705.

PROBLEMS AND SOLUTIONS.

[From the *Mathematical Questions with their Solutions from the Educational Times*, vols. XIV. to LXI. (1871—1894).]

[Vol. XIV., July to December, 1870, pp. 17—19.]

3002. (PROPOSED by MATTHEW COLLINS, B.A.)—If every two of five circles A, B, C, D, E touch each other, except D and E, prove that the common tangent of D and E is just twice as long as it would be if D and E touched each other.

Solution by PROFESSOR CAYLEY.

Consider the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, foci R, S; the coordinates of a point U on the ellipse may be taken to be $(a\cos u,\ b\sin u)$, and then the distances of this point from the foci will be

$$r=a(1-e\cos u),\quad s=a(1+e\cos u).$$

Taking k arbitrarily, with centre R describe a circle radius $a-k$, with centre S a circle radius $a+k$, and with centre U a circle radius $k-ae\cos u$: say these are the circles R, S, U respectively; the circle U will touch each of the circles R, S (viz. assuming $ae<k<a$, so that the foregoing radii are all positive, it will touch the circle R externally and the circle S internally).

Considering next a point V, coordinates $(a\cos v,\ b\sin v)$, and the circle described about this point with the radius $k-ae\cos v$, say the circle V; this will touch in like manner the circles R, S respectively. And the circles U, V may be made to touch each other externally; viz. this will be the case if squared sum of radii = squared distance of centres, or what is the same thing, squared difference of radii + 4 times the product of radii = squared distance of centres; that is,

$$a^2e^2(\cos u-\cos v)^2+4(k-ae\cos u)(k-ae\cos v)=a^2(\cos u-\cos v)^2+b^2(\sin u-\sin v)^2,$$

or

$$2(k-ae\cos u)(k-ae\cos v)=b^2\{1-\cos(u-v)\}.$$

If for a moment we write $\tan\frac{1}{2}u=x$, $\tan\frac{1}{2}v=y$, and therefore

$$\cos u=\frac{1-x^2}{1+x^2},\quad \cos v=\frac{1-y^2}{1+y^2},\quad \sin u=\frac{2x}{1+x^2},\quad \sin v=\frac{2y}{1+y^2},$$

$$\cos(u-v)=\frac{(1-x^2)(1-y^2)+4xy}{(1+x^2)(1+y^2)},\quad 1-\cos(u-v)=\frac{2(x-y)^2}{(1+x^2)(1+y^2)},$$

we have

$$\left\{k-\frac{ae(1-x^2)}{1+x^2}\right\}\left\{k-\frac{ae(1-y^2)}{1+y^2}\right\}=\frac{b^2(x-y)^2}{(1+x^2)(1+y^2)},$$

or

$$\{k-ae+(k+ae)x^2\}\{k-ae+(k+ae)y^2\}=b^2(x-y)^2,$$

which is readily identified with the circular relation

$$\tan^{-1}y\left(\frac{k+ae}{k-ae}\right)^{\frac{1}{2}}-\tan^{-1}x\left(\frac{k+ae}{k-ae}\right)^{\frac{1}{2}}=\tan^{-1}\left(\frac{k^2-a^2e^2}{a^2-k^2}\right)^{\frac{1}{2}}:$$

or, what is the same thing, in order that the circles U, V may touch, the relation between the parameters u, v must be

$$\tan^{-1}\left\{\left(\frac{k+ae}{k-ae}\right)^{\frac{1}{2}}\tan\tfrac{1}{2}v\right\}-\tan^{-1}\left\{\left(\frac{k+ae}{k-ae}\right)^{\frac{1}{2}}\tan\tfrac{1}{2}u\right\}=\tan^{-1}\left(\frac{k^2-a^2e^2}{a^2-k^2}\right)^{\frac{1}{2}}.$$

Considering in like manner a circle, centre the point W, coordinates $(a\cos w,\ b\sin w)$, and radius $k-ae\cos w$, say the circle W; this will, as before, touch the circles R, S; and we may make W touch each of the circles U, V; viz. we must have

$$\tan^{-1}\left\{\left(\frac{k+ae}{k-ae}\right)^{\frac{1}{2}}\tan\tfrac{1}{2}w\right\}-\tan^{-1}\left\{\left(\frac{k+ae}{k-ae}\right)^{\frac{1}{2}}\tan^{-1}\tfrac{1}{2}v\right\}=\tan^{-1}\left(\frac{k^2-a^2e^2}{a^2-k^2}\right)^{\frac{1}{2}},$$

$$\tan^{-1}\left\{\left(\frac{k+ae}{k-ae}\right)^{\frac{1}{2}}\tan\tfrac{1}{2}u\right\}-\tan^{-1}\left\{\left(\frac{k+ae}{k-ae}\right)^{\frac{1}{2}}\tan^{-1}\tfrac{1}{2}w\right\}=\tan^{-1}\left(\frac{k^2-a^2e^2}{a^2-k^2}\right)^{\frac{1}{2}},$$

where, in the last equation, $\tan^{-1}\left\{\left(\frac{k+ae}{k-ae}\right)^{\frac{1}{2}}\tan\frac{1}{2}u\right\}$ must be considered as denoting its value in the first equation increased by π. Hence, adding the three equations, we have

$$\pi=3\tan^{-1}\left(\frac{k^2-a^2e^2}{a^2-k^2}\right)^{\frac{1}{2}},$$

that is,

$$\left(\frac{k^2-a^2e^2}{a^2-k^2}\right)^{\frac{1}{2}}=\tan\tfrac{1}{3}\pi=\sqrt{3};$$

or

$$k^2 - a^2e^2 = 3(a^2 - k^2),$$

that is,

$$3a^2 - 4k^2 + a^2e^2 = 0;$$

viz. this is the condition for the existence of the three circles U, V, W, each touching the two others, and also the circles R, S.

The circle R lies inside the circle S, and the tangential distance is thus imaginary; but defining it by the equation

squared tangential dist. = squared dist. of centres − squared sum of radii,

the squared tangential distance is

$$= 4a^2e^2 - 4a^2.$$

But if the circles were brought into contact, the distance of the centres would be $2k$, and the value of the squared tangential distance $= 4k^2 - 4a^2$; hence, if this be = one-fourth of the former value, we have

$$4(k^2 - a^2) = a^2e^2 - a^2,$$

that is,

$$3a^2 - 4k^2 + a^2e^2 = 0,$$

the same condition as above. The solution might easily be varied in such wise that the circles R, S should be external to each other, and therefore the tangential distance real; but the case here considered, where the locus of the centres of the circles U, V, W is an ellipse, is the more convenient, and may be regarded as the standard case.

[Vol. XIV., p. 19.]

3144. (Proposed by Professor CAYLEY.)—If the extremities A, A' of a given line AA' describe given lines respectively, show that there is a point rigidly connected with AA' which describes a circle.

[Vol. XIV., pp. 67, 68.]

3120. (Proposed by Professor CAYLEY.)—To find the equation of the Jacobian of the quadric surfaces through the six points

$$(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), (1, 1, 1, 1), (\alpha, \beta, \gamma, \delta).$$

Solution by the PROPOSER.

Writing for shortness

$$a = \beta - \gamma, \quad b = \gamma - \alpha, \quad c = \alpha - \beta, \quad f = \alpha - \delta, \quad g = \beta - \delta, \quad h = \gamma - \delta,$$

(so that $a+h-g=0$, &c., $a+b+c=0$, $af+bg+ch=0$), the six points lie in each of the plane-pairs

$$x(hy-gz+aw)=0, \quad y(-hx+fz+bw)=0,$$
$$z(gx-fy+cw)=0, \quad w(-ax-by-cz)=0.$$

We cannot take these as the four quadrics, on account of the identical equation $0=0$, which is obtained by adding the four equations; but we may take the first three of them for three of the quadrics, and for the fourth quadric the cone, vertex $(0, 0, 0, 1)$, which passes through the other five points; viz. this is

$$a\alpha yz + b\beta zx + c\gamma xy = 0.$$

We write therefore

$$P = x(hy-gz+aw), \quad Q = y(-hx+fz+bw),$$
$$R = z(gx-fy+cw), \quad S = a\alpha yz + b\beta zx + c\gamma xy;$$

and we equate to zero the determinant formed with the derived functions of P, Q, R, S in regard to the coordinates (x, y, z, w) respectively. If, for a moment, we write A, B, C to denote $bg-ch$, $ch-af$, $af-bg$ respectively, it is easily found that the term containing d_xS is

$$(b\beta z + c\gamma y)\, x\,(-agh,\ bhf,\ cfg,\ abc,\ -af^2,\ -gB,\ hC,\ aA,\ b^2g,\ -c^2h \between x, y, z, w)^2:$$

the terms containing d_yS and d_zS are derived from this by a mere cyclical interchange of the letters (x, y, z), (A, B, C), (a, b, c), and (f, g, h). Collecting and reducing, it is found that the whole equation divides by $2abc$; and that, omitting this factor, the result is

$$\left.\begin{array}{r} ayz(\alpha w^2-\delta x^2)+fxw(\beta z^2-\gamma y^2) \\ +\,bzx(\beta w^2-\delta y^2)+gyw(\gamma x^2-\alpha z^2) \\ +\,cxy(\gamma w^2-\delta z^2)+hzw(\alpha y^2-\beta x^2) \end{array}\right\} = 0,$$

which, substituting for a, b, c, f, g, h their values, is the required form.

If, in the equation, we write for instance $x=0$, the equation becomes

$$\alpha yzw(hy-gz+aw)=0;$$

or, the section by the plane is made up of four lines. Calling the given points 1, 2, 3, 4, 5, 6, it thus appears that the surface contains the fifteen lines 12, 13, ..., 56, and also the ten lines 123.456, &c.; in all twenty-five lines. Moreover, since the surface contains the lines 12, 13, 14, 15, 16, it is clear that the point 1 is a node (conical point) on the surface; and the like as to the points 2, 3, 4, 5, 6.

[Vol. XIV., pp. 104, 105.]

3249. (Proposed by Professor CAYLEY.)—Given on a given conic two quadrangles $PQRS$ and $pqrs$, having the same centres, and such that P, p; Q, q; R, r; S, s are the corresponding vertices (that is, the four lines PQ, RS, pq, rs all pass through the same point; and similarly the lines PR, QS, pr, qs, and the lines PS, QR, ps, qr): it is required to show that a conic may be drawn, passing through the points p, q, r, s and touched at these points by the lines pP, qQ, rR, sS, respectively.

Solution by the PROPOSER.

Taking the centres for the vertices of the fundamental triangle, the equation of the given conic may be taken to be $x^2+y^2+z^2=0$; and then the coordinates of P, Q, R, S to be (A, B, C), $(A, -B, C)$, $(A, B, -C)$, $(A, -B, -C)$ respectively, where $A^2+B^2+C^2=0$; and those of p, q, r, s to be (α, β, γ), $(\alpha, -\beta, \gamma)$, $(\alpha, \beta, -\gamma)$, $(\alpha, -\beta, -\gamma)$ respectively, where $\alpha^2+\beta^2+\gamma^2=0$. The required conic, assuming it to exist, will be given by an equation of the form $lx^2+my^2+nz^2=0$. This must pass through the point (α, β, γ), and the tangent at this point must be

$$x(B\gamma-C\beta)+y(C\alpha-A\gamma)+z(A\beta-B\alpha)=0\,;$$

that is, we must have $l\alpha^2+m\beta^2+n\gamma^2=0$, and

$$l\alpha : m\beta : n\gamma = B\gamma - C\beta : C\alpha - A\gamma : A\beta - B\alpha.$$

The first condition is obviously included in the second; and the second condition remains unaltered if we reverse the signs of B, β, or of C, γ, or of B, β and C, γ. Hence the conic passing through p, and touched at this point by pP, will also pass through the points q, r, s, and be touched at these points by the lines qQ, rR, sS, respectively; that is, the equation of the required conic is

$$\frac{B\gamma-C\beta}{\alpha}x^2+\frac{C\alpha-A\gamma}{\beta}y^2+\frac{A\beta-B\alpha}{\gamma}z^2=0\,;$$

or, what is the same thing,

$$\begin{vmatrix} \beta\gamma x^2, & \gamma\alpha y^2, & \alpha\beta z^2 \\ A\ , & B\ , & C \\ \alpha\ , & \beta\ , & \gamma \end{vmatrix}=0.$$

[Vol. XV., January to June, 1871, pp. 17—20.]

3206. (Proposed by Professor CAYLEY.)—In how many geometrically distinct ways can nine points lie in nine lines, each through three points?

3278. (Proposed by Professor CAYLEY.)—It is required, with nine numbers each taken three times, to form nine triads containing twenty-seven distinct duads (or, what is the same thing, no duad twice), and to find in how many essentially distinct ways this can be done.

Solution by the PROPOSER.

Let the numbers be 1, 2, 3, 4, 5, 6, 7, 8, 9. Any number, say 1, enters into three triads, no two of which have any number in common. We may take these triads to be 123, 145, 167. There remain the two numbers 8, 9; and these are, or are not, a duad of the system.

First Case.—8 and 9 a duad. In the triad which contains 89, the remaining number cannot be 1; it must therefore be one of the numbers 2, 3; 4, 5; 6, 7; and it is quite immaterial which; the triad may therefore be taken to be 289. There is one other triad containing 2, the remaining two numbers thereof being taken from the numbers 4, 5; 6, 7. They cannot be 4, 5 or 6, 7; and it is indifferent whether they are taken to be 4, 6; 4, 7; 5, 6, or 5, 7: the triad is taken to be 247. We have thus the triads

123, 145, 167, 289, 247;

and we require two triads containing 8 and two triads containing 9. These must be made up with the numbers 3, 4, 5, 6, 7: but as no one of them can contain 47, it follows that, of the two pairs which contain 8 and 9 respectively, one pair must be made up with 3, 5, 6, 7, and the other pair with 3, 5, 6, 4; say, the pairs which contain 8 are made up with 3, 5, 6, 7, and those which contain 9 are made up with 3, 5, 6, 4 (since obviously no distinct case would arise by the interchange of the numbers 8, 9). The triads which contain 8 must contain each of the numbers 3, 5, 6, 7, and they cannot be 835, 867, since we have 67 in the triad 167; similarly the triads which contain 9 must contain each of the numbers 3, 5, 6, 4, and they cannot be 845, 836, since we have 45 in 145. Hence the triads can only be

836, 857	934, 956,
837, 856	935, 946;

and clearly the top row of 8 must combine with the top row of 9, and the bottom row of 8 with the bottom row of 9; that is, the system of the nine triads is

123, 145, 167, 289, 247,

in combination with

836, 857, 934, 956,

or else in combination with

837, 856, 935, 946.

These are really systems of the same form, that is, each of them is of the form

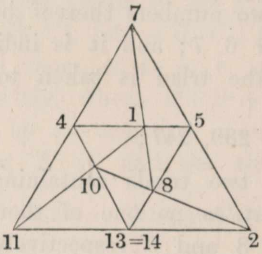
$BC\alpha$	$\beta\gamma a$	bcC
$CA\beta$	$\gamma\alpha b$	caA
$AB\gamma$	$\alpha\beta c$	abB;

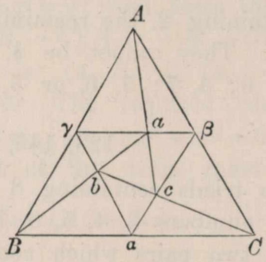
viz. in the first and second systems respectively we have

A	B	C	α	β	γ	a	b	c	
6	1	3	2	8	7	5	4	9	(First system),
5	1	3	2	9	4	6	7	8	(Second system),

as one out of many ways of effecting the identification. Observe that there is not in the system any triad of triads containing all the numbers. It thus appears that 8, 9, a duad, gives only a single form of the system.

Cor.—It is possible to find in a plane nine points such that the points belonging to the same triad lie *in lineâ*. The nine points are, in fact, on a cubic curve; and the figure is that belonging to a theorem of Prof. Sylvester's, according to which it is possible to find on a cubic curve a system of points 1, 2, 4, 5, 7, 8, &c., (a series of





numbers not divisible by 3), such that for any triad (such as 145) where the sum of the numbers, one taken negatively, $=0$, the three points are *in lineâ;* and so also that, if two of the points become identical, in the figure $13 = 14$, then there is not any new point, but the preceding points are indefinitely repeated; thus, 2, 14, 16 being *in lineâ*, and 14 being $=13$, 16 must be $=11$, and so on.

