# 703.

## ON THE ADDITION OF THE DOUBLE $\vartheta$-FUNCTIONS.

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I ASSUME in general

$$\Theta = a-\theta \,.\, b-\theta \,.\, c-\theta \,.\, d-\theta \,.\, e-\theta \,.\, f-\theta,$$

and I consider the variables $x$, $y$, $z$, $w$, $p$, $q$, connected by the equations

$$\begin{vmatrix} 1, & 1, & 1, & 1, & 1, & 1 \\ x, & y, & z, & w, & p, & q \\ x^2, & y^2, & z^2, & w^2, & p^2, & q^2 \\ x^3, & y^3, & z^3, & w^3, & p^3, & q^3 \\ \sqrt{X}, & \sqrt{Y}, & \sqrt{Z}, & \sqrt{W}, & \sqrt{P}, & \sqrt{Q} \end{vmatrix} = 0,$$

equivalent to two independent equations, which in fact serve to determine $p$, $q$, or say the symmetrical functions $p+q$ and $pq$, in terms of $x$, $y$, $z$, $w$.

These equations, it is well known, constitute a particular integral of the differential equations

$$\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} + \frac{dz}{\sqrt{Z}} + \frac{dw}{\sqrt{W}} + \frac{dp}{\sqrt{P}} + \frac{dq}{\sqrt{Q}} = 0,$$

$$\frac{x\,dx}{\sqrt{X}} + \frac{y\,dy}{\sqrt{Y}} + \frac{z\,dz}{\sqrt{Z}} + \frac{w\,dw}{\sqrt{W}} + \frac{p\,dp}{\sqrt{P}} + \frac{q\,dq}{\sqrt{Q}} = 0,$$

or what is the same thing, regarding $p$, $q$ as arbitrary constants, they constitute the general integral of the differential equations

$$\frac{dx}{\sqrt{X}} + \frac{dy}{\sqrt{Y}} + \frac{dz}{\sqrt{Z}} + \frac{dw}{\sqrt{W}} = 0,$$

$$\frac{x\,dx}{\sqrt{X}} + \frac{y\,dy}{\sqrt{Y}} + \frac{z\,dz}{\sqrt{Z}} + \frac{w\,dw}{\sqrt{W}} = 0.$$

I attach the numbers 1, 2, 3, 4, 5, 6 to the variables $x$, $y$, $z$, $w$, $p$, $q$, respectively: and write

$$A_{12} = \sqrt{a-x\,.\,a-y}\,;\quad A_{34} = \sqrt{a-z\,.\,a-w}\,;\quad A_{56} = \sqrt{a-p\,.\,a-q}\,;$$

$$\vdots$$

(six equations),

$$AB_{12} = \frac{1}{x-y}\{\sqrt{a-x\,.\,b-x\,.\,f-x\,.\,c-y\,.\,d-y\,.\,e-y} - \sqrt{a-y\,.\,b-y\,.\,f-y\,.\,c-x\,.\,d-x\,.\,e-x}\};\text{ etc.}$$

$$\vdots$$

(ten equations),

where it is to be borne in mind that $AB$ is an abbreviation for $ABF\,.\,CDE$, and so in other cases, the letter $F$ belonging always to the expressed duad: there are thus in all the sixteen functions $A$, $B$, $C$, $D$, $E$, $F$, $AB$, $AC$, $AD$, $AE$, $BC$, $BD$, $BE$, $CD$, $CE$, $DE$, these being functions of $x$ and $y$, of $z$ and $w$, and of $p$ and $q$, according as the suffix is 12, 34, or 56.

It is to be shown that the 16 functions $A_{56}$, $AB_{56}$ of $p$ and $q$ can be by means of the given equations expressed as proportional to rational and integral functions of the 16 functions $A_{12}$, $AB_{12}$, $A_{34}$, $AB_{34}$ of $x$ and $y$, and of $z$ and $w$ respectively: and it is clear that in so expressing them we have in effect the solution of the problem of the addition of the double ϑ-functions.

