697.

ON THE DOUBLE ϑ-FUNCTIONS.

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I HAVE sought to obtain, in forms which may be useful in regard to the theory of the double ϑ-functions, the integral of the elliptic differential equation

$$\frac{dx}{\sqrt{a-x\,.\,b-x\,.\,c-x\,.\,d-x}}+\frac{dy}{\sqrt{a-y\,.\,b-y\,.\,c-y\,.\,d-y}}=0:$$

the present paper has immediate reference only to this differential equation; but, on account of the design of the investigation, I have entitled it as above.

We may for the general integral of the above equation take a particular integral of the equation

$$\frac{dx}{\sqrt{a-x\,.\,b-x\,.\,c-x\,.\,d-x}}+\frac{dy}{\sqrt{a-y\,.\,b-y\,.\,c-y\,.\,d-y}}\pm\frac{dz}{\sqrt{a-z\,.\,b-z\,.\,c-z\,.\,d-z}}=0;$$

viz. this particular integral, regarding therein z as an arbitrary constant, will be the general integral of the first mentioned equation. And we may further assume that z is the value of y corresponding to the value a of x.

I write for shortness

$$a-x,\ b-x,\ c-x,\ d-x=\mathrm{a}\,,\ \mathrm{b}\,,\ \mathrm{c}\,,\ \mathrm{d}\,,$$
$$a-y,\ b-y,\ c-y,\ d-y=\mathrm{a}_1,\ \mathrm{b}_1,\ \mathrm{c}_1,\ \mathrm{d}_1;$$

and I write also $(xy,\ bc,\ ad)$, or more shortly $(bc,\ ad)$ to denote the determinant

$$\begin{vmatrix} 1, & x+y, & xy \\ 1, & b+c, & bc \\ 1, & a+d, & ad \end{vmatrix};$$

we have of course $(ad,\ bc)=-(bc,\ ad)$, and there are thus the three distinct determinants $(ad,\ bc)$, $(bd,\ ac)$ and $(cd,\ ab)$.

We have then for each of the functions

$$\sqrt{\frac{a-z}{d-z}},\ \sqrt{\frac{b-z}{d-z}},\ \sqrt{\frac{c-z}{d-z}}$$

a set of four equivalent expressions, the whole system being

$$\sqrt{\frac{a-z}{d-z}}=\frac{\sqrt{a-b\,.\,a-c}\,\{\sqrt{\mathrm{adb_1c_1}}+\sqrt{\mathrm{a_1d_1bc}}\}}{(bc,\ ad)}=\frac{\sqrt{a-b\,.\,a-c}\,(x-y)}{\sqrt{\mathrm{adb_1c_1}}-\sqrt{\mathrm{a_1d_1bc}}}$$

$$=\frac{\sqrt{a-b\,.\,a-c}\,\{\sqrt{\mathrm{abc_1d_1}}+\sqrt{\mathrm{a_1b_1cd}}\}}{(a-c)\sqrt{\mathrm{bdb_1d_1}}-(b-d)\sqrt{\mathrm{aca_1c_1}}}=\frac{\sqrt{a-b\,.\,a-c}\,\{\sqrt{\mathrm{acb_1d_1}}+\sqrt{\mathrm{a_1c_1bd}}\}}{(a-b)\sqrt{\mathrm{cdc_1d_1}}-(c-d)\sqrt{\mathrm{aba_1b_1}}};$$

$$\sqrt{\frac{b-z}{d-z}}=\frac{\sqrt{\frac{a-b}{a-d}}\,\{(a-c)\sqrt{\mathrm{bdb_1d_1}}+(b-d)\sqrt{\mathrm{aca_1c_1}}\}}{(bc,\ ad)}=\frac{\sqrt{\frac{a-b}{a-d}}\,\{\sqrt{\mathrm{abc_1d_1}}-\sqrt{\mathrm{a_1b_1cd}}\}}{\sqrt{\mathrm{adb_1c_1}}-\sqrt{\mathrm{a_1d_1bc}}}$$