Second and Third Cases.—8 and 9 do not form a duad. There are thus three triads composed of 8 with (2, 3; 4, 5; 6, 7), and three triads composed of 9 with (2, 3; 4, 5; 6, 7). If with these numbers (2, 3; 4, 5; 6, 7) we form all the arrangements of three duads other than those which contain all or any of the duads 23, 45, 67, there are the eight arrangements

$$\begin{aligned} A &= 24,\ 37,\ 56, & E &= 26,\ 35,\ 47, \\ B &= 24,\ 36,\ 57, & F &= 26,\ 34,\ 57, \\ C &= 25,\ 36,\ 47, & G &= 27,\ 34,\ 56, \\ D &= 25,\ 37,\ 46, & H &= 27,\ 35,\ 46, \end{aligned}$$

where A has a duad in common with B, with D, and with G: but it has no duad in common with C, E, F, or H. We have thus the sixteen pairs

$$\begin{array}{llll} AC, & AE, & AF, & AH, \\ BD, & BE, & BG, & BH, \\ CF, & CG, & CH, & \\ DE, & DF, & DG, & \\ EG, & FH, & & \end{array}$$

where each pair contains six different duads.

Combining AC with 8, 9, we have the triads 8 (24, 37, 56) and 9 (24, 36, 57), that is, the triads

$$824,\ 837,\ 856:\ 924,\ 936,\ 957:$$

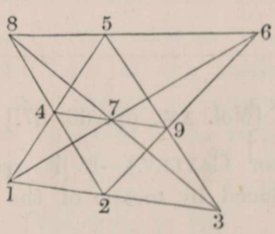
which, with the original three triads 123, 145, 167, form a system of nine triads; 8 and 9 might, of course, be interchanged, but no essentially distinct system would arise thereby. Hence we have a system of nine triads by combining the original three triads 123, 145, 167, with any one of the sixteen pairs AC, AE, &c. But it is sufficient to consider the combinations of the three triads with each of the pairs AC, AE, AF, AH; in fact, these are the only systems which contain the triad 824; and since there is no distinction between the two pairs 4, 5 and 6, 7, or between the two numbers of the same pair, it is allowable to take 824 as a triad of the system. Hence—

Second Case.—The system consists of the three triads combined with AE; viz. it is

$$123,\ 145,\ 167:\ 824,\ 837,\ 856:\ 926,\ 935,\ 947:$$

which, it is to be observed, consists of three triads of triads, each triad of triads containing all the nine numbers; viz. the system is

$$123,\ 479,\ 568:\ 145,\ 269,\ 378:\ 167,\ 248,\ 359.$$



Cor.—We may have nine points such that the points belonging to the same triad lie *in lineâ*, viz. the figure is that of Pascal's hexagon when the conic is a line-pair.

Third Case.—Combining the three triads with AC, AF, or AH, it is readily seen that we obtain in each case a system of the form

$$\begin{array}{lll} A\alpha\alpha', & A\beta\gamma, & A\beta'\gamma', \\ B\beta\beta', & B\gamma\alpha, & B\gamma'\alpha', \\ C\gamma\gamma', & C\alpha\beta, & C\alpha'\beta', \end{array}$$

viz. in the case where the pair is AC; that is, the system is

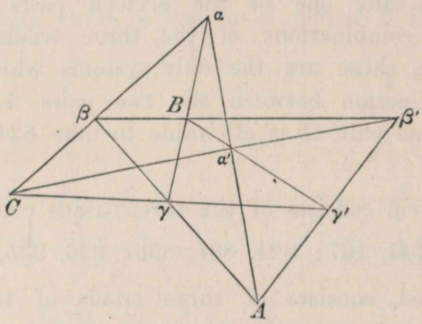
$$123,\ 145,\ 167:\ 824,\ 837,\ 856:\ 925,\ 936,\ 947;$$

and in the cases where the pair is AF or AH, the identifications may be taken to be

A	B	C	α	β	γ	α'	β'	γ'	
9,	8,	1;	4,	5,	2;	7,	6,	3	...... (AC),
9,	8,	1;	2,	3,	4;	6,	7,	5	...... (AF),
9,	8,	1;	5,	4,	6;	3,	2,	7	...... (AH).

Observe that there is in the system a single triad of triads $A\alpha\alpha'$, $B\beta\beta'$, $C\gamma\gamma'$, containing all the numbers; viz. for the system with AC, this is 123, 856, 947; for the system with AF, it is 145, 837, 926; and for the system with AH, it is 167, 824, 935.

Cor.—It is possible to find a system of nine points such that the points belonging to the same triad lie *in lineâ.* Such a figure is this:—



The solution shows that these are the only systems of nine points satisfying the prescribed conditions.

[Vol. xv., pp. 66, 67.]

3329. (Proposed by Professor Cayley.)—It is required to show that every permutation of 12345 can be produced by means of the cyclical substitution (12345), and the interchange (12).

Solution by the Proposer.

It is sufficient to show that the interchanges (13), (14), (15) can be so produced; for then, with the interchanges (12), (13), (14), (15), we can, by at most two such interchanges, bring any number into any place.

Writing $P = (12345)$, $\alpha = (12)$, we have

$$(12) = \alpha,$$
$$(13) = \alpha P \alpha P^4 \alpha,$$
$$(14) = \alpha P \alpha P^4 \alpha P^2 \alpha P^3 \alpha P \alpha P^4 \alpha,$$
$$(15) = P^4 \alpha P,$$

as can be at once verified; and the theorem is thus proved.

I remark that, starting with any two or more substitutions, and combining them in every possible manner (each of them being repeatable an indefinite number of times), we obtain a "group"; viz. this is either (as in the problem proposed) the system of all the substitutions (or say the entire group), or else it is a system the number of whose terms is a submultiple of the whole number of substitutions. The interesting question is, to determine those two or more substitutions, which, by their combination as above, do *not* give the entire group; for in this way we should arrive at all the forms of a submultiple group.

[Vol. XV., p. 80.]

3356. (Proposed by Professor CAYLEY.)—If the roots (α, β, γ, δ) of the equation $(a, b, c, d, e)(u, 1)^4 = 0$ are no two of them equal; and if there exist unequal magnitudes θ and ϕ, such that

$$(\theta+\alpha)^4 : (\theta+\beta)^4 : (\theta+\gamma)^4 : (\theta+\delta)^4 = (\phi+\alpha)^4 : (\phi+\beta)^4 : (\phi+\gamma)^4 : (\phi+\delta)^4;$$

show that the cubinvariant

$$ace - ad^2 - b^2e - c^3 + 2bcd = 0;$$

and find the values of θ, ϕ.

[Vol. XVI., June to December, 1871, p. 65.]

3507. (Proposed by Professor CAYLEY.)—Show that, for the quadric cones which pass through six given points, the locus of the vertices is a quartic surface having upon it twenty-five right lines; and, thence or otherwise, that for the quadric cones passing through seven given points the locus of the vertices is a sextic curve.

[Vol. XVI., p. 90.]

3536. (Proposed by Professor CAYLEY.)—A particle describes an ellipse under the simultaneous action of given central forces, each varying as (distance)$^{-2}$, at the two foci respectively: find the differential relation between the time and the excentric anomaly.

[Vol. XVII., January to June, 1872, p. 35.]

3591. (Proposed by Professor CAYLEY.)—If in a plane A, B, C, D are fixed points and P a variable point, find the linear relation

$$\alpha . PAB + \beta . PBC + \gamma . PCD + \delta . PDA = 0,$$

which connects the areas of the triangles PAB, &c.

[Vol. XVII., p. 49.]

2652. (Proposed by Professor CAYLEY.)—Find the differential equation of the parallel surfaces of an ellipsoid.

[Vol. XVII., p. 60.]

3677. (Proposed by Professor CAYLEY.)—Find at any point of a plane curve the angle between the normal and the line drawn from the point to the centre of the chord parallel and indefinitely near to the tangent at the point; and examine whether a like question applies to a point on a surface and the indicatrix section at such point.

[Vol. XVII., p. 72.]

3564. (Proposed by Professor CAYLEY.)—To determine the least circle enclosing three given points.

[Vol. XVIII., July to December, 1872, p. 68.]

3875. (Proposed by Professor CAYLEY.)—Given the constant a and the variables x, y, to construct mechanically $\frac{a^2-x^2}{y}$; or what is the same thing, given the fixed points A, B, and the moving point P, to mechanically connect therewith a point P' such that PP' shall be always at right angles to AB, and the point P' in the circle APB.

[Vol. XX., July to December, 1873, pp. 106, 107.]

3430. (Proposed by W. J. C. MILLER.)—Find the equation of the first negative focal pedal of (1) an ellipsoid, and (2) an ellipse.

Solution by PROFESSOR CAYLEY.

1. It is easily seen that if a sphere be drawn, passing through the centre of the given quadric and touching it at any point (x', y', z'), then the point (x, y, z) on the required surface, which corresponds to (x', y', z'), is the extremity of the diameter of this sphere which passes through the centre of the quadric. We thus easily find the expressions

$$x = x'\left(2-\frac{t}{a^2}\right),\quad y = y'\left(2-\frac{t}{b^2}\right),\quad z = z'\left(2-\frac{t}{c^2}\right);$$

where

$$t = x'^2+y'^2+z'^2.$$

Solving these equations for x', y', z', and substituting in the two equations

$$xx'+yy'+zz' = x'^2+y'^2+z'^2,\quad \frac{x'^2}{a^2}+\frac{y'^2}{b^2}+\frac{z'^2}{c^2} = 1,$$

we get

$$\frac{x^2}{\left(2-\dfrac{t}{a^2}\right)}+\frac{y^2}{\left(2-\dfrac{t}{b^2}\right)}+\frac{z^2}{\left(2-\dfrac{t}{c^2}\right)}=t \quad \text{.........................(1)},$$

$$\frac{x^2}{a^2\left(2-\dfrac{t}{a^2}\right)^2}+\frac{y^2}{b^2\left(2-\dfrac{t}{b^2}\right)^2}+\frac{z^2}{c^2\left(2-\dfrac{t}{c^2}\right)^2}=1 \quad \text{.........................(2)}.$$

Since (2) is the differential with respect to t of (1), the result of eliminating t between these two equations is the discriminant of (1). Hence the equation of the required surface is the discriminant of (1) with respect to t. Since (1) is only of the fourth degree, this discriminant is easily formed. If (1) be written in the form

$$At^4+4Bt^3+6Ct^2+4Dt+E=0,$$

it will be found that A and B do not contain x, y, z, while C, D, E contain them, each in the second degree. Now the discriminant is of the sixth degree in the coefficients, and of the form $A\phi+B^2\psi$ (see Salmon's *Higher Algebra*, § 107); hence it contains x, y, z only in the tenth degree. This is therefore the degree of the required surface.

The section of this derived surface by the principal plane z consists of the discriminant of

$$\frac{x^2}{2-\dfrac{t}{a^2}}+\frac{y^2}{2-\dfrac{t}{b^2}}=t \quad \text{.................................. (3)},$$

$\left(\text{which is of the sixth degree, and is the first negative pedal of } \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1\right)$, together with the conic (taken twice), which is obtained by putting $t=2c^2$ in (3).

This conic, which is a double curve on the surface, touches the curve of the sixth degree in four points.

2. The formulæ for the conic are quite analogous to those for the ellipsoid, viz. we have

$$x=X\left\{2-\frac{1}{a^2}(X^2+Y^2)\right\},\quad y=Y\left\{2-\frac{1}{b^2}(X^2+Y^2)\right\},$$

leading to the equations

$$\theta=\frac{x^2}{2-\dfrac{\theta}{a^2}}+\frac{y^2}{2-\dfrac{\theta}{b^2}},$$

and its derived equation, from which to eliminate θ. The first is the cubic equation $(A, B, C, D)(\theta, 1)^3=0$, where

$$A=1,\quad B=-\tfrac{2}{3}(a^2+b^2),\quad C=\tfrac{1}{3}(a^2x^2+b^2y^2+4a^2b^2),\quad D=-2a^2b^2(x^2+y^2).$$

Equating the discriminant to zero, this is

$$0 = A^2 \nabla = 4(AC - B^2)^3 - (3ABC - A^2D - 2B^3)^2.$$

Or finally

$$(3a^2x^2 + 3b^2y^2 - 4a^4 + 4a^2b^2 - 4b^4)^3$$
$$+ \{9(a^2 - 2b^2)a^2x^2 + 9(b^2 - 2a^2)b^2y^2 - 8a^6 + 12a^4b^2 + 12a^2b^4 - b^6\}^2 = 0,$$

which is the required equation.

[Vol. XXI., January to June, 1874, pp. 29, 30.]

4298. (Proposed by J. W. L. GLAISHER, B.A.)—With four given straight lines to form a quadrilateral inscribable in a circle.

Solution by PROFESSOR CAYLEY.

Let the sides of the quadrilateral taken in order be a, b, c, d; and let its diagonals be x, y; viz. x the diagonal joining the intersection of the sides a, b with that of the sides c, d; y the diagonal joining the intersection of the sides a, d with that of the sides b, c; then, the quadrilateral being inscribed in a circle, the opposite angles are supplementary to each other. Suppose for a moment that the angles subtended by the diagonal x are θ, $\pi - \theta$, we have

$$x^2 = b^2 + c^2 + 2bc\cos\theta, \quad x^2 = a^2 + d^2 - 2ad\cos\theta;$$

and thence

$$(ad + bc)x^2 = ad(b^2 + c^2) + bc(a^2 + d^2) = (ac + bd)(ab + cd),$$

that is,

$$x^2 = (ac + bd)\frac{ab + cd}{ad + bc},$$

and similarly

$$y^2 = (ac + bd)\frac{ad + bc}{ab + cd},$$

agreeing as they should do with the known relation $xy = ac + bd$: the quadrilateral is thus determined by means of either of its diagonals. It is however interesting to treat the question in a different manner.

Considering a, b, c, d, x, y as the sides and diagonals of a quadrilateral, we have between these quantities a given relation, say

$$F(a, b, c, d, x, y) = 0,$$

and the quadrilateral being inscribed in a circle, we have also the relation $xy = ac + bd$; which two equations determine x, y; and thus give the solution of the problem.

The expression of the function F is in effect given in my paper, "Note on the value of certain determinants, &c.," *Quarterly Mathematical Journal*, t. III. (1860), pp. 275—277, [286]; viz. a, b, c being the edges of any face, and f, g, h the remaining edges of a tetrahedron, then

$$\begin{aligned}\text{volume} = \tfrac{1}{144}\ \{&b^2c^2\ (g^2+h^2) + c^2a^2(h^2+f^2) + a^2b^2(f^2+g^2)\\ +\ &g^2h^2\ (b^2+c^2) + h^2f^2(c^2+a^2) + f^2g^2(a^2+b^2)\\ -\ &a^2f^2\ (a^2+f^2) - b^2g^2(b^2+g^2) + c^2h^2(c^2+h^2)\\ -\ &a^2g^2h^2 - b^2h^2f^2 - c^2f^2g^2 - a^2b^2c^2\},\end{aligned}$$

where, when the tetrahedron becomes a quadrilateral, the volume is $=0$.

In this formula, changing c, b, h, g, f, a into a, h, c, d, x, y, we have the required equation $F=0$; viz. this is found to be

$$\begin{aligned}a^2b^2c^2 + b^2c^2d^2 + c^2d^2a^2 + d^2a^2b^2 - b^2d^2(b^2+d^2) - a^2c^2(a^2+c^2) + x^2y^2(a^2+b^2+c^2+d^2-x^2-y^2)\\ + x^2(a^2c^2+b^2d^2-a^2d^2-b^2c^2) + y^2(a^2c^2+b^2d^2-a^2b^2-c^2d^2) = 0,\end{aligned}$$

which, with $xy = ac+bd$, determines x, y. Substituting in the foregoing equation for xy its value, the equation becomes

$$(ad+bc)^2x^2 + (ab+cd)^2y^2 = 2\{a^2b^2c^2 + b^2c^2d^2 + c^2d^2a^2 + d^2a^2b^2 + abcd(a^2+b^2+c^2+d^2)\},$$

or

$$(ad+bc)^2x^2 + (ab+cd)^2y^2 = 2(ad+bc)(ab+cd)(ac+bd).$$

To show more clearly how this equation arises, I observe that we have identically

$$\begin{aligned}F - (a^2+b^2+c^2+d^2-x^2-y^2)(xy+ac+bd)(xy-ac-bd) - 2(ad+bc)(ab+cd)(xy-ac-bd)\\ = \{(ad+bc)x - (ab+cd)y\}^2.\end{aligned}$$

The resulting equation $(ad+bc)x-(ab+cd)y=0$, together with $xy=ac+bd$, gives for x, y the foregoing values.

[Vol. XXI., pp. 81, 82.]

4392. (Proposed by S. ROBERTS, M.A.)—If N_p denotes the number of terms in a symmetrical determinant of p rows and columns, show that the successive numbers are given by the equation

$$N_k - N_{k-1} - (k-1)^2N_{k-2} + \tfrac{1}{2}(k-1)(k-2)\{N_{k-3} + (k-3)N_{k-4}\} = 0,$$

k being positive and N_0 being taken equal to unity.

Solution by PROFESSOR CAYLEY.