I use when convenient the abbreviated notations

$$a-x=\mathrm{a}_1,\quad a-y=\mathrm{a}_2,\quad \text{etc.},$$

$$b-x=\mathrm{b}_1,\quad \text{etc.},$$

$$\theta_{12}=x-y,\quad \theta_{34}=z-w,\quad \theta_{56}=p-q\,;$$

we have of course

$$X=\mathrm{a}_1\mathrm{b}_1\mathrm{c}_1\mathrm{d}_1\mathrm{e}_1\mathrm{f}_1,$$

$$A_{12}=\sqrt{\mathrm{a}_1\mathrm{a}_2},$$

$$AB_{12}=\frac{1}{\theta_{12}}\{\sqrt{\mathrm{a}_1\mathrm{b}_1\mathrm{f}_1\mathrm{c}_2\mathrm{d}_2\mathrm{e}_2}-\sqrt{\mathrm{a}_2\mathrm{b}_2\mathrm{f}_2\mathrm{c}_1\mathrm{d}_1\mathrm{e}_1}\},\ \text{etc.}$$

Proceeding to the investigation, the equations between the variables are obviously those obtained by the elimination of the arbitrary multipliers $\alpha$, $\beta$, $\gamma$, $\delta$, $\epsilon$ from the six equations obtained from

$$\alpha\theta^3+\beta\theta^2+\gamma\theta+\delta=\epsilon\sqrt{\Theta},$$

by writing therein for $\theta$ the values $x$, $y$, $z$, $w$, $p$, $q$ successively; we may consider the four equations

$$\alpha x^3+\beta x^2+\gamma x+\delta=\epsilon\sqrt{X},$$

$$\alpha y^3+\beta y^2+\gamma y+\delta=\epsilon\sqrt{Y},$$

$$\alpha z^3+\beta z^2+\gamma z+\delta=\epsilon\sqrt{Z},$$

$$\alpha w^3+\beta w^2+\gamma w+\delta=\epsilon\sqrt{W},$$

as serving to determine the ratios $\alpha : \beta : \gamma : \delta : \epsilon$ in terms of $x, y, z, w$; and we have then for the determination of $p, q$ the remaining two equations

$$\alpha p^3 + \beta p^2 + \gamma p + \delta = \epsilon \sqrt{P},$$
$$\alpha q^3 + \beta q^2 + \gamma q + \delta = \epsilon \sqrt{Q},$$

which two equations may be replaced by the identity

$$(\alpha\theta^3 + \beta\theta^2 + \gamma\theta + \delta)^2 - \epsilon^2\Theta = \alpha^2 - \epsilon^2 . \theta - x . \theta - y . \theta - z . \theta - w . \theta - p . \theta - q.$$

Writing herein $\theta =$ any one of the values $a, b, c, d, e, f$, for instance $\theta = a$, and taking the square root of each side, we have

$$\alpha a^3 + \beta a^2 + \gamma a + \delta = \sqrt{\alpha^2 - \epsilon^2}\sqrt{a - x . a - y}\sqrt{a - z . a - w}\sqrt{a - p . a - q},$$

or as this may be written

$$\alpha a^3 + \beta a^2 + \gamma a + \delta = \sqrt{\alpha^2 - \epsilon^2}\, A_{12} . A_{34} . A_{56},$$

which equation when reduced gives the proportional value of $A_{56}$.