$$=\frac{\sqrt{\frac{a-b}{a-d}}\,(cd,\ ab)}{(a-c)\sqrt{\mathrm{bdb_1d_1}}-(b-d)\sqrt{\mathrm{aca_1c_1}}}=\frac{\sqrt{\frac{a-b}{a-d}}\,\{(a-d)\sqrt{\mathrm{bcb_1c_1}}+(b-c)\sqrt{\mathrm{ada_1d_1}}\}}{(a-b)\sqrt{\mathrm{cdc_1d_1}}-(c-d)\sqrt{\mathrm{aba_1b_1}}};$$

$$\sqrt{\frac{c-z}{d-z}}=\frac{\sqrt{\frac{a-c}{a-d}}\,\{(a-b)\sqrt{\mathrm{cdc_1d_1}}+(c-d)\sqrt{\mathrm{aba_1b_1}}\}}{(bc,\ ad)}=\frac{\sqrt{\frac{a-c}{a-d}}\,\{\sqrt{\mathrm{acb_1d_1}}-\sqrt{\mathrm{a_1c_1bd}}\}}{\sqrt{\mathrm{adb_1c_1}}-\sqrt{\mathrm{a_1d_1bc}}}$$

$$=\frac{\sqrt{\frac{a-c}{a-d}}\,\{(a-d)\sqrt{\mathrm{bcb_1c_1}}-(b-c)\sqrt{\mathrm{ada_1d_1}}\}}{(a-c)\sqrt{\mathrm{bdb_1d_1}}-(b-d)\sqrt{\mathrm{aca_1c_1}}}=\frac{\sqrt{\frac{a-c}{a-d}}\,(bd,\ ac)}{(a-b)\sqrt{\mathrm{cdc_1d_1}}-(c-d)\sqrt{\mathrm{aba_1b_1}}}.$$

The expressions in the like fourfold form for the functions $\mathrm{sn}\,(u+v)$, $\mathrm{cn}\,(u+v)$, $\mathrm{dn}\,(u+v)$ are given p. 63 of my *Treatise on Elliptic Functions.*

It is easy to verify first that the four expressions for the same function of z are identical, and next that the expressions for the three several functions

$$\sqrt{\frac{a-z}{d-z}},\ \sqrt{\frac{b-z}{d-z}},\ \sqrt{\frac{c-z}{d-z}},$$

are consistent with each other. For instance, comparing the first and second expressions of $\sqrt{\frac{a-z}{d-z}}$, the equation to be verified is

$$\mathrm{adb_1c_1}-\mathrm{a_1d_1bc}=(x-y)\,(bc,\ ad),$$

which is at once shown to be true. Again comparing the first and second expressions for $\sqrt{\frac{b-z}{d-z}}$, we ought to have

$$\{(a-c)\sqrt{\mathrm{bdb_1d_1}}+(b-d)\sqrt{\mathrm{aca_1c_1}}\}\,\{\sqrt{\mathrm{adb_1c_1}}-\sqrt{\mathrm{a_1d_1bc}}\}=(bc,\ ad)\,\{\sqrt{\mathrm{abc_1d_1}}-\sqrt{\mathrm{a_1b_1cd}}\}.$$

Here the product on the left-hand side is

$$= (a-c)\{\mathrm{b_1 d}\sqrt{\mathrm{abc_1d_1}} - \mathrm{bd_1}\sqrt{\mathrm{a_1b_1cd}}\} + (b-d)\{-\mathrm{a_1c}\sqrt{\mathrm{abc_1d_1}} + \mathrm{ac_1}\sqrt{\mathrm{a_1b_1cd}}\},$$

viz. this is

$$= \sqrt{\mathrm{abc_1d_1}}\{(a-c)\,\mathrm{b_1d} - (b-d)\,\mathrm{a_1c}\} - \sqrt{\mathrm{a_1b_1cd}}\{(a-c)\,\mathrm{bd_1} - (b-d)\,\mathrm{ac_1}\},$$

and in this last expression the two terms in { } are at once shown to be each $=(bc,\ ad)$; whence the identity in question.