It is a curious coincidence that the question of determining the number of distinct terms in a symmetrical determinant has been recently solved by Captain Allan

Cunningham in a paper in the last number of the *Quarterly Journal of Science**; and the question having been proposed to me by Mr Glaisher, I have also solved it in a paper [580] printed in the April Number of the *Monthly Notices of the Royal Astronomical Society.* I there obtain

$$N_k = 1 \,.\, 2 \ldots k \text{ coeff. } x^k \text{ in } \frac{e^{\frac{1}{2}x+\frac{1}{4}x^2}}{(1-x)^{\frac{1}{2}}},$$

viz. writing

$$u = N_0 + N_1 \frac{x}{1} + N_2 \frac{x^2}{1\,.\,2} + \ldots,$$

I show that u satisfies the differential equation

$$2\frac{du}{dx} = \left\{1 + x + \frac{1}{1-x}\right\} u,$$

giving when the constant is determined

$$u = \frac{e^{\frac{1}{2}x+\frac{1}{4}x^2}}{(1-x)^{\frac{1}{2}}}.$$

Writing the differential equation in the form

$$2(1-x)\frac{du}{dx} = (2-x^2)\,u,$$

we at once obtain for N_k the equation of differences

$$N_k - kN_{k-1} + \tfrac{1}{2}(k-1)(k-2)\,N_{k-3} = 0,$$

which is in fact a particular first integral of Mr Roberts's equation; viz. from the above equation we have

$$N_{k-1} - (k-1)\,N_{k-2} + \tfrac{1}{2}(k-2)(k-3)\,N_{k-4} = 0,$$

and multiplying this last by $k-1$ and adding, we have

$$N_k - N_{k-1} - (k-1)^2 N_{k-2} + \tfrac{1}{2}(k-1)(k-2)\{N_{k-3} + (k-3)\,N_{k-4}\} = 0,$$

which is the equation obtained by Mr Roberts. It thence appears that the *general* first integral of his equation is

$$N_k - kN_{k-1} + \tfrac{1}{2}(k-1)(k-2)\,N_{k-3} = (-)^k\,C1\,.\,2\ldots(k-1).$$

The equation

$$N_k = kN_{k-1} - \tfrac{1}{2}(k-1)(k-2)\,N_{k-3}$$

gives very readily the numerical values, viz.

$1 = 1\,.\,1 - 0$	$17 = 4\,.\,5 - 3\,.\,1$	$2461 = 7\,.\,388 - 15\,.\,17$
$2 = 2\,.\,1 - 0$	$73 = 5\,.\,17 - 6\,.\,2$	$18155 = 8\,.\,2461 - 21\,.\,73.$
$5 = 3\,.\,2 - 1\,.\,1$	$388 = 6\,.\,73 - 10\,.\,5$	

* I have not the volume at hand to refer to, but he obtains an equation of differences, and gives the numbers 1, 2, 5, 73, 398 (should be 388), ...

[Vol. XXII., July to December, 1874, pp. 20, 21.]

4354. (Proposed by R. TUCKER, M.A.)—Solve the equations

$$-x^2+xy+xz=a=4 \quad\ldots\ldots\ldots\ldots(1),$$
$$-y^2+xy+yz=b=-20 \quad\ldots\ldots\ldots\ldots(2),$$
$$-z^2+xz+yz=c=-8 \quad\ldots\ldots\ldots\ldots(3).$$

Note on Question 4354. *By* PROFESSOR CAYLEY.

A question of simple algebra such as this, becomes more interesting when interpreted geometrically: thus, writing the equations in the form

$$-x^2+xy+xz=aw^2,\quad yx-y^2+yz=bw^2,\quad zx+zy-z^2=cw^2,$$

and then putting for shortness

$$\alpha=-a+b+c,\quad \beta=a-b+c,\quad \gamma=a+b-c,$$

the solutions obtained are

$$x : y : z : w=a\alpha : b\beta : c\gamma : (\alpha\beta\gamma)^{\frac{1}{2}},$$
$$x : y : z : w=a\alpha : b\beta : c\gamma : -(\alpha\beta\gamma)^{\frac{1}{2}};$$

say these are

$$\{a\alpha,\ b\beta,\ c\gamma,\ (\alpha\beta\gamma)^{\frac{1}{2}}\} \text{ and } \{a\alpha,\ b\beta,\ c\gamma,\ -(\alpha\beta\gamma)^{\frac{1}{2}}\}.$$

But the equations are also satisfied by

$$(x=0,\ y=z,\ w=0),\quad (y=0,\ z=x,\ w=0),\quad (z=0,\ x=y,\ w=0),$$

or what is the same thing, (0, 1, 1, 0), (1, 0, 1, 0), (1, 1, 0, 0). The three equations represent quadric surfaces, each two of them intersecting in a proper quadric curve, and the three having in common 8 points; viz. these are made up of the first mentioned two points each once, and the last mentioned three points each twice: $2+3.2,=8$.

To verify this, observe that, at each of the three points, the tangent planes of the surfaces have a common line of intersection; this line is the tangent of the curve of intersection of any two of the surfaces, and the curve of intersection therefore touches the third surface; wherefore the point counts for two intersections. In fact, taking (X, Y, Z, W) as current coordinates, the equations of the tangent planes at the point (x, y, z, w) are

$$\begin{array}{llll} X(2x-y-z)-Yx & & -Zx & +2aWw=0, \\ -Xy & +Y(-x+2y-z) & -Zy & +2bWw=0, \\ -Xz & -Yz & +Z(-x-y+2z) & +2cWw=0: \end{array}$$

hence at the point (0, 1, 1, 0) these equations are

$$-2X=0,\quad X+Y-Z=0,\quad -X-Y+Z=0,$$

which three planes meet in the line $X=0$, $Y-Z=0$; and similarly for the other two of the three points.

[Vol. XXII., pp. 60—64.]

4458. (Proposed by Professor CAYLEY.)—Find (1) the intersections of the two quartic curves

$$\lambda (ab-xy)^2 = abx(a-y)(b-y), \quad \mu(ab-xy)^2 = aby(a-x)(b-x);$$

and (2) trace the curves in some particular cases; for instance, when $a=1$, $b=2$, $\lambda=1$, $\mu=-2$.

Solution by the PROPOSER.

1. The 16 intersections are made up as follows: 5 points at infinity on the line $x=0$, 5 at infinity on the line $y=0$, the two points $(x=a,\ y=b)$, $(x=b,\ y=a)$, and 4 other points, $16=5+5+2+4$. As to the points at infinity, observe that, as regards the first curve, the point at infinity on the line $x=0$ is a flecnode having this line for a tangent to the flecnodal branch; and, as regards the second curve, the same point is a cusp, having this line for its tangent; hence the point in question counts as $2+3,\ =5$ intersections; and the like as to the point at infinity on the line $y=0$. It remains to find the coordinates of the 4 points of intersection. Assume $xy=ab\omega$, then the equations become

$$\lambda(1-\omega)^2 = x+\omega y-(a+b)\omega, \quad \mu(1-\omega)^2 = \omega x + y-(a+b)\omega;$$

hence, eliminating successively y and x, the factor $1-\omega$ divides out,—this factor belongs to the points $(x=a,\ y=b)$, $(x=b,\ y=a)$ for which obviously $\omega=1$—, and the equations become

$$(\lambda-\mu\omega)(1-\omega)+(a+b)\omega = (1+\omega)x, \quad (\mu-\lambda\omega)(1-\omega)+(a+b)\omega=(1+\omega)y.$$

Multiplying these two equations together, and substituting for xy its value $ab\omega$, we find

$$\{(\lambda-\mu\omega)(\mu-\lambda\omega)+(a+b)(\lambda+\mu)\omega\}(1-\omega)^2+(a+b)^2\omega^2-(1+\omega)^2\omega ab=0.$$

Write, for shortness, $p=(\lambda+\mu)(a+b)-\lambda^2-\mu^2$, then, dividing by ω^2, and writing $\omega+\dfrac{1}{\omega}=\Omega$, the equation is

$$(\lambda\mu\Omega+p)(\Omega-2)+(a+b)^2-ab(\Omega+2)=0;$$

viz. this is a quadric equation for Ω. But, instead of Ω, it is convenient to introduce the quantity $\theta,\ =\dfrac{\Omega-2}{\Omega+2},\ =\left(\dfrac{\omega-1}{\omega+1}\right)^2$. The equation thus becomes

$$\left\{2\lambda\mu\,\frac{1+\theta}{1-\theta}+p\right\}\frac{4\theta}{1-\theta}+(a+b)^2-ab\,\frac{4}{1-\theta}=0,$$

or

$$\{2\lambda\mu(1+\theta)+p(1-\theta)\}\,4\theta+(a+b)^2(1-\theta)^2-4ab(1-\theta)=0,$$

or

$$\theta^2\{(a+b)^2-4(p-2\lambda\mu)\}+\theta\{-2a^2-2b^2+4(p+2\lambda\mu)\}+(a-b)^2=0;$$

viz. substituting for p its values, this is

$$\theta^2(a+b-2\lambda-2\mu)^2+2\theta\{-a^2-b^2+2(\lambda+\mu)(a+b)-2(\lambda-\mu)^2\}^2+(a-b)^2=0\,;$$

or if we write

$$A=a^2-2a(\lambda+\mu)+(\lambda-\mu)^2,\quad B=b^2-2b(\lambda+\mu)+(\lambda-\mu)^2,$$

this is

$$\theta^2(a+b-2\lambda-2\mu)^2-2(A+B)\theta+(a-b)^2=0,$$

whence

$$\begin{aligned}\{(a-b)^2-(A+B)\theta\}^2&=\theta^2\{(A+B)^2-(a-b)^2(a+b-2\lambda-2\mu)^2\}\\&=\theta^2\{(A+B)^2-(A-B)^2\}=4AB\theta^2\,;\end{aligned}$$

viz. taking for convenience the sign $-$ on the right-hand side, this is

$$(a-b)^2-(A+B)\theta=-2\theta\sqrt{AB}\,;$$

and we have thus

$$\theta=\frac{(a-b)^2}{(\sqrt{A}-\sqrt{B})^2},$$

that is,

$$\theta,=\frac{\omega-1}{\omega+1}=\frac{a-b}{\sqrt{A}-\sqrt{B}};\quad \omega=\frac{\sqrt{A}-\sqrt{B}+a-b}{\sqrt{A}-\sqrt{B}-a+b}.$$

We may write

$$x=\mu(\omega-1)+\tfrac{1}{2}(a+b)+\tfrac{1}{2}(a+b-2\lambda-2\mu)\frac{\omega-1}{\omega+1},$$

$$y=\lambda(\omega-1)+\tfrac{1}{2}(a+b)+\tfrac{1}{2}(a+b-2\lambda-2\mu)\frac{\omega-1}{\omega+1};$$

whence also $x-y=(\mu-\lambda)(\omega-1)$, as is also seen at once from the original equations; then we have

$$\begin{aligned}\tfrac{1}{2}(a+b-2\lambda-2\mu)\frac{\omega-1}{\omega+1}&=\frac{\tfrac{1}{2}(a-b)(a+b-2\lambda-2\mu)}{\sqrt{A}-\sqrt{B}}\\&=\frac{\tfrac{1}{2}(A-B)}{\sqrt{A}-\sqrt{B}},=\tfrac{1}{2}(\sqrt{A}+\sqrt{B});\end{aligned}$$

and the values are

$$\begin{aligned}x&=\frac{2\mu(a-b)}{\sqrt{A}-\sqrt{B}-a+b}+\tfrac{1}{2}(\sqrt{A}+\sqrt{B}+a+b)\\&=\frac{(a-b)(\mu-\lambda)+b\sqrt{A}-a\sqrt{B}}{\sqrt{A}-\sqrt{B}-a+b},\\y&=\frac{2\lambda(a-b)}{\sqrt{A}-\sqrt{B}-a+b}+\tfrac{1}{2}(\sqrt{A}+\sqrt{B}+a+b)\\&=\frac{(a-b)(\lambda-\mu)+b\sqrt{A}-a\sqrt{B}}{\sqrt{A}-\sqrt{B}-a+b},\end{aligned}$$

which may be expressed in the more simple form

$$x=\frac{1}{4\lambda}(a+\lambda-\mu+\sqrt{A})(b+\lambda-\mu+\sqrt{B}),$$

$$y=\frac{1}{4\mu}(a-\lambda+\mu+\sqrt{A})(b-\lambda+\mu+\sqrt{B}),$$

the transformations depending on the identity

$$\frac{8\lambda\mu(a-b)}{\sqrt{A}-\sqrt{B}-a+b}=ab-(\lambda+\mu)(a+b)+(\lambda-\mu)^2+\sqrt{A}(b-\lambda-\mu)+\sqrt{B}(a-\lambda-\mu)+\sqrt{AB},$$

which is easily verified. Of course, since the signs of $\sqrt{A}$, $\sqrt{B}$ are arbitrary, we have 4 systems of values of (x, y), which is right.

In the original equations, for a, b, λ, μ, x, y, write $1, k^{-2}, \lambda^2, -\mu^2, x^2, -y^2$; then the equations become

$$\lambda^2(1+k^2x^2y^2)^2=x^2(1+y^2)(1+k^2y^2),\quad \mu^2(1+k^2x^2y^2)^2=y^2(1-x^2)(1-k^2x^2),$$

and we thence have

$$\lambda+\mu i=\frac{x\sqrt{(1+y^2)(1+k^2y^2)}+iy\sqrt{(1-x^2)(1-k^2x^2)}}{1+k^2x^2y^2};$$

viz. assuming $x=\operatorname{sn}\alpha\,(\operatorname{sinam}\alpha)$, $iy=\operatorname{sn}i\beta$, this is $\lambda+\mu i=\operatorname{sn}(\alpha+\beta i)$; viz. the problem is (for a given modulus k, assumed as usual to be real, positive, and less than 1) to reduce a given imaginary quantity $\lambda+\mu i$ to the form $\operatorname{sn}(\alpha+\beta i)$. The proper solution is that in which the signs of the radicals are each $-$, viz. it may in this case be shown that the value of x^2 is positive and less than 1, that of y^2 positive. The values thus are

$$x^2=\frac{1}{4\lambda^2}(1+\lambda^2+\mu^2-\sqrt{A})\left(\frac{1}{k^2}+\lambda^2+\mu^2-\sqrt{B}\right),$$

$$y^2=\frac{1}{4\mu^2}(1-\lambda^2-\mu^2-\sqrt{A})\left(\frac{1}{k^2}-\lambda^2-\mu^2-\sqrt{B}\right),$$

where

$$A=1-2\lambda^2+2\mu^2+(\lambda^2+\mu^2),\quad B=\frac{1}{k^2}-\frac{2}{k^2}\lambda^2+\frac{2}{k^2}\mu^2+(\lambda^2+\mu^2)^2.$$

The solution is really equivalent to that given by Richelot (*Crelle*, t. XLV., 1853, p. 225). To verify this partially, observe that, writing σ, τ for Richelot's $\tan\frac{1}{2}\phi$, $\tan\frac{1}{2}\psi$, we have

$$\left(\sigma+\frac{1}{\sigma}\right)\lambda=1+\lambda^2+\mu^2,$$

giving

$$\left(\sigma-\frac{1}{\sigma}\right)\lambda=-\sqrt{A};$$

$$\left(\tau+\frac{1}{\tau}\right)\frac{\lambda}{k}=\frac{1}{k^2}+\lambda^2+\mu^2,$$

giving

$$\left(\tau-\frac{1}{\tau}\right)\frac{\lambda}{k}=-\sqrt{B};$$

whence

$$2\sigma\lambda=1+\lambda^2+\mu^2-\sqrt{A},\quad 2\tau\frac{\lambda}{k}=\frac{1}{k^2}+\lambda^2+\mu^2-\sqrt{B},$$

or the above value of x^2 is $=k^{-1}\sigma\tau$, agreeing with his; the value of y^2 is, however, presented under a somewhat different form.

2. The curves are

$$(2-xy)^2 = 2x(1-y)(2-y), \quad -(2-xy)^2 = y(1-x)(2-x) \ldots\ldots\ldots (1,\ 2),$$

each passing through the points (1, 2) and (2, 1); the four points of intersection found by the foregoing general theory are all real, viz. these are

$x = \frac{1}{2}(2+\sqrt{3})(5+\sqrt{17})$,		$y = -\frac{1}{8}(-1+\sqrt{3})(-1+\sqrt{17})$,	say	$+17{\cdot}00$ and	$-0{\cdot}57$,
$-\sqrt{3}$,	$+\sqrt{17}$	„	„	$+\ 1{\cdot}65$	$+0{\cdot}94$,
$+\sqrt{3}$,	$-\sqrt{17}$	„	„	$+\ 1{\cdot}22$	$+2{\cdot}13$,
$-\sqrt{3}$,	$-\sqrt{17}$	„	„	$-\ 0{\cdot}12$	$-3{\cdot}49$.