For the reduction we require the value of $\alpha a^3 + \beta a^2 + \gamma a + \delta$: calling this for the moment $\Omega$, we join to the four equations a fifth equation

$$\alpha a^3 + \beta a^2 + \gamma a + \delta = \Omega.$$

Eliminating $\alpha, \beta, \gamma, \delta$, we find

$$\begin{vmatrix} x^3, & x^2, & x, & 1, & \epsilon\sqrt{X} \\ y^3, & y^2, & y, & 1, & \epsilon\sqrt{Y} \\ z^3, & z^2, & z, & 1, & \epsilon\sqrt{Z} \\ w^3, & w^2, & w, & 1, & \epsilon\sqrt{W} \\ a^3, & a^2, & a, & 1, & \Omega \end{vmatrix} = 0,$$

or, what is the same thing,

$$\Omega \begin{vmatrix} x^3, & x^2, & x, & 1 \\ y^3, & y^2, & y, & 1 \\ z^3, & z^2, & z, & 1 \\ w^3, & w^2, & w, & 1 \end{vmatrix} + \epsilon \begin{vmatrix} \sqrt{X}, & x^3, & x^2, & x, & 1 \\ \sqrt{Y}, & y^3, & y^2, & y, & 1 \\ \sqrt{Z}, & z^3, & z^2, & z, & 1 \\ \sqrt{W}, & w^3, & w^2, & w, & 1 \\ & a^3, & a^2, & a, & 1 \end{vmatrix} = 0;$$

viz. this is

$$\begin{aligned} \Omega . x-y . x-z . x-w . y-z . y-w . z-w = -\epsilon \{ & \sqrt{X} . y-z . y-w . y-a . z-w . z-a . w-a \\ & + \sqrt{Y} . z-w . z-a . z-x . w-a . w-x . a-x \\ & + \sqrt{Z} . w-a . w-x . w-y . a-x . a-y . x-y \\ & + \sqrt{W} . a-x . a-y . a-z . x-y . x-z . y-z \}, \end{aligned}$$

or as it may be written

$$\Omega . x-z . x-w . y-z . y-w=\frac{\epsilon . a-z . a-w}{x-y}\{y-z . y-w . a-y . \sqrt{X}-x-z . x-w . a-x . \sqrt{Y}\}$$

$$+\frac{\epsilon . a-x . a-y}{z-w}\{w-x . w-y . a-w . \sqrt{Z}-z-x . z-y . a-z . \sqrt{W}\},$$

an equation for the determination of $\Omega$.

Consider first the expression which multiplies $\epsilon . a-z . a-w$; this is

$$=\frac{1}{\theta_{12}}\{y-z . y-w . \mathrm{a}_2 \sqrt{X}-x-z . x-w . \mathrm{a}_1 \sqrt{Y}\};$$

we have

$$BE_{12}=\frac{1}{\theta_{12}}\{\sqrt{\mathrm{b}_1\mathrm{e}_1\mathrm{f}_1\mathrm{a}_2\mathrm{c}_2\mathrm{d}_2}-\sqrt{\mathrm{b}_2\mathrm{e}_2\mathrm{f}_2\mathrm{a}_1\mathrm{c}_1\mathrm{d}_1}\},$$

and multiplying this by

$$A_{12} . C_{12} . D_{12}, =\sqrt{\mathrm{a}_1\mathrm{c}_1\mathrm{d}_1\mathrm{a}_2\mathrm{c}_2\mathrm{d}_2},$$

we derive

$$BE_{12} . C_{12} . D_{12} . A_{12}=\frac{1}{\theta_{12}}\{\mathrm{c}_2\mathrm{d}_2\mathrm{a}_2 \sqrt{X}-\mathrm{c}_1\mathrm{d}_1\mathrm{a}_1 \sqrt{Y}\},$$

and similarly two other equations; the system may be written

$$BE . C . D . A=\frac{1}{\theta_{12}}\{\mathrm{c}_2\mathrm{d}_2\mathrm{a}_2 \sqrt{X}-\mathrm{c}_1\mathrm{d}_1\mathrm{a}_1 \sqrt{Y}\},$$

$$CE . D . B . A= \text{,,} \ \{\mathrm{d}_2\mathrm{b}_2 \text{,,} \ \text{,,} \ -\mathrm{d}_1\mathrm{b}_1 \text{,,} \ \text{,,} \ \},$$