Comparing in like manner the first expressions for $\sqrt{\dfrac{a-z}{d-z}}$ and $\sqrt{\dfrac{b-z}{d-z}}$ respectively, we have

$$(b-d)(bc,\ ad)^2\frac{a-z}{d-z} = (a-b)(a-c)(b-d)\{\mathrm{adb_1c_1} + \mathrm{a_1d_1bc} + 2\sqrt{\mathrm{abcda_1b_1c_1d_1}}\},$$

$$(d-a)(bc,\ ad)^2\frac{b-z}{d-z} =$$

$$-(a-b)\{(a-c)^2\,\mathrm{bdb_1d_1} + (b-d)^2\,\mathrm{aca_1c_1} + 2(a-c)(b-d)\sqrt{\mathrm{abcda_1b_1c_1d_1}}\},$$

whence, adding, the radical on the right-hand side disappears; the whole equation divides by $-(a-b)$, and omitting this factor, the relation to be verified is

$$(bc,\ ad)^2 = (a-c)^2\,\mathrm{bdb_1d_1} + (b-d)^2\,\mathrm{aca_1c_1} - (a-c)(b-d)(\mathrm{adb_1c_1} + \mathrm{a_1d_1bc});$$

the right-hand side is here

$$= \{(a-c)\,\mathrm{b_1d} - (b-d)\,\mathrm{a_1c}\}\{(a-c)\,\mathrm{bd_1} - (b-d)\,\mathrm{ac_1}\},$$

and each of the two factors being $=(bc,\ ad)$, the identity is verified. It thus appears that the twelve equations are in fact equivalent to a single equation in x, y, z.

Writing in the several formulæ $x=a,\ b,\ c,\ d$ successively, they become

$$\begin{array}{llll}
x=a, & x=b, & x=c, & x=d,\\[4pt]
\dfrac{a-z}{d-z}=\dfrac{\mathrm{a_1}}{\mathrm{d_1}}, & -\dfrac{c-a}{d-b}\cdot\dfrac{\mathrm{b_1}}{\mathrm{c_1}}, & -\dfrac{b-a}{d-c}\cdot\dfrac{\mathrm{c_1}}{\mathrm{b_1}}, & \dfrac{a-b\,.\,a-c}{d-b\,.\,d-c}\cdot\dfrac{\mathrm{d_1}}{\mathrm{a_1}},\\[8pt]
\dfrac{b-z}{d-z}=\dfrac{\mathrm{b_1}}{\mathrm{d_1}}, & -\dfrac{c-b}{d-a}\cdot\dfrac{\mathrm{a_1}}{\mathrm{c_1}}, & \dfrac{b-a\,.\,b-c}{d-a\,.\,d-c}\cdot\dfrac{\mathrm{d_1}}{\mathrm{b_1}}, & -\dfrac{a-b}{d-c}\cdot\dfrac{\mathrm{c_1}}{\mathrm{a_1}},\\[8pt]
\dfrac{c-z}{d-z}=\dfrac{\mathrm{c_1}}{\mathrm{d_1}}, & \dfrac{c-a\,.\,c-b}{d-a\,.\,d-b}\cdot\dfrac{\mathrm{d_1}}{\mathrm{c_1}}, & -\dfrac{b-c}{d-a}\cdot\dfrac{\mathrm{a_1}}{\mathrm{b_1}}, & -\dfrac{a-c}{d-b}\cdot\dfrac{\mathrm{b_1}}{\mathrm{a_1}},
\end{array}$$

viz. for $x=a$, the relation is $z=y$, but in the other three cases respectively the relation is a linear one, $z=\dfrac{\alpha y+\beta}{\gamma y+\delta}$.