The equation of the curve (1) may also be written in the forms

$$y^2(x^2-2x) + 2yx - 4x + 4 = 0, \quad x^2y^2 + x(-2y^2+2y-4) + 4 = 0.$$

The original form shows that, if y is between 1 and 2, x is negative—(but by a further examination it appears that there is not in fact any branch of the curve between these limits of y)—but y being outside these limits, then x is positive; in fact, the whole curve lies on the positive side of the axis of y. And then the inspection of the first quadric equation shows that the lines $x=0$ and $x=2$ are each an asymptote.

The point at infinity on the axis of y is in fact a flecnode, the tangent to the flecnodal branch being $x=0$, and that of the ordinary branch $x=2$.

Similarly, from the second quadric equation, it appears that the line $y=0$ is an asymptote; the point at infinity on the axis of x is in fact a cusp, the axis in question $y=0$ being the cuspidal tangent.

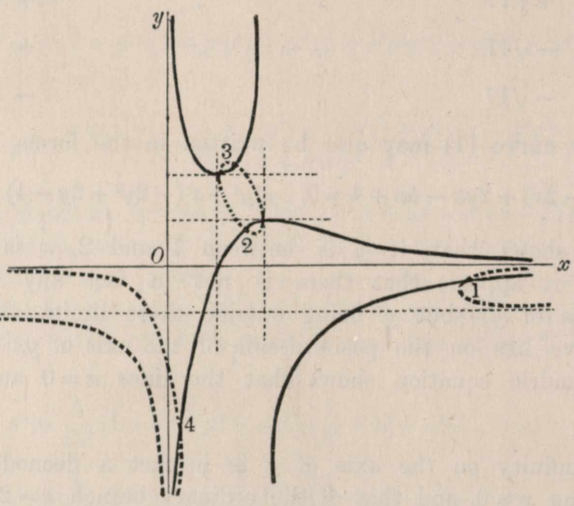
The equation of the curve (2) may also be written in the forms

$$x^2y^2 + (x^2-7x+2)y + 4 = 0, \quad (y^2+y)x^2 - 7yx + 2y + 4 = 0.$$

The original form shows that, if x is between 1 and 2, y is positive; but that x being beyond these limits, y is negative; and as regards the first case, x between 1 and 2, we at once establish the existence of an oval, meeting the line $y=1$ in the points $x=2$ and $\frac{3}{2}$, and the line $y=2$ in the points $x=1$ and $\frac{4}{3}$; it is further easy to see that the horizontal tangents of the oval are $y=\frac{1}{16}(25 \pm \sqrt{113})$, $=$ say $2{\cdot}2$ and $0{\cdot}9$.

The remainder of the curve lies wholly below the line $y=0$. The first quadric equation shows the asymptote $x=0$; the point at infinity on the axis of y is in fact a cusp, having the axis itself for the cuspidal tangent. The second quadric equation shows the asymptotes $y=0$, $y=-1$; the point at infinity on the axis of x is in fact a flecnode, having the line $y=0$ for the tangent to the flecnodal branch, and $y=-1$ for that of the other branch. It is further seen that there are two vertical tangents $x=\frac{1}{2}(11 \pm \sqrt{113}) = 10{\cdot}8$ or $0{\cdot}2$; the former of these touches a branch lying wholly between the two asymptotes $y=0$, $y=-1$; the latter one of the branches belonging to the cuspidal asymptote $x=0$; this last branch cuts the asymptote $x=0$ at $y=-2$, and then, cutting the asymptote $y=-1$ and $x=-\frac{2}{7}(=-0\cdot3)$, goes on to touch at infinity the asymptote $y=0$. It is now easy to trace the curve.

The figure shows the two curves. The curve (1) is shown by a continuous line, the curve (2) by a thick dotted line; the points 1, 2, 3, 4 show the above mentioned



four intersections of the curves; the point 1 and the dotted branch through it are of necessity drawn considerably out of their true positions; viz. as above appearing, the x-coordinate of 1 is $=17\cdot00$, and the equation of the vertical tangent to the branch is $x=10\cdot8$.

[Vol. XXII., pp. 78, 79.]

4520. (Proposed by A. B. EVANS, M.A.)—Find the least integral values of x and y that will satisfy the equation $x^2-953y^2=-1$.

Solution by PROFESSOR CAYLEY.

The values are given in Degen's Tables, viz.

$$x=2746864744, \quad y=88979677.$$

The work referred to is entitled "Canon Pellianus, sive Tabula simplicissimam æquationis celebratissimæ $y^2=ax^2+1$ solutionem pro singulis numeri dati valoribus ab 1 usque ad 1000 in numeris rationalibus iisdemque integris exhibens. Auctore C. F. Degen, Hafniæ (Copenhagen), 1817."

Table I., pp. 3—106 gives, for all numbers 1 to 1000, the denominators, (?) the quotients of the convergent fraction of $\sqrt{a}$, and also the least values of x, y which will satisfy the equation $x^2 - ay^2 = +1$. Thus

953	30, 1, 6, 1, 2, 1, 3, 8, 1, 1, (4, 4),
	1, 53, 8, 41, 17, 37, 16, 7, 32, 29, (13, 13),
	488830275367615376, 15090531843660371073.

Table II., pp. 109—112, is described as giving for all those values of a between 1 and 1000, for which there exists a solution of the equation $x^2 - ay^2 = -1$, the least values of x and y which satisfy this equation: thus 953, x and y as above. It is, however, to be noticed that the values of $a = \beta^2 + 1$, for which there is the obvious solution $x = \beta$, $y = 1$, are omitted from the table. The reason for this appears, but the heading should have been different.

[Vol. XXIII., January to July, 1875, pp. 18, 19.]

4528. (Proposed by Professor CAYLEY.)—A lottery is arranged as follows:—There are n tickets representing a, b, c pounds respectively. A person draws once; looks at his ticket; and if he pleases, draws again (out of the remaining $n-1$ tickets); looks at his ticket, and if he pleases draws again (out of the remaining $n-2$ tickets); and so on, drawing in all not more than k times; and he receives the value of the last drawn ticket. Supposing that he regulates his drawings in the manner most advantageous to him according to the theory of probabilities, what is the value of his expectation?

Solution by the PROPOSER.

Let the expression "a or α" signify "a or α, whichever of the two is greatest," and let $M_1(a, b, c, \ldots)$ denote the mean of the quantities $(a, b, c, \ldots)$, viz. their sum, divided by the number of them.

To fix the ideas, consider five quantities a, b, c, d, e, and write

$$M_1(a, b, c, d, e) = M_1(a, b, c, d, e),$$

$$M_2(a, b, c, d, e) = M_1\{a \text{ or } M_1(b, c, d, e),\ b \text{ or } M_1(a, c, d, e), \ldots, e \text{ or } M_1(a, b, c, d)\},$$

$$M_3(a, b, c, d, e) = M_1\{a \text{ or } M_2(b, c, d, e),\ b \text{ or } M_2(a, c, d, e), \ldots, e \text{ or } M_2(a, b, c, d)\},$$

and so on. And the like in the case of any number of quantities a, b, c, ...

Then the value of the expectation is $= M_k(a, b, c, \ldots)$.

For, when $k=1$, the value is obviously $= M_1(a, b, c, \ldots)$.

When $k=2$, if a is drawn, the adventurer will be satisfied or he will draw again, according as a or $M_1(b, c, \ldots)$ is greatest, viz. in this case the value of the expectation is "a or $M_1(b, c, \ldots)$."

So if b is drawn, the adventurer will be satisfied or he will draw again, according as b or $M_1(a, c, \ldots)$ is greatest; viz. in this case the value of the expectation is "b or $M_1(a, c, \ldots)$"; and so on: and the several cases being equally probable, the value of the total expectation is

$$=M_1\{a \text{ or } M_1(b, c, \ldots),\ \ b \text{ or } M_1(a, c, \ldots), \ldots\} = M_2(a, b, c, \ldots):$$

and the like for $k=3$, $k=4$, &c.

For instance, $a, b, c, d = 1, 2, 3, 4$, $M_1(1, 2, 3, 4) = \frac{10}{4}$,

$$M_2(1, 2, 3, 4) = M_1(1 \text{ or } \tfrac{9}{3}, 2 \text{ or } \tfrac{8}{3}, 3 \text{ or } \tfrac{7}{3}, 4 \text{ or } \tfrac{6}{3}) = M_1(\tfrac{9}{3}, \tfrac{8}{3}, \tfrac{9}{3}, \tfrac{12}{3}) = \tfrac{38}{12},$$

$$M_2(2, 3, 4) = M_1(2 \text{ or } \tfrac{7}{2}, 3 \text{ or } \tfrac{6}{2}, 4 \text{ or } \tfrac{5}{2}) = M_1(\tfrac{7}{2}, \tfrac{6}{2}, \tfrac{8}{2}) = \tfrac{21}{6},$$

$$M_2(1, 3, 4) = M_1(1 \text{ or } \tfrac{7}{2}, 3 \text{ or } \tfrac{5}{2}, 4 \text{ or } \tfrac{4}{2}) = M_1(\tfrac{7}{2}, \tfrac{6}{2}, \tfrac{8}{2}) = \tfrac{21}{6},$$

$$M_2(1, 2, 4) = M_1(1 \text{ or } \tfrac{6}{2}, 2 \text{ or } \tfrac{5}{2}, 4 \text{ or } \tfrac{3}{2}) = M_1(\tfrac{6}{2}, \tfrac{5}{2}, \tfrac{8}{2}) = \tfrac{19}{6},$$

$$M_2(1, 2, 3) = M_1(1 \text{ or } \tfrac{5}{2}, 2 \text{ or } \tfrac{4}{2}, 3 \text{ or } \tfrac{3}{2}) = M_1(\tfrac{5}{2}, \tfrac{4}{2}, \tfrac{6}{2}) = \tfrac{15}{6},$$

$$M_3(1, 2, 3, 4) = M_1(1 \text{ or } \tfrac{21}{6}, 2 \text{ or } \tfrac{21}{6}, 3 \text{ or } \tfrac{19}{6}, 4 \text{ or } \tfrac{15}{6}) = M_1(\tfrac{21}{6}, \tfrac{21}{6}, \tfrac{19}{6}, \tfrac{24}{6}) = \tfrac{85}{24},$$

$$M_3(1, 2, 3) = 3 \text{ \&c.},$$

$$M_4(1, 2, 3, 4) = M_1(1 \text{ or } 4, 2 \text{ or } 4, 3 \text{ or } 4, 4 \text{ or } 3) = M_1(4, 4, 4, 4) = 4.$$

Or finally

$$M_1, M_2, M_3, M_4 = \tfrac{10}{4}, \tfrac{38}{12}, \tfrac{85}{24}, 4 = \tfrac{60}{24}, \tfrac{76}{24}, \tfrac{85}{24}, \tfrac{96}{24}.$$

Cor. If the $a, b, c, \ldots$ denote penalties instead of prizes, then the solution is the same, except that "a or α" must now denote "a or α, whichever of them is least."

[Vol. XXIII., pp. 47, 48.]

4581. (Proposed by the Rev. M. M. U. Wilkinson.)—A witness, whose statement is what he opines once in m times, and whose opinion is correct once in n times, asserted that the number of a note, issued by a bank universally known to have issued notes numbered from B to $B+A-1$ inclusive, was $B+P$, where P is either 0, 1, 2, ..., or $A-1$. Prove (1) that the probability that the note in question was that note is

$$\frac{1}{mn}\left\{1+\frac{(m-1)(n-1)}{A-1}\right\}.$$

The above witness also said that the note was signed by X, it being universally known that X has signed one note, and Y the remaining $A-1$ notes; find (2) the probability that this last statement was correct.

Remark by PROFESSOR CAYLEY.

There is a serious difficulty in the question, or the answer; I think, in the question. Try the answer in numbers $m=10$, $n=10$. The witness says what he opines once out of 10 times—he is in fact an atrocious liar; and he opines rightly once out of 10 times, that is, wrongly 9 times out of 10; he is therefore a blunderer—but a remarkably ingenious one, in that the chances are so greatly against his blundering upon a right result.

He says that the note was signed by X, and the chance of this being so is found to be $\frac{1}{100}+\frac{81}{100}=\frac{82}{100}$, or more than $\frac{8}{10}$; the larger part $\frac{81}{100}$ of this is obtained as follows:—the witness having said that the note was signed by X, the chances are 9 out of 10 that he thought the reverse; and, thinking the reverse, the chance is 9 out of 10 that he thought wrongly, viz. that the note was signed by X. But can the statement of such a witness create any probability in favour of the event?

The fallacy seems to consist in the assumption that n can have a determinate value irrespective of the nature of the opinion. Suppose there are 500 notes, and that the opinion is that the note was a definite number 99; it is quite conceivable that, in forming a series of such opinions, the witness may be wrong 9 times out of 10. But let the opinion be that the note was *not* 99; no amount of ingenuity of blundering can make him wrong 9 times out of 10 in a series of such opinions. If it could, a friend who knew the true opinion of the witness, would be able 9 times out of 10 to know the number of the note, from the mere fact that the witness opines that the note is not a named number.

[Vol. XXIII., p. 58.]

4638. (Proposed by Professor CAYLEY.)—Find the equation of the surface which is the envelope of the quadric surface $ax^2+by^2+cz^2+dw^2=0$, where a, b, c, d are variable parameters connected by the equation $Abc+Bca+Cab+Fad+Gbd+Hcd=0$; and consider in particular the case in which the constants A, B, C, F, G, H satisfy the condition

$$(AF)^{\frac{1}{2}}+(BG)^{\frac{1}{2}}+(CH)^{\frac{1}{2}}=0.$$

[Vol. XXIV., July to December, 1875, p. 41.]

4694. (Proposed by Professor CAYLEY.)—Taking F, F' a pair of reciprocal points in respect to a circle, centre O; then if F, F' are centres of force, each force varying as (distance)$^{-n}$, prove that (1) the resultant force upon any point P on the circle is in the direction of a fixed point S on the axis OFF'; and if, moreover, the forces at the unit of distance are as $(OF)^{\frac{1}{2}(n-1)}$ to $(OF')^{\frac{1}{2}(n-1)}$, then (2) the resultant force is proportional to

$$(SP)^{-\frac{1}{2}(n-1)}\,.\,(PV)^{-\frac{1}{2}(n+1)},$$

where PV is the chord through S.

[Vol. XXIV., pp. 72—74.]

4793. (Proposed by Professor WOLSTENHOLME, M.A.)—If $y = x^n (\log x)^r$, where n and r are integers, prove that

$$x^r \frac{d^{n+r}y}{dx^{n+r}} + \frac{r(r-1)}{2} x^{r-1} \frac{d^{n+2-1}y}{dx^{n+2-1}} + \frac{r(r-1)(r-2)(3r-5)}{24} x^{r-2} \frac{d^{n+r-2}y}{dx^{n+r-2}} + \dots$$

$$\dots + (2^{r-1}-1) x^2 \frac{d^{n+2}y}{dx^{n+2}} + x \frac{d^{n+1}y}{dx^{n+1}} = \underline{|r}\ \underline{|n},$$

the coefficients being

$$\frac{\Delta^{r-1}1^{r-1}}{\underline{|r-1}}, \quad \frac{\Delta^{r-2}1^{r-1}}{\underline{|r-2}}, \quad \frac{\Delta^{r-3}1^{r-1}}{\underline{|r-3}}, \quad \dots\dots, \quad \frac{\Delta 1^{r-1}}{\underline{|1}}, \text{ and } 1;$$

so that the result may be symbolically written

$$\epsilon^{xD\Delta}\left(1^{r-1}\frac{d^{n+1}y}{dx^{n+1}}\right) = \frac{\underline{|r}\ \underline{|n}}{x},$$

where D denotes $\frac{d}{dx}$ and operates on $\frac{d^n y}{dx^n}$ only, and Δ operates on 1^{r-1} only, the terms after the rth all vanishing since $\Delta^m x^n = 0$, when m is an integer $> n$. The calculations involved prove that, when $x = 1$,

$$\Delta^{n-1}x^n = \frac{\underline{|n+1}}{2}, \quad \Delta^{n-2}x^n = \underline{|n+1}\,.\frac{3n-2}{24},$$

$$\Delta^{n-3}x^n = \underline{|n+1}\,.\frac{(n-1)(n-2)}{48}.$$

Solution by PROFESSOR CAYLEY.

Since $y = x^n (\log x)^r$, therefore $(xd_x - n)\,y = rx^n (\log x)^{r-1}$; by repeating the same operation, we have

$$(xd_x - n)^r y = [r]^r x^n; \text{ whence } d_x^{\,n}(xd_x - n)^r y = [r]^r [n]^n.$$

Now, for any value whatever of the function y, we have

$$d_x^{\,n}(xd_x - n)^r y = Ax^r d_x^{\,r+n} y + Bx^{r-1} d_x^{\,r+n-1} y + Cx^{r-2} d_x^{\,r+n-2} y + \&c.,$$

the coefficients A, B, C, ... being functions, presumably of r, n, but independent of the form of the function y. It will, however, appear that A, B, C, ... are, in fact, functions of r only.