$$DE . B . C . A= \text{,,} \ \{\mathrm{b}_2\mathrm{c}_2 \text{,,} \ \text{,,} \ -\mathrm{b}_1\mathrm{c}_1 \text{,,} \ \text{,,} \ \},$$

the suffixes on the left-hand side being always 12. The letters $b$, $c$, $d$ which enter cyclically into these equations are any three of the five letters other than $a$; the remaining two letters $e$ and $f$ enter symmetrically, for $BE$ is a mere abbreviation for the double triad $BEF . ACD$; and the like for $CE$, and $DE$.

Multiplying these equations by

$$\frac{b-z . b-w}{b-c . b-d}, \quad \frac{c-z . c-w}{c-d . c-b}, \quad \frac{d-z . d-w}{d-b . d-c},$$

respectively, and then adding, the right-hand side becomes

$$=\frac{1}{\theta_{12}}\{y-z . y-w . \mathrm{a}_2 \sqrt{X}-x-z . x-w . \mathrm{a}_1 \sqrt{Y}\}.$$

Writing

$$\frac{b-z . b-w}{b-c . b-d}=\frac{-1}{c-d . d-b . b-c} . c-d . B_{34}^2, \text{ etc.},$$

the left-hand side becomes

$$=\frac{-A_{12}}{c-d.d-b.b-c}\{c-d.B_{34}^2.BE_{12}.C_{12}.D_{12}+d-b.C_{34}^2.CE_{12}.D_{12}.B_{12}+b-c.D_{34}^2.DE_{12}.B_{12}.C_{12}\},$$

which for shortness may be written

$$=\frac{-A_{12}}{c-d.d-b.b-c}\Sigma\{c-d.B_{34}^2.BE_{12}.C_{12}.D_{12}\},$$

the summation referring to the three terms obtained by the cyclical interchange of the letters $b$, $c$, $d$. The result thus is

$$\frac{1}{\theta_{12}}\{y-z.y-w.\mathrm{a}_2\sqrt{X}-x-z.x-w.\mathrm{a}_1\sqrt{Y}\}$$

$$=\frac{-A_{12}}{c-d.d-b.b-c}\Sigma\{c-d.B_{34}^2.BE_{12}.C_{12}.D_{12}\}.$$

Interchanging $x$, $y$ with $z$, $w$ respectively, we have of course to interchange the suffixes 1, 2 and 3, 4; we thus find

$$\frac{1}{\theta_{34}}\{w-x.w-y.\mathrm{a}_4\sqrt{Z}-z-x.z-y.\mathrm{a}_3\sqrt{W}\}$$

$$=\frac{-A_{34}}{c-d.d-b.b-c}\Sigma\{c-d.B_{12}^2.BE_{34}.C_{34}.D_{34}\},$$

and we hence find the value of $\Omega.x-z.x-w.y-z.y-w$. But $\Omega$, $=\alpha a^3+\beta a^2+\gamma a+\delta$, is $=\sqrt{\alpha^2-\epsilon^2}.A_{12}.A_{34}.A_{56}$: the resulting equation divides by $A_{12}.A_{34}$: throwing out this factor, we have

$$-\frac{\sqrt{\alpha^2-\epsilon^2}}{\epsilon}(x-z.x-w.y-z.y-w)(c-d.d-b.b-c)A_{56}$$

$$=A_{34}\Sigma\{c-d.B_{34}^2.BE_{12}.C_{12}.D_{12}\}+A_{12}\Sigma\{c-d.B_{12}^2.BE_{34}.C_{34}.D_{34}\},$$

where, as before, the summations refer to the three terms obtained by the cyclical interchange of the letters $b$, $c$, $d$; these being any three of the five letters other than $a$; and the remaining two letters $e$, $f$ enter into the formula symmetrically. The formula gives thus for $A_{56}$ ten values which are of course equal to each other.