Rationalising the first equation for $\sqrt{\dfrac{a-z}{d-z}}$, we have

$$(bc,\ ad)^2(a-z) = (a-b)(a-c)(d-z)\{\mathrm{adb_1c_1} + \mathrm{a_1d_1bc} + 2\sqrt{\mathrm{abcda_1b_1c_1d_1}}\},$$

and thence

$$\{(bc,\ ad)^2(a-z) - (a-b)(a-c)(d-z)(\mathrm{adb_1c_1} + \mathrm{a_1d_1bc})\}^2$$
$$= (a-b)^2(a-c)^2(d-z)^2\,.\,4\mathrm{abcda_1b_1c_1d_1}.$$

Expanding, and observing that

$$(\mathrm{adb_1c_1} + \mathrm{a_1d_1bc})^2 = (\mathrm{adb_1c_1} - \mathrm{a_1d_1bc})^2 + 4\mathrm{abcda_1b_1c_1d_1} = (bc,\ ad)^2 (x-y)^2 + 4\mathrm{abcda_1b_1c_1d_1},$$

the whole equation becomes divisible by $(bc,\ ad)^2$, and omitting this factor, the equation is

$$(bc,\ ad)^2 (a-z)^2 - 2(a-b)(a-c)(a-z)(d-z)(\mathrm{adb_1c_1} + \mathrm{a_1d_1bc}) \\ + (a-b)^2 (a-c)^2 (d-z)^2 (x-y)^2 = 0,$$

or, as this may also be written,

$$\begin{aligned} &z^2 \{(bc,\ ad)^2 - 2(a-b)(a-c)(\mathrm{adb_1c_1} + \mathrm{a_1d_1bc}) + (a-b)^2(a-c)^2(x-y)^2\} \\ -\,&2z\,\{(bc,\ ad)\,a - (a-b)(a-c)(\mathrm{adb_1c_1} + \mathrm{a_1d_1bc})(a+d) + (a-b)^2(a-c)^2(x-y)^2 d\} \\ +\,&\{(bc,\ ad)a^2 - 2(a-b)(a-c)(\mathrm{adb_1c_1} + \mathrm{a_1d_1bc})\,ad + (a-b)^2(a-c)^2(x-y)^2 d^2\} = 0. \end{aligned}$$

This is really a symmetrical equation in x, y, z of the form

$$\begin{aligned} &A \\ &+ 2B(x+y+z) \\ &+ \ C(x^2+y^2+z^2) \\ &+ 2D(yz+zx+xy) \\ &+ 2E(y^2z+yz^2+z^2x+zx^2+x^2y+xy^2) \\ &+ 4Fxyz \\ &+ 2G(x^2yz+xy^2z+xyz^2) \\ &+ \ H(y^2z^2+z^2x^2+x^2y^2) \\ &+ 2I(xy^2z^2+x^2yz^2+x^2y^2z) \\ &+ \ Jx^2y^2z^2 = 0\,; \end{aligned}$$

the several coefficients being symmetrical as regards b, c, d, but the a entering unsymmetrically: the actual values are

$$\begin{aligned} A &= \quad a^4\{b^2c^2+b^2d^2+c^2d^2-2bcd(b+c+d)\} + 2a^3bcd(bc+bd+cd) - 3a^2b^2c^2d^2, \\ B &= \quad 2a^4bcd - a^3(b^2c^2+b^2d^2+c^2d^2) + ab^2c^2d^2, \\ C &= -4a^3bcd + a^2(bc+bd+cd)^2 - 2abcd(bc+bd+cd) + b^2c^2d^2, \\ D &= -\ a^4(bc+bd+cd) + a^3(b^2c+bc^2+b^2d+bd^2+c^2d+cd^2-2bcd) \\ &\qquad + a^2\{b^2c^2+b^2d^2+c^2d^2-bcd(b+c+d)\} - b^2c^2d^2, \\ E &= \quad a^3(bc+bd+cd) - a^2(b^2c+bc^2+b^2d+bd^2+c^2d+cd^2) + abcd(b+c+d), \\ F &= \quad a^4(b+c+d) - a^3(b^2+c^2+d^2+bc+bd+cd) + 6a^2bcd \\ &\qquad - a\{b^2c^2+b^2d^2+c^2d^2+bcd(b+c+d)\} + bcd(bc+bd+cd), \\ G &= -\ a^4 + a^2(b^2+c^2+d^2-bc-bd-cd) + a(b^2c+bc^2+b^2d+bd^2+c^2d+cd^2-2bcd) \\ &\qquad - bcd(b+c+d), \\ H &= \quad a^4 - 2a^3(b+c+d) + a^2(b+c+d)^2 - 4abcd, \\ I &= \quad a^3 - a(b^2+c^2+d^2) + 2bcd, \\ J &= -3a^2 + 2a(b+c+d) + b^2+c^2+d^2 - 2(bc+bd+cd). \end{aligned}$$