To see how this is, observe that $(xd_x - n)^r$ consists of a set of terms

$$(xd_x)^\theta, \quad (\theta = 0 \text{ to } r),$$

where $(xd_x)^\theta$ denotes θ repetitions of the operation xd_x; by a well-known theorem, this is $= [xd_x + \theta - 1]^\theta$, where, after expansion of the factorial, $(xd_x)^s$ is to be replaced by $x^s d_x^{\,s}$, thus

$$(xd_x)^2 = [xd_x + 1]^2 = x^2 d_x^{\,2} + xd_x, \quad (xd_x)^3 = [xd_x + 2]^3 = x^3 d_x^{\,3} + 3x^2 d_x^{\,2} + 2xd_x, \ \&c.;$$

thus $(xd_x - n)^r$ consists of a series of terms $x^\theta d_x{}^\theta$, $(\theta = 0$ to $r)$, and, operating with $d_x{}^n$, this last, $= (d_x + d_x')^n$, consists of a series of terms such as $d_x{}^\alpha d_x'{}^{n-\alpha}$, where the unaccented symbol operates on the x^θ, and the accented symbol on the y; the term is thus $x^{\theta-\alpha} d_x{}^{n+\theta-\alpha}$, or observing that $\theta - \alpha$ is at most $= r$, and putting it $= r - k$, the term is $x^{r-k} d_x{}^{n+k}$, viz. $d_x{}^n (xd_x - n)^r$ consists of a series of terms of the form $x^{r-k} d_x{}^{n+k}$; or, what is the same thing, $d_x{}^n (xd_x - n)^r y$ is a series of the form in question.

To understand how it can be that the coefficients A, B, C, ... are independent of n, take the particular case $r = 2$; then we have here

$$d_x{}^n (xd_x - n)^2 y = Ax^2 d_x{}^{n+2} y + Bxd_x{}^{n+1} y + Cd_x{}^n y.$$

The right-hand side is

$$d_x{}^n \{x^2 d_x{}^2 - (2n-1)\, xd_x + n^2\}\, y,$$

which is

$$\begin{aligned} = \qquad & \{x^2 d^{n+2} + 2nxd_x{}^{n+1} + (n^2 - n)\, d_x{}^n\}\, y \\ -(2n-1)\,& \{\qquad\qquad xd_x{}^{n+1} + \qquad nd_x{}^n\}\, y \\ + n^2\, & \{\qquad\qquad\qquad\qquad\qquad\quad d_x{}^n\}\, y; \end{aligned}$$

hence

$$A = 1,\quad B = 2n - (2n-1),\ = 1,\quad C = (n^2 - n) - n\,(2n-1) + n^2,\ = 0;$$

and we thus see also how in this particular case the last coefficient is $= 0$, viz. that we have

$$d_x{}^n (xd_x - n)^2 y = x^2 d_x{}^{n+2} y + xd_x{}^{n+1} y,$$

without any term in $d_x{}^n y$.

To find the coefficients A, B, C, ... generally, write $y = x^{r+n+\theta}$, then $xd_x - n = r + \theta$, and consequently

$$d_x{}^n (xd_x - n)^r y,\quad = (r+\theta)^r d_x{}^n x^{r+n+\theta},\quad = (r+\theta)^r\, [r+n+\theta]^n x^{r+\theta};$$

whence

$$(r+\theta)^r\, [r+n+\theta]^n = A\, [r+\theta+n]^{n+r} + B\, [r+\theta+n]^{n+r-1} + \ldots\ldots;$$

or, what is the same thing,

$$(r+\theta)^r = A\, [r+\theta]^r + B\, [r+\theta]^{r-1} + \ldots\ldots.$$

Since the left-hand side, and every term $[r+\theta]^s$ on the right-hand side, contains the factor $r + \theta$, there is not on the right-hand side any term $[r+\theta]^0$; dividing the equation by $r + \theta$, it then becomes

$$(r+\theta)^{r-1} = A\, [r+\theta-1]^{r-1} + B\, [r+\theta-1]^{r-2} + \ldots\ldots,$$

and we thus have

$$A = \frac{\Delta^{r-1} 1^{r-1}}{[r-1]^{r-1}}\,(=1),\quad B = \frac{\Delta^{r-2} 1^{r-1}}{[r-2]^{r-2}};$$

viz. writing $r + \theta = 1 + x$, $u_x = (1+x)^{r-1}$, and taking the terms in the reverse order, the series is the well-known one

$$u_x = u_0 + \frac{x}{1}\,\Delta u_0 + \frac{x\,.\,x-1}{1\,.\,2}\,\Delta^2 u_0 + \&\text{c}.$$

Hence, in general,

$$d_x{}^n (xd_x - n)^r y = \frac{\Delta^{r-1} 1^{r-1}}{[r-1]^{r-1}} x^r d_x{}^{r+n} y + \frac{\Delta^{r-2} 1^{r-1}}{[r-2]^{r-2}} x^{r-1} d_x{}^{r+n-1} y + \&c.,$$

where observe that the last term is $= xd_x{}^{n+1}y$.

For the function $y = x^n (\log x)^n$, the value of each side is $= [r]^r [n]^n$.

[Vol. XXIV., pp. 89—91.]

4752. (Proposed by Professor CAYLEY.)—Mr Wolstenholme's Question 3067 may evidently be stated as follows:—

If (a, b, c) are the coordinates of a point on the cubic curve

$$a^3 + b^3 + c^3 = (b + c)(c + a)(a + b),$$

and if

$$(b^2 + c^2 - a^2) x = (c^2 + a^2 - b^2) y = (a^2 + b^2 - c^2) z;$$

then (x, y, z) are the coordinates of a point on the same cubic curve.

This being so, it is required to find the geometrical relation of the two points to each other.

Solution by PROFESSOR CAYLEY.

1. On referring to Professor Wolstenholme's Solution of the original Question 3067 (*Reprint*, Vol. XIII., p. 70), it appears that the coordinates (x, y, z) of the point in question may be expressed in the more simple form

$$x : y : z = a(-a + b + c) : b(a - b + c) : c(a + b - c);$$

viz. the given relation between (a, b, c) being equivalent to

$$4abc + (-a + b + c)(a - b + c)(a + b - c) = 0,$$

we have

$$a^2 - (b - c)^2 = \frac{-4abc}{-a + b + c},$$

and thence

$$b^2 + c^2 - a^2 = 2bc\left(1 + \frac{2a}{-a + b + c}\right) = 2bc\left(\frac{a + b + c}{-a + b + c}\right);$$

and consequently

$$(b^2 + c^2 - a^2) x = \frac{2abc(a + b + c) x}{a(-a + b + c)},$$

whence the transformation in question.

2. Writing for greater symmetry (x, y, z) in place of (a, b, c), and (x', y', z') in place of (x, y, z), the coordinates (x, y, z) and (x', y', z') of the two points are connected by the relation

$$x' : y' : z' = x(-x + y + z) : y(x - y + z) : z(x + y - z),$$

and we thence at once deduce the converse relation

$$x : y : z = x'(-x'+y'+z') : y'(x'-y'+z') : z'(x'+y'-z').$$

Hence, writing

$$(-x+y+z,\quad x-y+z,\quad x+y-z) = (\xi,\ \eta,\ \zeta),$$

and similarly

$$(-x'+y'+z',\quad x'-y'+z',\quad x'+y'-z') = (\xi',\ \eta',\ \zeta'),$$

we have

$$x' : y' : z' = x\xi : y\eta : z\zeta,\quad x : y : z = x'\xi' : y'\eta' : z'\zeta',$$

and thence also $\xi\xi' = \eta\eta' = \zeta\zeta'$; so that, regarding (ξ, η, ζ), (ξ', η', ζ') as the coordinates of the two points, we see that these are inverse points one of the other in regard to the triangle $\xi = 0$, $\eta = 0$, $\zeta = 0$.

To complete the solution, we must introduce these new coordinates into the equation of the cubic curve. Writing this under the form

$$8xyz + 2(-x+y+z)(x-y+z)(x+y-z) = 0,$$

and observing that

$$(2x,\ 2y,\ 2z) = (\eta+\zeta,\ \zeta+\xi,\ \xi+\eta),$$

the equation is

$$(\eta+\zeta)(\zeta+\xi)(\xi+\eta) + 2\xi\eta\zeta = 0;$$

viz. this is a cubic curve inverting into itself. And the two points in question are thus any two inverse points on this cubic curve.

3. In regard to the original form, that the point (x, y, z) defined by the equations

$$x(-a^2+b^2+c^2) = y(a^2-b^2+c^2) = z(a^2+b^2-c^2),$$

lies on the cubic curve

$$a^3+b^3+c^3-(b+c)(c+a)(a+b) = 0,$$

Professor Sylvester proceeds as follows:—Writing

$$(x, y, z) = \{a^4-(b^2-c^2)^2,\ b^4-(c^2-a^2)^2,\ c^4-(a^2-b^2)^2\},\ = (A, B, C),$$

suppose; and

$$F(a, b, c) = a^3+b^3+c^3-(b+c)(c+a)(a+b),$$

he observes that the truth of the theorem depends on the identity

$$F(A, B, C) + F(a, b, c)F(a, -b, c)F(a, b, -c)F(a, -b, -c) = 0,$$

and that, in order to prove the identity generally, it is sufficient to prove it for the three cases $a^2 = 0$, $a^2 = b^2+c^2$, $a^2 = b^2$, which may be effected without difficulty.

4. But, for a general proof of the identity, write

$$\lambda = b^2+c^2,\quad \mu = b^2-c^2,$$

so that

$$A = a^4-\mu^2,\quad B = (a^2+\mu)(-a^2+\lambda),\quad C = (-a^2+\lambda)(a^2-\mu),$$

whence

$$-F(A, B, C) = -(a^4-\mu^2)^3 + 2(a^2-\lambda)^3(a^6+3a^2\mu^2) - 8a^2b^2c^2(a^4-\mu^2)(a^2-\lambda),$$
$$= a^{12} - 6\lambda a^{10} + (6\lambda^2+9\mu^2-8b^2c^2)a^8 + \lambda(-2\lambda^2-18\mu^2+8b^2c^2)a^6$$
$$+\mu^2(18\lambda^2-3\mu^2+8b^2c^2)a^4 + \lambda\mu^2(-6\lambda^2-8b^2c^2)a^2+\mu^6.$$

Moreover

$$F(a, b, c) = a\{a^2-(b+c)^2\} - (b+c)\{a^2-(b-c)^2\},$$

therefore

$$F(a, -b, -c) = a\{a^2-(b+c)^2\} + (b+c)\{a^2-(b-c)^2\};$$

whence

$$F(a, b, c)F(a, -b, -c) = a^2\{a^2-(b+c)^2\}^2 - (b+c)^2\{a^2-(b-c)^2\}^2$$
$$= a^6 - 3\gamma^2a^4 + \gamma^2(\gamma^2+2\delta^2)a^2 - \gamma^2\delta^2,$$

if $\gamma = b+c$, $\delta = b-c$. By changing the sign of c, we interchange γ and δ, and we thus have

$$F(a, b, -c)F(a, -b, c) = a^6 - 3\delta^2a^4 + \delta^2(2\gamma^2+\delta^2)a^2 - \gamma^4\delta^2,$$

and the identity to be verified is thus

$$\{a^6-3\gamma^2a^4+\gamma^2(\gamma^2+2\delta^2)a^2-\gamma^2\delta^4\}\{a^6-3\delta^2a^4+\delta^2(2\gamma^2+\delta^2)a^2-\gamma^4\delta^2\}$$
$$= a^{12}-6\lambda a^{10}+\ldots\ldots+\mu^6, \textit{ ut suprà};$$

the values of λ, μ in terms of γ, δ are $\lambda = \frac{1}{2}(\gamma^2+\delta^2)$, $\mu = \gamma\delta$; substituting these values on the right-hand side, the verification can be completed without difficulty.

[Vol. XXV., January to July, 1876, p. 82.]

4946. (Proposed by Professor CAYLEY.)—Show that the attraction of an indefinitely thin double convex lens on a point at the centre of one of its faces is equal to that of the infinite plate included between the tangent plane at the point and the parallel tangent plane of the other face of the lens.

[Vol. XXVI., July to December, 1876, pp. 41, 42.]

5020. (Proposed by W. S. B. WOOLHOUSE, F.R.A.S.)—Let 1, δ_1, δ_2, δ_3, ..., δ_n be the first differences of the coefficients of the expansion of the binomial $(1+x)^{2n}$ taken as far as the central or maximum coefficient; also let

$$\nu = \tfrac{1}{2}(n+1)n, \quad \nu' = \tfrac{1}{2}n(n-1), \quad \nu'' = \tfrac{1}{2}(n-1)(n-2), \text{ \&c.};$$

then show that the algebraic function

$$x^\nu - \delta_1x^{\nu'} + \delta_2x^{\nu''} - \delta_3x^{\nu'''} + \text{\&c.}$$

is divisible by $(x-1)^n$ without a remainder; and that the sum of the numerical coefficients of the quotient is equal to $1.3.5\ldots 2n-1$.

[See Solution to Question 1894, *Reprint*, vol. V., p. 113.]

Solution by PROFESSOR CAYLEY.

Mr Woolhouse's elegant theorem depends ultimately on the property of triangular numbers $\phi(n), =\frac{1}{2}(n^2-n)$; then $\phi(n+1)=\phi(-n)$, so that, writing down the series of triangular numbers backwards and forwards,

$$\begin{array}{cccccccccccc} \ldots, & 10, & 6, & 3, & 1, & 0, & 0, & 1, & 3, & 6, & 10, \ldots \\ \ldots & & , & a, & b, & c, & d, & e, & f, & g, & h, & \ldots \end{array}$$

we have, in fact, a continuous single series obtained by giving to n the different negative and positive integer values, zero included.

Thus a particular case is

$$(1+x)^6-5(1+x)^3+9(1+x)-5 \equiv 0 \,(\text{mod. } x^3) = 1 \,.\, 3 \,.\, 5x^3 + \&\text{c. } x^4 + \ldots,$$

where, on the left-hand side, the exponents are the triangular numbers $\phi(n+1)$, $n=0$ to 3; and the coefficients, after the first, are the differences of the binomial coefficients of the power $2n$ (in the particular case, $n=3$); viz. the binomial coefficients being

$$1, \;\; 6, \;\; 15, \;\; 20, \;\; 15, \;\; 6, \;\; 1,$$

the differences taken as far as they are positive are

$$5, \;\; 9, \;\; 5.$$

Expanding the several terms and writing down only the coefficients, we have a diagram

$$\begin{array}{r|lllllll} & 1, & 6, & 15, & 20, & 15, & 6, & 1, \\ -5 & 1, & 3, & 3, & 1, & & & \\ +9 & 1, & 1, & & & & & \\ -5 & 1, & & & & & & \end{array}$$

The theorem in the particular case depends on the identities

$$\begin{aligned} 1-\;\;5+9-5 &= 0, \\ 6-15+9 \quad\;\; &= 0, \\ 5-15 \qquad\quad\;\; &= 0, \\ 20-\;\;5 \qquad\quad\;\; &= 1 \,.\, 3 \,.\, 5; \end{aligned}$$

or writing, as above, h, g, f, e, to denote the triangular numbers 6, 3, 1, 0, these may be replaced by

$$\begin{array}{llll} h^0 & -5g^0 & +9f^0-5e^0 & =0, \\ h & -5g & +9f\;-5e & =0, \\ \frac{1}{2}h(h-1) & -5\,.\,\frac{1}{2}g(g-1) & +\ldots & =0, \\ \frac{1}{6}h(h-1)(h-2) & -5\,.\,\frac{1}{6}g(g-1)(g-2) & +\ldots & =1\,.\,3\,.\,5; \end{array}$$

or, reducing each equation by those which precede it, these become

$$h^0 - 5g^0 + 9f^0 - 5e^0 = 0,$$
$$h^1 - 5g^1 + 9f^1 - 5e^1 = 0,$$
$$h^2 - 5g^2 + 9f^2 - 5e^2 = 0,$$
$$h^3 - 5g^3 + 9f^3 - 5e^3 = 1.2.3.1.3.5.$$

Consider any one of these, for instance the third; the function on the left-hand is

$$1h^2 - (6-1)\,g^2 + (15-6)\,f^2 - (20-15)\,e^2,$$

or, introducing the values b, c, d as above,

$$1h^2 - 6g^2 + 15f^2 - 20e^2 + 15d^2 - 6c^2 + 1b^2,$$

which is, in fact, $=0$, if b, c, d, e, f, g, h are *any* successive triangular numbers; viz. this is an immediate consequence of the well-known theorem

$$1\,(\theta+6)^m - 6\,(\theta+5)^m + 15\,(\theta+4)^m - 20\,(\theta+3)^m + 15\,(\theta+2)^m - 6\,(\theta+1)^m + \theta^m$$
$$= \Delta^6\theta^m, \ = 0 \text{ for any value of } m \text{ up to } m=5, \text{ and}$$
$$= 1.2.3.4.5.6 \text{ for } m = 6.$$

We have thus all the equations except the last; and as regards the last equation, observe that the equation to be verified is

$$1\,[\tfrac{1}{2}\,(\theta+6)(\theta+5)]^3 - 6\,[\tfrac{1}{2}\,(\theta+5)(\theta+4)]^3 + \ldots = 1.2.3.1.3.5,$$

viz. this may be replaced by

$$1\,(\theta+6)^6 - 6\,(\theta+5)^6 + \ldots = 2^3.1.2.3.1.3.5 = 2.4.6.1.3.5 = 1.2.3.4.5.6,$$

which is right.