Writing for $a$ each letter in succession, we obtain formulæ for each of the six single-letter functions $A_{56}$ of $p$ and $q$; and the factor

$$-\frac{\sqrt{\alpha^2-\epsilon^2}}{\epsilon}(x-z.x-w.y-z.y-w)$$

is the same in all the formulæ.

We require further the expressions for the double-letter functions of $p$, $q$. Considering for example the function $DE_{56}$, which is

$$=\frac{1}{\theta_{56}}\{\sqrt{\mathrm{d}_5\mathrm{e}_5\mathrm{f}_5\mathrm{a}_6\mathrm{b}_6\mathrm{c}_6}-\sqrt{\mathrm{d}_6\mathrm{e}_6\mathrm{f}_6\mathrm{a}_5\mathrm{b}_5\mathrm{c}_5}\},$$

then multiplying by

$$A_{56}\,.\,B_{56}\,.\,C_{56},\ = \sqrt{\mathrm{a}_5\mathrm{b}_5\mathrm{c}_5\mathrm{a}_6\mathrm{b}_6\mathrm{c}_6},$$

we have

$$DE_{56}\,.\,A_{56}\,.\,B_{56}\,.\,C_{56} = \frac{1}{\theta_{56}}\{\mathrm{a}_6\mathrm{b}_6\mathrm{c}_6\sqrt{P} - \mathrm{a}_5\mathrm{b}_5\mathrm{c}_5\sqrt{Q}\},$$

$$= \frac{1}{p-q}\{a-q\,.\,b-q\,.\,c-q\,.\sqrt{P} - a-p\,.\,b-p\,.\,c-p\,.\sqrt{Q}\},$$

or recollecting that $\epsilon\sqrt{P}$, $\epsilon\sqrt{Q}$ are $=\alpha p^3+\beta p^2+\gamma p+\delta$ and $\alpha q^3+\beta q^2+\gamma q+\delta$ respectively, this is

$$\epsilon\,.\,DE_{56}\,.\,A_{56}\,.\,B_{56}\,.\,C_{56}$$

$$= \frac{1}{p-q}\{a-q\,.\,b-q\,.\,c-q\,.\,(\alpha p^3+\beta p^2+\gamma p+\delta) - a-p\,.\,b-p\,.\,c-p\,.\,(\alpha q^3+\beta q^2+\gamma q+\delta)\}.$$

Using the well-known identity

$$\begin{aligned}\alpha p^3+\beta p^2+\gamma p+\delta = {} & \alpha a^3+\beta a^2+\gamma a+\delta\,.\,\frac{b-p\,.\,c-p\,.\,d-p}{b-a\,.\,c-a\,.\,d-a}\\ & + \alpha b^3+\beta b^2+\gamma b+\delta\,.\,\frac{c-p\,.\,d-p\,.\,a-p}{c-b\,.\,d-b\,.\,a-b}\\ & + \alpha c^3+\beta c^2+\gamma c+\delta\,.\,\frac{d-p\,.\,a-p\,.\,b-p}{d-c\,.\,a-c\,.\,b-c}\\ & + \alpha d^3+\beta d^2+\gamma d+\delta\,.\,\frac{a-p\,.\,b-p\,.\,c-p}{a-d\,.\,b-d\,.\,c-d},\end{aligned}$$

and the like expression for $\alpha q^3+\beta q^2+\gamma q+\delta$, there will be on the right-hand side terms involving

$$\alpha a^3+\beta a^2+\gamma a+\delta,\quad \alpha b^3+\beta b^2+\gamma b+\delta,\quad \alpha c^3+\beta c^2+\gamma c+\delta:$$

but the term in $\alpha d^3+\beta d^2+\gamma d+\delta$ will disappear of itself.