It may be remarked by way of verification that the equation remains unaltered on substituting for x, y, z, a, b, c, d their reciprocals and multiplying the whole by $a^4b^2c^2d^2x^2y^2z^2$.

I further remark that, writing $a=0$, we have

$$A=0,\quad B=0,\quad C=b^2c^2d^2,\quad D=-b^2c^2d^2,\quad E=0,\quad F=bcd\,(bc+bd+cd),$$

$$G=-bcd\,(b+c+d),\quad H=0,\quad I=2bcd,\quad J=b^2+c^2+d^2-2\,(bc+bd+cd);$$

and writing also

$$\epsilon=1,\quad -\delta=(b+c+d),\quad \gamma=bc+bd+cd,\quad -\beta=bcd,$$

(whence

$$a-x\,.\,b-x\,.\,c-x\,.\,d-x=\beta x+\gamma x^2+\delta x^3+\epsilon x^4),$$

we have the formula

$$\begin{aligned}&\beta^2\,(x^2+y^2+z^2-2yz-2zx-2xy)\\ &-4\beta\gamma\,xyz\\ &-2\beta\delta\,xyz\,(x+y+z)\\ &-4\beta\epsilon\,xyz\,(yz+zx+xy)\\ &+\ (\delta^2-4\gamma\epsilon)\,x^2y^2z^2=0,\end{aligned}$$

given p. 348 of my *Elliptic Functions* as a particular integral of the differential equation when the radical is $\sqrt{\beta x+\gamma x^2+\delta x^3+\epsilon x^4}$.

Let the equation in $(x,\ y,\ z)$ be called $u=0$; u has been given in the form $u=\mathfrak{C}z^2-2\mathfrak{B}z+\mathfrak{A}$, and we thence have $\tfrac{1}{2}\dfrac{du}{dz}=\mathfrak{C}z-\mathfrak{B}$ which, in virtue of the equation $u=0$ itself, becomes $\tfrac{1}{2}\dfrac{du}{dz}=\sqrt{\mathfrak{B}^2-\mathfrak{A}\mathfrak{C}}$; we find easily

$$\mathfrak{B}^2-\mathfrak{A}\mathfrak{C}=(a-b)^2\,(a-c)^2\,(a-d)^2\,\{(\mathrm{adb_1c_1}+\mathrm{a_1d_1bc})^2-(bc,\ ad)^2\,(x-y)^2\},$$

or, attending to the relation

$$\begin{aligned}(\mathrm{adb_1c_1}+\mathrm{a_1d_1bc})^2&=(\mathrm{adb_1c_1}-\mathrm{a_1d_1bc})^2+4\mathrm{abcda_1b_1c_1d_1}\\ &=(bc,\ ad)^2\,(x-y)^2+4\mathrm{abcda_1b_1c_1d_1},\end{aligned}$$

this is

$$\mathfrak{B}^2-\mathfrak{A}\mathfrak{C}=4\,(a-b)^2\,(a-c)^2\,(a-d)^2\,\mathrm{abcda_1b_1c_1d_1},$$

or we have

$$\tfrac{1}{4}\frac{du}{dz}=(a-b)\,(a-c)\,(a-d)\,\sqrt{\mathrm{abcd}}\,\sqrt{\mathrm{a_1b_1c_1d_1}}.$$