It is clear that the proof, although worked out on a particular case, is perfectly general; and Mr Woolhouse's theorem is thus proved.

[Vol. XXVI., pp. 77, 78.]

5079. (Proposed by Professor CAYLEY.)—Show that the curve

$$\{(\beta-\gamma i)^2-\delta^2\}^{\frac{1}{2}}\,\{(x-\beta i)^2+y^2\}^{\frac{1}{2}} + q\,\{(\beta+\gamma i)^2-\delta^2\}^{\frac{1}{2}}\,\{(x+\beta i)^2+y^2\}^{\frac{1}{2}}$$
$$= \left\{(1-q^2)\frac{\beta}{\delta}\right\}^{\frac{1}{2}}\{\beta^2-(\gamma-\delta i)^2\}^{\frac{1}{2}}\,\{(x-\gamma-\delta i)^2+y^2\}^{\frac{1}{2}},$$

where $i = (\sqrt{-1})$ as usual, is a real bicircular quartic having the axial foci

$$\beta i,\ -\beta i,\ \gamma+\delta i,\ \gamma-\delta i.$$

Solution by the PROPOSER.

Consider the equation

$$(l+mi)^{\frac{1}{2}}[(x-\beta i)^2+y^2]^{\frac{1}{2}}+q(l-mi)^{\frac{1}{2}}[(x+\beta i)^2+y^2]^{\frac{1}{2}}=(\lambda+\mu i)^{\frac{1}{2}}[(x-\gamma-\delta i)^2+y^2]^{\frac{1}{2}}.$$

This is

$$\begin{aligned}(l+mi)\{x^2+y^2-\beta^2-2\beta xi\}+q^2(l-mi)\{x^2+y^2-\beta^2+2\beta xi\}\\ -(\lambda+\mu i)\{x^2+y^2-\beta^2+\beta^2+\gamma^2-\delta^2-2\gamma x-2(x-\gamma)\delta i\}\\ +2q(l^2+m^2)^{\frac{1}{2}}[(x^2+y^2-\beta^2)^2+4\beta^2x^2]^{\frac{1}{2}}=0,\end{aligned}$$

where, putting the imaginary part equal to zero, we have

$$m(1-q^2)(x^2+y^2-\beta^2)-2l(1-q^2)\beta x-\mu\{x^2+y^2-\beta^2+(\beta^2+\gamma^2-\delta^2)-2\gamma x\}+2\lambda(x-\gamma)\delta=0,$$

which will be true identically if

$$\begin{aligned}m(1-q^2)-\mu&=0,\\ -l(1-q^2)\beta+\mu\gamma+\lambda\delta&=0,\\ -\mu(\beta^2+\gamma^2-\delta^2)-2\lambda\gamma\delta&=0.\end{aligned}$$

The last gives

$$\lambda=\theta(\beta^2+\gamma^2-\delta^2),\quad \mu=-2\theta\gamma\delta,\quad \theta \text{ arbitrary};$$

and then

$$l(1-q^2)\beta=\theta\delta(\beta^2+\gamma^2-\delta^2-2\gamma^2)=\theta\delta(\beta^2-\gamma^2-\delta^2),$$

$$m(1-q^2)=-2\theta\delta\gamma;$$

so that, putting

$$\theta\delta=(1-q^2)\beta,\ \text{or}\ \theta=(1-q^2)\frac{\beta}{\delta},$$

we have

$$l=\beta^2-\gamma^2-\delta^2,\quad m=-2\beta\gamma,$$

$$\lambda=(1-q^2)\frac{\beta}{\delta}(\beta^2+\gamma^2-\delta^2),\quad \mu=(1-q^2)\frac{\beta}{\delta}(-2\gamma\delta).$$

Therefore

$$l\pm mi=(\beta\mp\gamma i)^2-\delta^2,$$

$$\lambda\pm\mu i=(1-q^2)\frac{\beta}{\delta}[\beta^2+(\gamma\mp\delta i)^2];$$

and the equation is

$$\begin{aligned}\{(\beta-\gamma i)^2-\delta^2\}^{\frac{1}{2}}\{(x-\beta i)^2+y^2\}^{\frac{1}{2}}+q\{(\beta+\gamma i)^2-\delta^2\}^{\frac{1}{2}}\{(x+\beta i)^2+y^2\}^{\frac{1}{2}}\\ =\left\{(1-q^2)\frac{\beta}{\delta}\right\}^{\frac{1}{2}}\{\beta^2-(\gamma-\delta i)^2\}^{\frac{1}{2}}\{(x-\gamma-\delta i)^2+y^2\}^{\frac{1}{2}},\end{aligned}$$

which is a real curve having the axial foci $+\beta i$, $-\beta i$; $\gamma+\delta i$; $\gamma-\delta i$; viz. $\gamma+\delta i$ being a focus, and the curve being real, it is clear that $\gamma-\delta i$ is also a focus.

[Vol. XXVII., January to June, 1877, p. 20.]

5130. (Proposed by Professor CAYLEY.)—Show that the envelope of a variable circle, having its centre on a given conic and cutting at right angles a given circle, is a bicircular quartic; which, when the given conic and the circle have double contact, becomes a pair of circles; and, by means of the last-mentioned particular case of the theorem, connect together the porisms arising out of the two problems—

(i) Given two conics, to find a polygon of n sides inscribed in the one and circumscribed about the other.

(ii) Given two circles, to find a closed series of n circles each touching the two circles and the two adjacent circles of the series.

[Vol. XXVII., pp. 81—83.]

5208. (Proposed by Professor SYLVESTER.)—Let the *magnitude* of any ramification signify the number of its branches, and let its partial magnitudes in respect to any node signify the magnitudes of the ramifications which come together at that node. If at any node the largest magnitude exceeds by k the sum of the other magnitudes, let the node be called superior by k, or be said to be of superiority k; but if no magnitude exceeds the sum of the other magnitudes, let the node be called subequal. Then the theorem is, in any ramification, *either* there is one and only one subequal node; *or else* there are two and only two nodes each superior by unity, these two nodes being contiguous.

Solution by PROFESSOR CAYLEY.

The proof consists in showing that (1) there cannot be more than one subequal node; (2) there cannot be more than two nodes each superior by unity: and if there is one such node, then there is, contiguous to it, another such node; (3) starting from a node which is superior by more than unity, there is always a contiguous node which is either of smaller superiority, or else subequal; for, these theorems holding good, we can, by (3), always arrive at a node which is either subequal or else superior by unity; in the former case, by (1), the subequal node thus arrived at is unique; in the latter case, by (2), we have, contiguous to the node arrived at, a second node superior by unity; and we have thus a unique pair of nodes each superior by unity.

I will prove only (3), as it is easy to see that the like process applies to the proof of (1) and (2).

Let the whole magnitude be n; and suppose at a node P which is superior by k, the largest magnitude is α, and that the other magnitudes are, say, β, γ, δ. We have $\alpha = \beta + \gamma + \delta + k$; and since $n = \alpha + \beta + \gamma + \delta$, we have thence $n = 2\alpha - k$, or $\alpha = \frac{1}{2}(n+k)$, $\beta + \gamma + \delta = \frac{1}{2}(n-k)$: clearly k is even or odd, according as n is even or odd.

Suppose now that we pass from P, along the branch of magnitude α, to a contiguous node Q; and let the magnitudes for Q be α', β', γ', δ', ϵ', where α' denotes the magnitude for the branch QP. We have $\alpha' = \beta + \gamma + \delta + 1$: for the ramification consists of the branch QP and of the ramifications of magnitudes β, γ, δ which meet in P. We have thus

$$\alpha' = \tfrac{1}{2}(n-k) + 1 = \tfrac{1}{2}n - \tfrac{1}{2}(k-2);$$

and thence

$$\beta' + \gamma' + \delta' + \epsilon' = \tfrac{1}{2}n + \tfrac{1}{2}(k-2).$$

Supposing here that k is greater than 1, viz. that it is $=$ or >2, $k-2$ is 0 or positive; and if α' be the greatest magnitude belonging to the node Q, this is a subequal node. But it may be that α' is not the greatest magnitude; supposing then that the greatest magnitude is β', we have

$$\beta' = \tfrac{1}{2}n + \tfrac{1}{2}(k-2) - \gamma' - \delta' - \epsilon',$$

$$\alpha' + \gamma' + \delta' + \epsilon' = \tfrac{1}{2}n - \tfrac{1}{2}(k-2) + \gamma' + \delta' + \epsilon',$$

and thence

$$\beta' - (\alpha' + \gamma' + \delta' + \epsilon') = k - 2 - 2(\gamma' + \delta' + \epsilon');$$

viz. either the node is subequal, or else, being superior, the superiority is at most $=k-2$; that is, if from the node P, of superiority $=$ or >2, we pass along the branch of greatest magnitude to the contiguous node Q, this is either subequal, or else of superiority less than that of P; which is the foregoing proposition (3).

The subequal node, and the two nodes of superiority 1, in the cases where they respectively exist, may be termed the centre and the bicentre respectively; and the theorem thus is, every ramification has either a centre or else a bicentre. But the centre and the bicentre here considered, due (as remarked by Professor Sylvester) to M. Camille Jordan, and which may for distinction be termed the centre and the bicentre of *magnitude*, are quite distinct from the centre and the bicentre discovered by Professor Sylvester, and considered in my researches upon trees, *British Association Report*, 1875, [610]. These last may for distinction be termed the centre and the bicentre of *distance:* viz. we here consider, not the magnitude, but the length of a ramification, such length being measured by the number of branches to be travelled over in order to reach the most distant terminal node. The ramification has either a centre or else a bicentre of distance: viz. the centre is a node such that, for two or more of the ramifications which proceed from it, the lengths are equal and superior to those of the other ramifications, if any; the bicentre a pair of contiguous nodes such that, disregarding the branch which unites the two nodes, there are from the two nodes respectively (one at least from each of them) two or more ramifications the lengths of which are equal to each other and superior to those of the other ramifications, if any.

It is very noticeable how close the agreement is between the proofs for the existence of the two kinds of centre or bicentre respectively. Say, first as regards distance, if at any node the length of the longest branch exceeds by k the length of the next longest branch or branches, then the node is superior by k, or is of the superiority k; but, if there are two or more longest branches, then the node is subequal. And say next, in regard to magnitude, if at any node the largest magnitude exceeds by k the sum of all the other magnitudes, the node is superior by k, or has a superiority k; but if the largest magnitude does not exceed the sum of the other magnitudes, then the node is subequal. Then, whether we attend to distance or to magnitude, the three propositions hold good: (1) there cannot be more than one subequal node; (2) there cannot be more than two nodes each superior by unity: and if there is one such node, there is contiguous to it another such node; (3) starting from a node which is superior by more than unity, there is always a contiguous node which is of smaller superiority or else subequal; whence, as in the solution just referred to, there is always, as regards distance, a centre or a bicentre; and there is always, as regards magnitude, a centre or a bicentre.

[Vol. XXVII., pp. 89, 90.]

On Mr Artemas Martin's First Question in Probabilities. By PROFESSOR CAYLEY.

The question was, "A says that B says that a certain event took place: required the probability that the event did take place, p_1 and p_2 being A's and B's respective probabilities of speaking the truth."

The solutions, referred to or given on pp. 77—79 of volume XXVII. of the *Reprint*, give the following values for the probability in question:—

Todhunter's *Algebra* .... $p_1p_2 + (1-p_1)(1-p_2)$.

Artemas Martin .... $p_1[p_1p_2 + (1-p_1)(1-p_2)]$.

American Mathematicians and Woolhouse... p_1p_2.

It seems to me that the true answer cannot be expressed in terms of only p_1 and p_2, but that it involves two other constants β and k; and my value is—

Cayley ....$p_1p_2 + \beta(1-p_1)(1-p_2) + k(1-\beta)(1-p_1)$.

In obtaining this I introduce, *but I think of necessity*, elements which Mr Woolhouse calls extraneous and imperfect.

B told A that the event happened, or he did not tell A this; the only evidence is A's statement that B told him that the event happened: and the chances are p_1 and $1-p_1$. But, in the latter case, either B told A that the event did not happen, or he did not tell him at all; the chances (on the supposition of the incorrectness of A's statement) are β and $1-\beta$; and the chances of the three cases are thus p_1, $\beta(1-p_1)$, and $(1-\beta)(1-p_1)$. On the suppositions of the first and second cases respectively, the chances for the event having happened are p_2 and $1-p_2$; on the supposition of the third case (viz. here there is no information as to the event) the chance is k, the antecedent probability; and the whole chance in favour of the event is

$$p_1p_2+\beta(1-p_1)(1-p_2)+k(1-\beta)(1-p_1).$$

If $\beta=1$, we have Todhunter's solution; if $\beta=0$, and also $k=0$, we have the solution preferred by Woolhouse; but we do not (otherwise than by establishing between k and β a relation which is quite arbitrary) obtain Martin's solution. The error in this seems to be as follows:—A says that B told him as to the event, and says further that B told him that the event did happen; the probability of the truth of the compound statement is taken to be $=p_1^2$; whereas, in calling the probability of A's speaking the truth p_1, we mean that if A makes the statement, "B says that the event took place," this is to be regarded as a simple statement, and the probability of the truth of the statement is $=p_1$; viz. I think that Martin introduces into his solution a hypothesis contradictory to the assumptions of the question.

I remark further that in my solution I assume that the event is of such a nature that, when there is *any* testimony in regard to it, the probability is determined by that testimony, irrespectively of the antecedent probability. This is quite consistent with the antecedent probability being, not zero, but as small as we please; so that, if k is (as it may very well be) indefinitely small, the whole probability is the same as if k were $=0$. But there is absolutely no reason for assigning any determinate value to β; so that the solutions $p_1p_2+(1-p_1)(1-p_2)$ and p_1p_2, which assume respectively $\beta=1$ and $\beta=0$, seem to me on this ground erroneous.

[Vol. XXVIII., June to December, 1877, p. 17.]

5306. (Proposed by Professor CAYLEY.)—If α, β, γ, δ; α_1, β_1, γ_1, δ_1, are such that

$$(\alpha_1-\delta_1)(\beta_1-\gamma_1)=(\alpha-\delta)(\beta-\gamma),$$

$$(\beta_1-\delta_1)(\gamma_1-\alpha_1)=(\beta-\delta)(\gamma-\alpha),\quad (\gamma_1-\delta_1)(\alpha_1-\beta_1)=(\gamma-\delta)(\alpha-\beta);$$

show that the three equations

$$\frac{x_1-\alpha_1}{x_1-\delta_1}=\frac{1}{(\beta_1-\delta_1)(\gamma_1-\delta_1)}\{(x-\alpha)^{\frac{1}{2}}(x-\delta)^{\frac{1}{2}}-(x-\beta)^{\frac{1}{2}}(x-\gamma)^{\frac{1}{2}}\}^2,$$

$$\frac{x_1-\beta_1}{x_1-\delta_1}=\frac{1}{(\gamma_1-\delta_1)(\alpha_1-\delta_1)}\{(x-\beta)^{\frac{1}{2}}(x-\delta)^{\frac{1}{2}}-(x-\gamma)^{\frac{1}{2}}(x-\alpha)^{\frac{1}{2}}\}^2,$$

$$\frac{x_1-\gamma_1}{x_1-\delta_1}=\frac{1}{(\alpha_1-\delta_1)(\beta_1-\delta_1)}\{(x-\gamma)^{\frac{1}{2}}(x-\delta)^{\frac{1}{2}}-(x-\alpha)^{\frac{1}{2}}(x-\beta)^{\frac{1}{2}}\}^2,$$

are equivalent to each other; and show also that, consistently with the foregoing relations between the constants, the differences $\alpha_1-\delta_1$, $\beta_1-\delta_1$, $\gamma_1-\delta_1$ may be so determined that the equations in (x, x_1) constitute a particular integral of the differential equation

$$\frac{dx}{\{(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)\}^{\frac{1}{2}}}=\frac{dx_1}{\{(x_1-\alpha_1)(x_1-\beta_1)(x_1-\gamma_1)(x_1-\delta_1)\}^{\frac{1}{2}}}.$$

[Vol. XXIX., January to June, 1878, p. 20.]