The term in $\alpha a^3+\beta a^2+\gamma a+\delta$ is

$$\frac{1}{p-q}\,\frac{\alpha a^3+\beta a^2+\gamma a+\delta}{b-a\,.\,c-a\,.\,d-a}\,.\,b-q\,.\,c-q\,.\,b-p\,.\,c-p\,.\,(a-q\,.\,d-p-a-p\,.\,d-q),$$

where the expression in ( ) is $=d-a\,.\,p-q$: hence the term is

$$= \frac{\alpha a^3+\beta a^2+\gamma a+\delta}{b-a\,.\,c-a}\,.\,b-q\,.\,c-q\,.\,b-p\,.\,c-p,$$

which is

$$= \frac{\alpha a^3+\beta a^2+\gamma a+\delta}{b-a\,.\,c-a}\,{B_{56}}^2\,.\,{C_{56}}^2.$$

Forming the two other like terms, the equation is

$$\begin{aligned}\epsilon\,.\,DE_{56}\,.\,A_{56}\,.\,B_{56}\,.\,C_{56} = {} & \frac{\alpha a^3+\beta a^2+\gamma a+\delta}{b-a\,.\,c-a}\,{B_{56}}^2\,.\,{C_{56}}^2\\ & + \frac{\alpha b^3+\beta b^2+\gamma b+\delta}{c-b\,.\,a-b}\,{C_{56}}^2\,.\,{A_{56}}^2\\ & + \frac{\alpha c^3+\beta c^2+\gamma c+\delta}{a-c\,.\,b-c}\,{A_{56}}^2\,.\,{B_{56}}^2.\end{aligned}$$

But the expressions

$$\alpha a^3+\beta a^2+\gamma a+\delta,\quad \alpha b^3+\beta b^2+\gamma b+\delta,\quad \alpha c^3+\beta c^2+\gamma c+\delta,$$

are

$$=\sqrt{\alpha^2-\epsilon^2}\,A_{12}\,.\,A_{34}\,.\,A_{56},\quad \sqrt{\alpha^2-\epsilon^2}\,B_{12}\,.\,B_{34}\,.\,B_{56},\quad \sqrt{\alpha^2-\epsilon^2}\,C_{12}\,.\,C_{34}\,.\,C_{56},$$

respectively: the whole equation thus divides by $A_{56}\,.\,B_{56}\,.\,C_{56}$; throwing out this factor, and then multiplying each side by $-\dfrac{\sqrt{\alpha^2-\epsilon^2}}{\epsilon}$, we find

$$-\frac{\sqrt{\alpha^2-\epsilon^2}}{\epsilon}DE_{56}=\frac{1}{b-c\,.\,c-a\,.\,a-b}\left(-\frac{\sqrt{\alpha^2-\epsilon^2}}{\epsilon}\right)^2\{\;b-c\,.\,A_{12}\,.\,A_{34}\,.\,B_{56}\,.\,C_{56}$$
$$+c-a\,.\,B_{12}\,.\,B_{34}\,.\,C_{56}\,.\,A_{56}$$
$$+a-b\,.\,C_{12}\,.\,C_{34}\,.\,A_{56}\,.\,B_{56}\},$$

in which formula if we imagine

$$-\frac{\sqrt{\alpha^2-\epsilon^2}}{\epsilon}A_{56},\quad -\frac{\sqrt{\alpha^2-\epsilon^2}}{\epsilon}B_{56},\quad -\frac{\sqrt{\alpha^2-\epsilon^2}}{\epsilon}C_{56}$$

each replaced by its value in terms of the *xy*- and *zw*-functions, we have an equation of the form

$$-\frac{\sqrt{\alpha^2-\epsilon^2}}{\epsilon}(x-z\,.\,x-w\,.\,y-z\,.\,y-w)\,DE_{56}=\frac{1}{x-z\,.\,x-w\,.\,y-z\,.\,y-w}M,$$

where $M$ is a given rational and integral function of the 16 and 16 functions $A_{12}$, $AB_{12}$ and $A_{34}$, $AB_{34}$ of $x$ and $y$ and of $z$ and $w$ respectively. The factor