Writing

$$a-z,\ b-z,\ c-z,\ d-z=\mathrm{a_2},\ \mathrm{b_2},\ \mathrm{c_2},\ \mathrm{d_2},$$

we have of course the like formulæ

$$\tfrac{1}{4}\frac{du}{dx}=(a-b)\,(a-c)\,(a-d)\,\sqrt{\mathrm{a_1b_1c_1d_1}}\,\sqrt{\mathrm{a_2b_2c_2d_2}},$$

$$\tfrac{1}{4}\frac{du}{dy}=(a-b)\,(a-c)\,(a-d)\,\sqrt{\mathrm{abcd}}\,\sqrt{\mathrm{a_2b_2c_2d_2}};$$

and the equation $du=0$ then gives

$$\frac{dx}{\sqrt{\mathrm{abcd}}}+\frac{dy}{\sqrt{\mathrm{a_1b_1c_1d_1}}}+\frac{dz}{\sqrt{\mathrm{a_2b_2c_2d_2}}}=0,$$

as it should do. The differential equation might also have been verified directly from any one of the expressions for

$$\sqrt{\frac{a-z}{d-z}},\quad \sqrt{\frac{b-z}{d-z}}\ \text{or}\ \sqrt{\frac{c-z}{d-z}}.$$

Writing for shortness

$$X=a-x\,.\,b-x\,.\,c-x\,.\,d-x,\ \text{etc.},$$

then the general integral of the differential equation

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}+\frac{dz}{\sqrt{Z}}=0$$

by Abel's theorem is

$$\begin{vmatrix} x^2, & x, & 1, & \sqrt{X} \\ y^2, & y, & 1, & \sqrt{Y} \\ z^2, & z, & 1, & \sqrt{Z} \\ w^2, & w, & 1, & \sqrt{W} \end{vmatrix}=0,$$

where w is the constant of integration: and it is to be shown that the value of w which corresponds to the integral given in the present paper is $w=a$. Observe that writing in the determinant $w=a$, the determinant on putting therein $x=a$, would vanish whether z were or were not $=y$; but this is on account of an extraneous factor $a-w$, so that we do not thus prove the required theorem that (w being $=a$) we have $y=z$ when $x=a$.

An equivalent form of Abel's integral is that there exist values A, B, C such that

$$\begin{aligned} Ax^2+Bx+C&=\sqrt{X},\\ Ay^2+By+C&=\sqrt{Y},\\ Az^2+Bz+C&=\sqrt{Z},\\ Aw^2+Bw+C&=\sqrt{W},\end{aligned}$$

or, what is the same thing, that we have identically

$$(A\theta^2+B\theta+C)^2-\Theta=(A^2-1)\,.\,\theta-x\,.\,\theta-y\,.\,\theta-z\,.\,\theta-w.$$

We have therefore

$$C^2-abcd=(A^2-1)\,xyzw,$$

or say

$$xyzw=\frac{C^2-abcd}{A^2-1};$$

which equation, regarding therein A, B, C as determined by the three equations

$$Ax^2+Bx+C=\sqrt{X},$$
$$Ay^2+By+C=\sqrt{Y},$$
$$Aw^2+Bw+C=\sqrt{W},$$

is a form of Abel's integral, giving z rationally in terms of x, y, w.

Supposing that, when $x=a$, $z=y$: then the last-mentioned integral gives

$$ay^2w=\frac{C^2-abcd}{A^2-1},$$

where A, C are now determined by the equations

$$Aa^2+Ba+C=0,$$
$$Ay^2+By+C=\sqrt{Y},$$
$$Aw^2+Bw+C=\sqrt{W},$$

and, imagining these values actually substituted, it is to be shown that the equation

$$ay^2w=\frac{C^2-abcd}{A^2-1}$$

is satisfied by the value $w=a$.

