 1397	− 1632	− 1877	− 2172	− 2480	− 2846	− 3228	− 3677
	5942	6510	7104	7760	8442	9192	9975	10829
	− 4541	− 4828	− 5164	− 5481	− 5850	− 6204	− 6609	− 6998
$X_{18}=$	13	+ 63	+ 85	+ 137	+ 157	+ 203	+ 223	+ 265

Calculation of the Y's.

Ind. x even or odd as suffix X is even or odd.

	0_1	2_3	4_5	6_7	8_9	10_{11}	12_{13}	14
	1							
$Y_0=$	1							
				1				
	−1	−1	−1	−1				
$Y_1=$	−1	−1	−1	0	0	0	0	
	0	1	0	1	0	1	0	1
					−1	−1	−1	−1
			1	1	2	1	1	
$Y_2=$	0	1	1	2	1	1	0	0
		1	1	1	2	1	1	
		−1	−1	−2	−2	−2	−2	
						1	1	
					−1	−1	−1	
$Y_3=$	0	0	0	−1	−1	−1	−1	
	1	0	3	1	3	2	3	2
			−1	−2	−3	−5	−5	−5
				1	1	3	2	4
$Y_4=$	1	0	+2	0	+1	0	0	+1
	1	2	3	4	4	5	4	
	−1	−1	−4	−5	−7	−9	−9	
				1	2	4	6	
						−1	−1	
$Y_5=$	0	+1	−1	0	−1	−1	0	
		4	2	7	5	8	7	9
		−1	−3	−6	−10	−13	−16	−17
			1	1	5	5	11	10
							−1	−2
$Y_6=$	0	+3	0	+2	0	0	+1	0

53—2

	0_1	2_3	4_5	6_7	8_9	10_{11}	12_{13}	14
	3	4	7	9	10	11	13	
		− 4	− 6	− 13	− 18	− 22	− 27	
			1	3	7	12	17	
					− 1	− 1	− 4	
$Y_7 =$	3	0	+ 2	− 1	− 2	0	− 1	
	4	3	10	9	16	14	19	17
		− 3	− 7	− 14	− 23	− 30	− 37	− 43
				4	6	17	20	33
						− 1	− 3	− 6
$Y_8 =$	4	0	+ 3	− 1	− 1	0	− 1	+ 1
	4	10	13	17	21	24	25	
	− 4	− 7	− 17	− 26	− 38	− 49	− 58	
			3	7	17	27	40	
						− 4	− 6	
$Y_9 =$	0	+ 3	− 1	− 2	0	− 2	+ 1	
		12	13	23	23	34	31	40
		− 4	− 14	− 27	− 44	− 61	− 75	− 87
			4	7	21	29	52	61
						− 3	− 7	− 14
$Y_{10} =$	0	+ 8	+ 3	+ 3	0	− 1	+ 1	0
	8	16	24	31	38	43	49	
		− 12	− 25	− 48	− 71	− 93	− 111	
			4	14	31	54	78	
					− 4	− 7	− 17	
$Y_{11} =$	8	+ 4	+ 3	− 3	− 6	− 3	− 1	
	10	12	29	33	48	49	65	64
		− 8	− 24	− 48	− 79	− 109	− 136	− 161
				12	25	60	84	128
						− 4	− 14	− 27
$Y_{12} =$	10	+ 4	+ 5	− 3	− 6	− 4	− 1	+ 4

	0_1	2_3	4_5	6_7	8_9	10_{11}	12_{13}	14
	14	26	39	53	62	74	83	
	− 10	− 22	− 51	− 84	− 122	− 159	− 195	
			8	24	56	95	141	
						− 12	− 25	
$Y_{13}=$	4	+ 4	− 4	− 7	− 4	− 2	+ 4	
	4	30	37	62	68	91	95	116
		− 14	− 40	− 79	− 132	− 180	− 228	− 272
			10	22	61	96	161	204
						− 8	− 24	− 48
$Y_{14}=$	4	+ 16	+ 7	+ 5	− 3	− 1	+ 4	0
	20	42	62	80	101	116	132	
	− 4	− 34	− 71	− 133	− 197	− 258	− 316	
			14	40	93	158	233	
					− 10	− 22	− 51	
$Y_{15}=$	16	+ 8	+ 5	− 13	− 13	− 6	− 2	
	18	33	70	81	117	129	159	168
		− 20	− 62	− 124	− 204	− 285	− 359	− 429
			4	34	75	163	238	350
						− 14	− 40	− 79
$Y_{16}=$	18	+ 13	+ 12	− 9	− 12	− 7	− 2	+ 10
	31	62	92	122	149	177	200	
	− 18	− 51	− 121	− 202	− 301	− 397	− 486	
			20	62	144	246	367	
					− 4	− 34	− 71	
$Y_{17}=$	13	+ 11	− 9	− 18	− 12	− 8	+ 10	
	13	63	85	137	157	203	223	265
		− 31	− 93	− 185	− 307	− 425	− 540	− 648
			18	51	139	235	389	511
						− 20	− 62	− 124
$Y_{18}=$	13	+ 32	+ 10	+ 3	− 11	− 7	+ 10	+ 4

Cambridge, December 7th, 1878.