$$-\frac{\sqrt{\alpha^2-\epsilon^2}}{\epsilon}(x-z\,.\,x-w\,.\,y-z\,.\,y-w)$$

is retained on the left-hand side as being the same factor which enters into the equations for $A_{56}$, etc.: but on the right-hand side $x-z\,.\,x-w\,.\,y-z\,.\,y-w$ should be expressed in terms of the *xy*- and *zw*-functions. This can be done by means of the identity

$$x-z\,.\,x-w\,.\,y-z\,.\,y-w=\Sigma\,\frac{\begin{vmatrix}1, & x+y, & xy\\ 1, & z+w, & zw\\ 1, & a+b, & ab\end{vmatrix}\begin{vmatrix}1, & x+y, & xy\\ 1, & z+w, & zw\\ 1, & a+c, & ac\end{vmatrix}}{a-b\,.\,a-c},$$

where the summation refers to the three terms obtained by the cyclical interchange of the letters $a$, $b$, $c$. The first determinant, multiplied by $a-b$, is in fact

$$=\begin{vmatrix}a-z\,.\,a-w, & a-x\,.\,a-y\\ b-z\,.\,b-w, & b-x\,.\,b-y\end{vmatrix},$$

and the second determinant, multiplied by $a-c$, is

$$=\begin{vmatrix}a-z\,.\,a-w, & a-x\,.\,a-y\\ c-z\,.\,c-w, & c-x\,.\,c-y\end{vmatrix},$$

so that the formula may also be written

$$x-z.x-w.y-z.y-w=\Sigma\frac{\begin{vmatrix} a-z.a-w, & a-x.a-y \\ b-z.b-w, & b-x.b-y \end{vmatrix} . \begin{vmatrix} a-z.a-w, & a-x.a-y \\ c-z.c-w, & c-x.c-y \end{vmatrix}}{(a-b)^2(a-c)^2};$$

or, what is the same thing, it is

$$x-z.x-w.y-z.y-w=\Sigma\frac{(A_{34}^2B_{12}^2-A_{12}^2B_{34}^2)(A_{34}^2C_{12}^2-A_{12}^2C_{34}^2)}{(a-b)^2(a-c)^2},$$

which is the required expression for $x-z.x-w.y-z.y-w$; the letters $a$, $b$, $c$, which enter into the formula, are any three of the six letters.

As regards the verification of the identity, observe that it may be written

$$x-z.x-w.y-z.y-w=\Sigma\frac{\{L+M(a+b)+Nab\}\{L+M(a+c)+Nac\}}{a-b.a-c},$$

where $L$, $M$, $N$ are

$$=(x+y)zw-(z+w)xy,\quad xy-zw,\quad \text{and}\quad z+w-x-y:$$

this is readily reduced to

$$x-z.x-w.y-z.y-w=M^2-NL,$$

which can be at once verified.

*Cambridge, 12th March*, 1879.

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I take the opportunity of remarking that, in the double-letter formulæ, the sign of the second term is, not as I have in general written it −, but is +,

$$AB=\frac{1}{x-y}\{\sqrt{\mathrm{abfc_1d_1e_1}}+\sqrt{\mathrm{a_1b_1f_1cde}}\},\ \text{etc.}$$

In fact, introducing a factor $\omega$ which is a function of $x$ and $y$, the odd and even $\vartheta$-functions are $=\omega\sqrt{\mathrm{aa_1}}$, etc., and

$$\frac{\omega}{x-y}\{\sqrt{\mathrm{abfc_1d_1e_1}}+\sqrt{\mathrm{a_1b_1f_1cde}}\},\ \text{etc.},$$

respectively; $\omega$ is a function which on the interchange of $x$, $y$ changes only its sign; and this being so, then when $x$ and $y$ are interchanged, each single-letter function changes its sign, and each double-letter function remains unaltered.

*Cambridge, 29th July*, 1879.