4870. (Proposed by Professor CAYLEY.)—Given three conics passing through the same four points; and on the first a point A, on the second a point B, and on the third a point C. It is required to find on the first a point A', on the second a point B', and on the third a point C', such that the intersections of the lines

$A'B'$ and AC, $A'C'$ and AB, lie on the first conic;

$B'C'$ and BA, $B'A'$ and BC, lie on the second conic;

$C'A'$ and CB, $C'B'$ and CA, lie on the third conic.

[Vol. XXIX., pp. 96, 97.]

5625. (Proposed by Professor CAYLEY.)—The equation

$$\{q^2(x+y+z)^2-yz-zx-xy\}^2=4(2q+1)\,xyz\,(x+y+z)$$

represents a trinodal quartic curve having the lines $x=0$, $y=0$, $z=0$, $x+y+z=0$ for its four bitangents; it is required to transform to the coordinates X, Y, Z, where $X=0$, $Y=0$, $Z=0$ represent the sides of the triangle formed by the three nodes.

[Vol. XXXI., January to June, 1879, p. 38.]

5387. (Proposed by Professor CAYLEY.)—Show that a cubic surface has at most 4 conical points, and a quartic surface at most 16 conical points.

[Vol. XXXII., July to December, 1879, p. 35.]

5927. (Proposed by Professor CAYLEY.)—If $\{\alpha+\beta+\gamma+\ldots\}^p$, denote the expansion of $(\alpha+\beta+\gamma+\ldots)^p$, retaining those terms $N\alpha^a\beta^b\gamma^c\delta^d\ldots$ only in which

$$b+c+d+\ldots \not> p-1,\quad c+d+\ldots \not> p-2, \text{ \&c. \&c.};$$

prove that

$$x^n=(x+\alpha)^n-(n)_1\{\alpha\}^1(x+\alpha+\beta)^{n-1}+\frac{n(n-1)}{1.2}\{\alpha+\beta\}^2(x+\alpha+\beta+\gamma)^{n-2}$$

$$-\frac{n(n-1)(n-2)}{1.2.3}\{\alpha+\beta+\gamma\}^3(x+\alpha+\beta+\gamma+\delta)^{n-3}+\text{\&c.}\ldots(1).$$

[Vol. XXXIII., January to July, 1880, p. 17.]

6155. (Proposed by Professor CAYLEY.)—Given, by means of their metrical coordinates, any two lines; it is required to find their inclination, shortest distance, and the coordinates of the line of shortest distance.

N.B.—If λ, μ, ν are the inclinations of a line to three rectangular axes, and α, β, γ the coordinates to the same axes of a point on the line, then the metrical coordinates of the line are

$$\begin{array}{cccccc} a, & b, & c, & f, & g, & h, \\ = \cos\lambda, & \cos\mu, & \cos\nu, & \beta\cos\nu - \gamma\cos\mu, & \gamma\cos\lambda - \alpha\cos\nu, & \alpha\cos\mu - \beta\cos\lambda, \end{array}$$

satisfying identically the relations

$$a^2 + b^2 + c^2 = 1, \quad af + bg + ch = 0.$$

[Vol. XXXVI., 1881, p. 21.]

6470. (Proposed by Professor CAYLEY.)—It is required, by a real or imaginary linear transformation, to express the equation of a given cubic curve in the form

$$xy - z^2 = \{(z^2 - x^2)(z^2 - k^2x^2)\}^{\frac{1}{2}}.$$

[Vol. XXXVI., p. 64.]

6766. (Proposed by Professor CAYLEY.)—Find the stationary and the double tangents of the curve $x^4 + y^4 + z^4 = 0$.

Solution by the PROPOSER.

Take l a fourth root of -1; m and n fourth roots of $+1$; then the 28 double tangents are the lines $x = ly$, $x = lz$, $y = lz$, $(4 + 4 + 4 =)$ 12 lines; and the lines $x + my + nz = 0$, 16 lines; and the first 12 of these, each counted twice, are the 24 stationary tangents. In fact, any one of the 12 lines is an osculating tangent, or line meeting the curve in 4 coincident points; it counts therefore once as a double tangent, and twice as a stationary tangent. There should consequently be 16 other double tangents; and it only needs to be shown that these are the 16 lines $x + my + nz = 0$. Consider any one line $x + my + nz = 0$; for its intersections with the curve $x^4 + y^4 + z^4 = 0$, we have

$$(my + nz)^4 + y^4 + z^4 = 0,$$

or, as this may be written,

$$(my + nz)^4 + m^4y^4 + n^4z^4 = 0;$$

viz. this is

$$2(1,\ 2,\ 3,\ 2,\ 1 \between my,\ nz)^4 = 0,$$

or, what is the same thing,

$$2[(1,\ 1,\ 1 \between my,\ nz)^2]^2 = 0:$$

so that the line is a double tangent, the two points of contact being given by means of the equation $(1, 1, 1 \between my, nz)^2 = 0$; viz. ω being an imaginary cube root of unity, we have $nz = \omega my$ or $\omega^2 my$: and thence, for the points of contact,

$$x : y : z = 1 : \frac{\omega}{m} : \frac{\omega^2}{n}, \text{ or } = 1 : \frac{\omega^2}{m} : \frac{\omega}{n};$$

values which satisfy, as they should do, the two equations

$$x + my + nz = 0 \text{ and } x^4 + y^4 + z^4 = 0.$$

[Vol. XXXVI., pp. 106, 107.]

6800. (Proposed by W. J. C. MILLER, B.A.)—Prove that, if

$$\frac{ayz}{y^2+z^2} = \frac{bzx}{z^2+x^2} = \frac{cxy}{x^2+y^2} = 1,$$

then

$$a^2 + b^2 + c^2 = abc + 4.$$

Note on Question 6800. *By* PROFESSOR CAYLEY.

The identity given by the solution is a very interesting one. Instead of a, b, c, writing $(a, b, c) \div d$, we have

$$4d^3 - d(a^2 + b^2 + c^2) + abc = 0,$$

satisfied by

$$a : b : c : d = x(y^2 + z^2) : y(z^2 + x^2) : z(x^2 + y^2) : xyz;$$

or, considering (a, b, c, d) as the coordinates of a point in space, and (x, y, z) as the coordinates of a point in a plane, we have thus a correspondence between the points of the cubic surface $4d^3 - d(a^2 + b^2 + c^2) + abc = 0$, and the points of the plane. To a given system of values of (x, y, z) there corresponds, it is clear, a single system of values of (a, b, c, d); and it may be shown without difficulty that, to a given system of values of (a, b, c, d) satisfying the equation of the surface, there correspond two systems of values of (x, y, z); the plane and cubic surface have thus a $(1, 2)$ correspondence with each other.

[Vol. XXXVII., 1882, p. 74.]

5244. (Proposed by Professor CAYLEY.)—Writing for shortness

$$L = \beta^2\gamma^2 - \alpha^2\delta^2, \quad M = \gamma^2\alpha^2 - \beta^2\delta^2, \quad N = \alpha^2\beta^2 - \gamma^2\delta^2,$$

$$F = \alpha^2 + \delta^2 - \beta^2 - \gamma^2, \quad G = \beta^2 + \delta^2 - \gamma^2 - \alpha^2, \quad H = \gamma^2 + \delta^2 - \alpha^2 - \beta^2,$$

$$\Delta = \alpha^2 + \beta^2 + \gamma^2 + \delta^2;$$

show that the equation

$$LMN(x^4+y^4+z^4+w^4)+MN(F\Delta+2L)(y^2z^2+x^2w^2)+NL(G\Delta+2M)(z^2x^2+y^2w^2)$$
$$+LM(H\Delta+2N)(x^2y^2+z^2w^2)-2\alpha\beta\gamma\delta\, FGH\Delta\, xyzw=0$$

belongs to a 16-nodal quartic surface, having the nodes

$x=\alpha$	α	α	α	β	β	β	β	γ	γ	γ	γ	δ	δ	δ	δ
$y=\beta$	$-\beta$	$-\beta$	β	α	$-\alpha$	$-\alpha$	α	δ	$-\delta$	$-\delta$	δ	γ	$-\gamma$	$-\gamma$	γ
$z=\gamma$	$-\gamma$	γ	$-\gamma$	δ	$-\delta$	δ	$-\delta$	α	$-\alpha$	α	$-\alpha$	β	$-\beta$	β	$-\beta$
$w=\delta$	δ	$-\delta$	$-\delta$	γ	γ	$-\gamma$	$-\gamma$	β	β	$-\beta$	$-\beta$	α	α	$-\alpha$	$-\alpha$

and the sixteen singular tangent planes represented by the equations

$$(\alpha, \beta, \gamma, \delta)(x, y, z, w)=0, \text{ \&c.}$$

[Vol. XXXVIII., 1883, pp. 87—89.]

7190. (Proposed by Professor WOLSTENHOLME, M.A.)—If x, y, z be three quantities satisfying the two symmetrical equations

$$yz+zx+xy=0, \quad x^3+y^3+z^3+4xyz=0;$$

prove that (1) they will also satisfy one of the two pairs of semi-symmetrical expressions

$$y^2z+z^2x+x^2y=(y-z)(z-x)(x-y), \ =+xyz,$$
$$yz^2+zx^2+xy^2=(y-z)(z-x)(x-y), \ =-xyz;$$

and (2) one set of the following equations will also be satisfied:—

$$(x^2+yz-y^2=0, \quad y^2+zx-z^2=0, \quad z^2+xy-x^2=0);$$
$$(x^2+yz-z^2=0, \quad z^2+zx-x^2=0, \quad z^2+xy-y^2=0).$$

Solution by PROFESSOR CAYLEY.

The two symmetrical equations represent a conic and a cubic respectively; they intersect therefore in 6 points, and if we denote by α a root of the equation

$$u^3+u^2-2u-1=0,$$

then the other two roots of this equation are

$$\beta, =-1-\frac{1}{\alpha}, \quad \gamma=\frac{-1}{\alpha+1};$$

viz. if $\alpha^3+\alpha^2-2\alpha-1=0$, then we have

$$(u-\alpha)\left(u+1+\frac{1}{\alpha}\right)\left(u+\frac{1}{\alpha+1}\right)=u^3+u^2-2u-1,$$

an identity which is easily verified. It may be remarked that, if

$$\phi\alpha = -1 - \frac{1}{\alpha},$$

then

$$\phi^2\alpha = \frac{-1}{\alpha+1}, \quad \phi^3\alpha = \alpha;$$

the left-hand side of the last mentioned equation thus is $(u-\alpha)(u-\phi\alpha)(u-\phi^2\alpha)$, which remains unaltered when α is changed into $\phi\alpha$ or $\phi^2\alpha$. Then the coordinates of the six points of intersection can be expressed indifferently in terms of any one of the roots (α, β, γ), viz. the coordinates are

$$(\alpha^2-1,\ -\alpha,\ -1),\quad (-1,\ \alpha^2-1,\ -\alpha),\quad (-\alpha,\ -1,\ \alpha^2-1), \ldots (1,\ 2,\ 3),$$
$$(\alpha^2-1,\ -1,\ -\alpha),\quad (-\alpha,\ \alpha^2-1,\ -1),\quad (-1,\ -\alpha,\ \alpha^2-1), \ldots (4,\ 5,\ 6);$$

or they are equal to the like expressions in β and in γ respectively; say these are the coordinates of the points 1, 2, 3, 4, 5, 6 respectively, as shown by the attached numbers. Thus, writing

$$x, y, z = \alpha^2-1,\ -\alpha,\ -1,$$

we find

$$yz+zx+xy = \alpha-\alpha^2+1-\alpha^3+\alpha = -(\alpha^3+\alpha^2-2\alpha-1) = 0,$$

$$\begin{aligned} x^3+y^3+z^3+4xyz &= (\alpha^2-1)^3-\alpha^3-1+4\alpha(\alpha^2-1) \\ &= \alpha^6-3\alpha^4+3\alpha^3+3\alpha^2-4\alpha-2 = (\alpha^3+\alpha^2-2\alpha-1)(\alpha^3-\alpha^2+2) = 0, \end{aligned}$$

which verifies the formulæ for the six points of intersection. Take, again,

$$x, y, z = \alpha^2-1,\ -\alpha,\ -1;$$

then we find

$$yz^2+zx^2+xy^2 = -\alpha\ -(\alpha^2-1)^2+\alpha^2(\alpha^2-1) = \alpha^2-\alpha-1,$$
$$y^2z+z^2x+x^2y = -\alpha^2+(\alpha^2-1)\ -\alpha\ (\alpha^2-1)^2 = -\alpha^5+2\alpha^3-\alpha-1.$$

Or, since $\alpha^3 = -\alpha^2+2\alpha+1$, and thence

$$\alpha^4 = 3\alpha^2-\alpha-1, \quad \alpha^5 = -4\alpha^2+5\alpha+3,$$

the last equation becomes

$$y^2z+z^2x+x^2y = 2\alpha^2-2\alpha-2.$$

We have also

$$xyz = \alpha^3-\alpha,\ = -\alpha^2+\alpha+1;$$

hence the point in question is situate on each of the cubics

$$yz^2+zx^2+xy^2+xyz = 0, \quad y^2z+z^2x+x^2y+2xyz = 0,$$
$$y^2z+z^2x+x^2y-2(yz^2+zx^2+xy^2) = 0;$$

and this, of course, shows the points 1, 2, 3 are all three of them situate upon each of the three cubics; and in precisely the same manner it appears that the points 4, 5, 6 are all three of them situate on each of the three cubics

$$yz^2+zx^2+xy^2+2xyz = 0, \quad y^2z+z^2x+x^2y+xyz = 0,$$
$$yz^2+zx^2+xy^2-2(y^2z+z^2x+x^2y) = 0.$$

Again, from the values $x, y, z = \alpha^2 - 1, \ -\alpha, \ -1$, we have

$$x^2 + yz - y^2 = 0, \quad y^2 + zx - z^2 = 0, \quad z^2 + xy - x^2 = 0\,;$$

viz. the point 1 lies on each of these conics; similarly the point 2 lies on each of the same conics; and the point 3 lies on each of the same conics; that is, the conics in question have in common the points 1, 2, 3.

In like manner, the conics

$$x^2 + yz - z^2 = 0, \quad y^2 + zx - x^2 = 0, \quad z^2 + xy - y^2 = 0,$$

have in common the points 4, 5, 6.

The general result is that the given conic and the cubic meet in six points forming two groups of points (1, 2, 3) and (4, 5, 6); through the points (1, 2, 3) we have three cubics and three conics; and through the points (4, 5, 6) we have three cubics and three conics.

If in the equation $x^3 + x^2 - 2x - 1 = 0$, whose roots are α, $\phi(\alpha)$, $\phi^2(\alpha)$, we put $x = 2\cos\theta$, the equation becomes

$$2(3\cos\theta + \cos 3\theta) + 2(1 + \cos 2\theta) - 4\cos\theta - 2 = 0,$$

or

$$2\cos 3\theta + 2\cos 2\theta + 2\cos\theta = 0, \quad \text{or} \quad \frac{\sin\frac{7}{2}\theta}{\sin\frac{1}{2}\theta} = 0\,;$$

or the three roots are $2\cos\frac{2}{7}\pi$, $2\cos\frac{4}{7}\pi$, $2\cos\frac{6}{7}\pi$. The two equations

$$yz + zx + xy = 0, \quad x^3 + y^3 + z^3 + 3xyz = 0,$$

are satisfied if $x : y : z =$ these three roots in any order, giving the six solutions. The semi-symmetrical systems are satisfied, the one by

$$x : y : z, \ \text{ or } \ y : z : x, \ \text{ or } \ z : x : y, \ = \cos\tfrac{2}{7}\pi : \cos\tfrac{4}{7}\pi : \cos\tfrac{6}{7}\pi\,;$$

and the other by

$$z : y : x, \ \text{ or } \ y : x : z, \ \text{ or } \ x : z : y, \ = \cos\tfrac{2}{7}\pi : \cos\tfrac{4}{7}\pi : \cos\tfrac{6}{7}\pi.$$

[Vol. XXXIX., 1883, p. 31.]

5689. (Proposed by Professor CAYLEY.)—Show (1) that the apparent contour of a Steiner's surface $(2xyz + y^2z^2 + z^2x^2 + x^2y^2 = 0)$, as seen from an exterior point on a nodal line (say the axis of z), projected on the plane of the other two nodal lines, is an ellipse passing through the four points $(\pm 1, 0)$ and $(0, \pm 1)$; and (2) find the surface-contour, or curve of contact, of the cone and surface.

[Vol. XXXIX., p. 49.]

4722. (Proposed by Professor CAYLEY.)—1. Show that the conditions in regard to the reality of the roots of the equation

$$(x^2-\alpha)^2+16A(x-m)=0,$$

are, if

$$(4m^2-3\alpha)^3-(8m^3-9m\alpha-27A)^2=-,$$

then the roots are two real, two imaginary; but if

$$(4m^2-3\alpha)^3-(8m^3-9m\alpha-27A)^2=+,$$

then, if simultaneously

$$\alpha=+,\quad A(m\alpha-9A)=+,$$

the roots are all real, but otherwise they are all imaginary.