We have

$$A\,.\,a-y\,.\,a-w\,.\,w-y=\qquad(a-w)\sqrt{Y}-\qquad(a-y)\sqrt{W},$$
$$B\,.\,a-y\,.\,a-w\,.\,w-y=(a-w)(a+w)\sqrt{Y}-(a-y)(a+y)\sqrt{W},$$
$$C\,.\,a-y\,.\,a-w\,.\,w-y=(a-w)\,aw\qquad\sqrt{Y}-(a-y)\,ay\qquad\sqrt{W},$$

or writing as before

$$a-y,\ b-y,\ c-y,\ d-y=\mathrm{a}_1,\ \mathrm{b}_1,\ \mathrm{c}_1,\ \mathrm{d}_1,$$

and also

$$a-w,\ b-w,\ c-w,\ d-w=\mathrm{a}_3,\ \mathrm{b}_3,\ \mathrm{c}_3,\ \mathrm{d}_3,$$

then $Y=\mathrm{a}_1\mathrm{b}_1\mathrm{c}_1\mathrm{d}_1$, $W=\mathrm{a}_3\mathrm{b}_3\mathrm{c}_3\mathrm{d}_3$, and the formulæ become

$$A=\frac{1}{(w-y)\sqrt{\mathrm{a}_1\mathrm{a}_3}}\{\sqrt{\mathrm{a}_3\mathrm{b}_1\mathrm{c}_1\mathrm{d}_1}-\sqrt{\mathrm{a}_1\mathrm{b}_3\mathrm{c}_3\mathrm{d}_3}\},$$

$$B=\frac{1}{(w-y)\sqrt{\mathrm{a}_1\mathrm{a}_3}}\{-(a+w)\sqrt{\mathrm{a}_3\mathrm{b}_1\mathrm{c}_1\mathrm{d}_1}+(a+y)\sqrt{\mathrm{a}_1\mathrm{b}_3\mathrm{c}_3\mathrm{d}_3}\},$$

$$C=\frac{1}{(w-y)\sqrt{\mathrm{a}_1\mathrm{a}_3}}\{aw\sqrt{\mathrm{a}_3\mathrm{b}_1\mathrm{c}_1\mathrm{d}_1}-ay\sqrt{\mathrm{a}_1\mathrm{b}_3\mathrm{c}_3\mathrm{d}_3}\}.$$

If in these formulæ w is indefinitely nearly $=a$, then a_3 is indefinitely small, so that $\sqrt{\mathrm{a}_3\mathrm{b}_1\mathrm{c}_1\mathrm{d}_1}$ may be neglected in comparison with $\sqrt{\mathrm{a}_1\mathrm{b}_3\mathrm{c}_3\mathrm{d}_3}$: also $w-y$ may be put $=\mathrm{a}_1$; the formulæ thus become

$$A=-\frac{\sqrt{\mathrm{b}_3\mathrm{c}_3\mathrm{d}_3}}{\mathrm{a}_1\sqrt{\mathrm{a}_3}},\quad B=(a+y)\frac{\sqrt{\mathrm{b}_3\mathrm{c}_3\mathrm{d}_3}}{\mathrm{a}_1\sqrt{\mathrm{a}_3}},\quad C=-ay\frac{\sqrt{\mathrm{b}_3\mathrm{c}_3\mathrm{d}_3}}{\mathrm{a}_1\sqrt{\mathrm{a}_3}},$$

where the values of A, B, C are each of them indefinitely large on account of the factor $\sqrt{\mathrm{a}_3}$ in the denominator; the value of C is $C=ayA$, and substituting this value in the equation

$$ay^2w=\frac{C^2-abcd}{A^2-1},$$

and then considering A as indefinitely large, the equation becomes $ay^2w=a^2y^2$, that is, $w=a$; so that $w=a$ is a value of w satisfying this equation.

Cambridge, 3 *July*, 1878.