2. If the roots of the foregoing equation are all imaginary, then for any real value whatever of y, the roots of the equation

$$(x^2+y^2-\alpha)^2+16A(x-m)=0$$

are all imaginary.

[Vol. XXXIX., pp. 69, 70.]

4387. (Proposed by Professor CAYLEY.)—Using the term "Cassinian" to denote a bi-circular quartic having four foci in a right line; show that the equation of a Cassinian having for its four foci the points $x=a$, $x=b$, $x=c$, $x=d$ on the axis of x, may be written in the four equivalent forms

$$\left(\begin{array}{cccc} \cdot\ , & \tau(d-c), & \sigma(b-d), & \rho(c-b) \\ \tau(c-d), & \cdot\ , & \rho(d-a), & \sigma(a-c) \\ \sigma(d-b), & \rho(a-d), & \cdot\ , & \tau(b-a) \\ \rho(b-c), & \sigma(c-a), & \tau(a-b), & \cdot \end{array}\right)(A^{\frac{1}{2}},\ B^{\frac{1}{2}},\ C^{\frac{1}{2}},\ D^{\frac{1}{2}})=0,$$

that is,

$$\cdot\qquad \tau(d-c)B^{\frac{1}{2}}+\sigma(b-d)C^{\frac{1}{2}}+\rho(c-b)D^{\frac{1}{2}}=0,$$

$$\tau(c-d)A^{\frac{1}{2}}\qquad \cdot\qquad +\rho(d-a)C^{\frac{1}{2}}+\sigma(a-c)D^{\frac{1}{2}}=0,$$

&c., &c.,

where $A^{\frac{1}{2}}$, $B^{\frac{1}{2}}$, $C^{\frac{1}{2}}$, $D^{\frac{1}{2}}$ are the distances from the four foci respectively, and the parameters ρ, σ, τ are connected by the equation

$$\rho^2(a-d)(b-c)+\sigma^2(b-d)(c-a)+\tau^2(c-d)(a-b)=0.$$

Show also that the curve has, at right angles to the axis of x, two double tangents, the equation whereof is any one of the three equivalent forms

$$(a+d-2x)(b+c-2x)\ :\ (b+d-2x)(c+a-2x)\ :\ (c+d-2x)(a+b-2x)=\rho^2\ :\ \sigma^2\ :\ \tau^2.$$

[Vol. XL., 1884, p. 32.]

7376. (Proposed by Professor CAYLEY.)—Show how the construction of a regular heptagon may be made to depend on the trisection of the angle $\cos^{-1}\left(\frac{1}{2\sqrt{7}}\right)$.

[Vol. XL., p. 110.]

7352. (Proposed by Professor CAYLEY.)—Denoting by $x, y, z, \xi, \eta, \zeta$ homogeneous linear functions of four coordinates, such that identically

$$x+y+z+\xi+\eta+\zeta=0, \quad ax+by+cz+f\xi+g\eta+h\zeta=0,$$

where $af=bg=ch=1$; show that

$$\sqrt{(x\xi)}+\sqrt{(y\eta)}+\sqrt{(z\zeta)}=0$$

is the equation of a quartic surface having the sixteen singular tangent planes (each touching it along a conic)

$$x=0, \quad y=0, \quad z=0, \quad \xi=0, \quad \eta=0, \quad \zeta=0,$$

$$x+y+z=0, \quad x+\eta+z=0, \quad ax+by+cz=0, \quad ax+g\eta+cz=0,$$

$$\xi+y+z=0, \quad x+y+\zeta=0, \quad f\xi+by+cz=0, \quad ax+by+h\zeta=0,$$

$$\frac{x}{1-bc}+\frac{y}{1-ca}+\frac{z}{1-ab}=0, \quad \frac{\xi}{1-gh}+\frac{\eta}{1-hf}+\frac{\zeta}{1-fg}=0.$$

[Vol. XLI., 1884, p. 37.]

5421. (Proposed by Professor CAYLEY.)—Suppose

$$S_x=m_1(x-a_1),\ m_2(x-a_2),\ m_3(x-a_3),\ m_4(x-a_4);$$

where, for any given value of x, we write +, −, or 0, according as the linear function is positive, negative, or zero, and where the order of the terms is not attended to. If x is any one of the values a_1, a_2, a_3, a_4, the corresponding S is $0+++$, $0---$, $0++-$, or $0+--$: and if I denote indifferently the first or the second form, and R denote indifferently the third or the fourth form: then it is to be shown that the four S's are R, R, R, R, or else R, R, I, I.

[Vol. XLIV., 1886, p. 109.]

8340. (Proposed by F. MORLEY, B.A.)—Show that (1) on a chess-board the number of squares visible is 204, and the number of rectangles (including squares) visible is 1,296; and (2) on a similar board, with n squares in each side, the number of squares is the sum of the first n square numbers, and the number of rectangles (including squares) is the sum of the first n cube numbers.

Solution by PROFESSOR CAYLEY.

In a board of n^2 squares, the number of pairs of vertical lines at a distance from each other of $n-r+1$ squares is $=r$; and the number of pairs of horizontal

	4	3	2	1
4	1	2	3	4
3	2	4	6	8
2	3	6	9	12
1	4	8	12	16

lines at a distance from each other of $n-s$ squares is $=s$. Hence the number of rectangles, breadth $n-r+1$ and depth $n-s+1$, or say the number of

$$(n-r+1)(n-s+1)$$

rectangles, is $=rs$.

For instance, $n=4$, the number of rectangles 44, 43, 34, &c., is shown in the diagram; hence the whole number of rectangles is $(1+2+3+4)^2 = 1^3+2^3+3^3+4^3$, and so for any value of n.

The same diagram shows that the whole number of squares is $=1^2+2^2+3^2+4^2$; and so for any value of n.

[Vol. XLVI., 1887, pp. 49, 50.]

8636. (Proposed by Professor MAHENDRA NATH RAY, M.A., LL.B.)—Show that the following equations are satisfied by the *same* value of x, and find this value:—

$$ax(x^2-a^2)^{\frac{1}{2}} + bx(x^2-b^2)^{\frac{1}{2}} + cx(x^2-c^2)^{\frac{1}{2}} = 2abc,$$

$$2(x^2-a^2)^{\frac{1}{2}}(x^2-b^2)^{\frac{1}{2}}(x^2-c^2)^{\frac{1}{2}} = x(a^2+b^2+c^2-2x^2).$$

Solution by PROFESSOR CAYLEY.

The second equation rationalised gives

$$4x^6 - 4x^4(a^2+b^2+c^2) + 4x^2(b^2c^2+c^2a^2+a^2b^2) - 4a^2b^2c^2 = 4x^6 - 4x^4(a^2+b^2+c^2) + x^2(a^2+b^2+c^2)^2;$$

that is,

$$\nabla x^2 = 4a^2b^2c^2,$$

if, for shortness,

$$\nabla = -a^4 - b^4 - c^4 + 2b^2c^2 + 2c^2a^2 + 2a^2b^2.$$

We thence find

$$\nabla (x^2 - a^2) = a^2 (-a^2 + b^2 + c^2)^2,$$

$$\nabla (x^2 - b^2) = b^2 (a^2 - b^2 + c^2)^2, \quad \nabla (x^2 - c^2) = c^2 (a^2 + b^2 - c^2)^2,$$

and therefore also

$$\nabla^2 a^2 x^2 (x^2 - a^2) = 4a^6 b^2 c^2 (-a^2 + b^2 + c^2)^2, \text{ \&c.}$$

Or, assuming the sign of the square roots,

$$\nabla ax (x^2 - a^2)^{\frac{1}{2}} = 2abc (-a^4 + a^2b^2 + a^2c^2), \quad \nabla bx (x^2 - b^2)^{\frac{1}{2}} = 2abc (+a^2b^2 - b^4 + b^2c^2),$$
$$\nabla cx (x^2 - c^2)^{\frac{1}{2}} = 2abc (a^2c^2 + b^2c^2 - c^4),$$

whence, adding, the whole divides by ∇ and we have

$$ax (x^2 - a^2)^{\frac{1}{2}} + bx (x^2 - b^2)^{\frac{1}{2}} + cx (x^2 - c^2)^{\frac{1}{2}} = 2abc,$$

the second equation. Observe that the second equation rationalised gives an equation of the form $(x^2, 1)^4 = 0$; the foregoing value $x^2 = 4a^2b^2c^2/\Delta$ is thus one of the four values of x^2.

[Vol. XLVII., 1887, p. 141.]

5271. (Proposed by Professor CAYLEY.)—If ω be an imaginary cube root of unity, show that, if

$$y = \frac{(\omega - \omega^2) x + \omega^2 x^3}{1 - \omega^2 (\omega - \omega^2) x^2},$$

then

$$\frac{dy}{(1 - y^2)^{\frac{1}{2}} (1 + \omega y^2)^{\frac{1}{2}}} = \frac{(\omega - \omega^2) dx}{(1 - x^2)^{\frac{1}{2}} (1 + \omega x^2)^{\frac{1}{2}}};$$

and explain the general theory.

[Vol. L., 1889, p. 189.]

3105. (Proposed by Professor CAYLEY.)—The following singular problem of literal partitions arises out of the geometrical theory given in Professor Cremona's Memoir, "Sulle trasformazioni geometriche delle figure piane," *Mem. di Bologna*, tom. V. (1865). It is best explained by an example:—A number is made up in any manner with the parts 2, 5, 8, 11, &c., viz. the parts are always the positive integers $\equiv 2 \pmod 3$; for instance, $27 = 1.11 + 8.2$. Forming, then, the product of 27 factors $a^{11} (bcdefghi)^2$, this may be partitioned on the same type $1.11 + 8.2$ as follows,

$$a^3bcdefghi, \quad ab, \quad ac, \quad ad, \quad ae, \quad af, \quad ag, \quad ah, \quad ai.$$

(Observe that the partitionment is to be symmetrical as regards the letters which have a common index.) But, to take another example,

$$37 = 0.11 + 3.8 + 1.5 + 4.2 = 1.11 + 0.8 + 4.5 + 3.2.$$

The first of these gives the product $(abc)^8 d^5 (efgh)^2$, which cannot be partitioned (symmetrically as above) on its own type, though it may be on the second type; and the second gives the product $a^{11} (bcde)^5 (fgh)^2$, which cannot be partitioned (symmetrically as above) on its own type, though it may be on the first type; viz. the partitions of the two products respectively are:

First product on second type,

$$(abc)^2 defgh,\quad abcde,\quad abcdf,\quad abcdg,\quad abcdh,\quad ab,\quad ac,\quad bc;$$

Second product on first type,

$$a^2bcdefg,\quad a^2bcdefh,\quad a^2bcdegh,\quad abcde,\quad ab,\quad ac,\quad ad,\quad ae;$$

so that in the first example the type is sibi-reciprocal, but in the second example there are two conjugate types. Other examples are:

Parts	48	54	55	56	53		55		No.
2	14	3	1	0	3	6	0	2	Reciprocals.
5	0	2	3	0	6	0	5	0	
8	0	3	2	7	0	1	2	5	
11	0	0	2	0	0	3	0	1	
14	0	1	0	0	0	0	1	0	
17	0	0	0	0	1	0	0	0	
20	1	0	0	0	0	0	0	0	

viz. the first four columns give each of them a sibi-reciprocal type, but the last two double columns give conjugate types. It is required to investigate the general solution.

[Vol. L., p. 191.]

3304. (Proposed by Professor CAYLEY.)—The coordinates x, y, z being proportional to the perpendicular distances from the sides of an equilateral triangle, it is required to trace the curve

$$(y-z)\sqrt{x}+(z-x)\sqrt{y}+(x-y)\sqrt{z}=0.$$

[Prof. Cayley remarks that the curve in question is a particular case of that which presents itself in the following theorem, communicated to him (with a demonstration) several years ago by Mr J. Griffiths:—

The locus of a point (x, y, z) such that its pedal circle (that is, the circle which passes through the feet of the perpendiculars drawn from the point in question upon the sides of the triangle of reference) touches the nine-point circle, is the sextic curve

$$\left\{x \cos A\,(y \cos B - z \cos C)\left(\frac{y}{\cos B} - \frac{z}{\cos C}\right)\right\}^{\frac{1}{2}}$$
$$+\left\{y \cos B\,(z \cos C - x \cos A)\left(\frac{z}{\cos C} - \frac{x}{\cos A}\right)\right\}^{\frac{1}{2}}$$
$$+\left\{z \cos C\,(x \cos A - y \cos B)\left(\frac{x}{\cos A} - \frac{y}{\cos B}\right)\right\}^{\frac{1}{2}} = 0.$$

It would be an interesting problem to trace this more general curve.]

[Vol. L., p. 192.]

3481. (Proposed by Professor CAYLEY.)—Find, in the Hamiltonian form

$$\frac{d\eta}{dt} = \frac{dH}{d\varpi}, \quad \frac{d\varpi}{dt} = -\frac{dH}{d\eta}, \text{ \&c.,}$$

the equations for the motion of a particle acted on by a central force.

[Vol. LV., 1891, p. 27.]

10716. (Proposed by Professor CAYLEY.)—In a hexahedron $ABCDA'B'C'D'$ the plane faces of which are $ABCD$, $A'B'C'D'$, $A'ADD'$, $D'DCC'$, $C'CBB'$, $B'BAA'$, the edges AA', BB', CC', DD' intersect in four points, say AA', DD' in α; BB', CC' in β; CC', DD' in γ; AA', BB' in δ: that is, starting with the duad of lines $\alpha\beta$, $\gamma\delta$, the four edges AA', BB', CC', DD' are the lines $\alpha\delta$, $\beta\delta$, $\beta\gamma$, $\alpha\gamma$ which join the extremities of these duads. Similarly, the four edges AB, CD, $A'B'$, $C'D'$ are the lines joining the extremities of a duad; and the four edges AD, BC, $A'D'$, $B'C'$ are the lines joining the extremities of a duad. The question arises, "Given two duads, is it possible to place them in space so that the two tetrads of joining lines may be eight of the twelve edges of a hexahedron?" The duad $\alpha\beta$, $\gamma\delta$ is considered to be given when there is given the tetrahedron $\alpha\beta\gamma\delta$, which determines the relative position of the two finite lines $\alpha\beta$ and $\gamma\delta$.

[Vol. LXI., 1894, pp. 122, 123.]

3162. (Proposed by Professor CAYLEY.)—By a proper determination of the coordinates, the skew cubic through any six given points may be taken to be

$$x : y : z = y : z : w;$$

or, what is the same thing, the coordinates of the six given points may be taken to be

$$(1,\ t_1,\ t_1^2,\ t_1^3),\ \ldots,(1,\ t_6,\ t_6^2,\ t_6^3).$$

Assuming this, it is required to show that if

$$p_1=\Sigma t_1,\ p_2=\Sigma t_1t_2,\ldots,\ p_6=t_1t_2t_3t_4t_5t_6,$$

and if

$$\nabla=6xyzw-4xz^3-4y^3w+3y^2z^2-x^2w^2;$$

then the equation of the Jacobian surface of the six points is

$$\left.\begin{array}{l} 3(\ xp_3 \qquad\quad +\ zp_1-2w\ )\,\delta_x\nabla \\ +\ (\qquad\qquad\qquad 2zp_2-wp_1)\,\delta_y\nabla \\ +\ (\ xp_5-2yp_4 \qquad\qquad\ )\,\delta_z\nabla \\ +\ (2xp_6-\ yp_5 \qquad\quad -wp_3)\,\delta_w\nabla \end{array}\right\}=0.$$

[Vol. LXI., p. 123.]

3185. (Proposed by Professor CAYLEY.)—An unclosed polygon of $(m+1)$ vertices is constructed as follows: viz. the abscissæ of the several vertices are 0, 1, 2, …, m, and corresponding to the abscissa k, the ordinate is equal to the chance of $m+k$ heads in $2m$ tosses of a coin; and m then continually increases up to any very large value; what information in regard to the successive polygons, and to the areas of any portions thereof, is afforded by the general results of the Theory of Probabilities?

[Vol. LXI., p. 124.]

3229. (Proposed by Professor CAYLEY.)—It is required to find the value of the elliptic integral $F(c,\ \theta)$ when c is very nearly $=1$ and θ very nearly $=\frac{1}{2}\pi$; that is, the value of

$$\int_0^{\frac{1}{2}\pi-\alpha}\frac{d\theta}{\{1-(1-b^2)\sin^2\theta\}^{\frac{1}{2}}},$$

where α, b are each of them indefinitely small.

N.B.—Observe that, for $\alpha=0$, b small, the value is equal $\log 4/b$, and for $b=0$, α small, the value is $-\log\cot\frac{1}{2}\alpha$.

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