895.

A THEOREM ON TREES.

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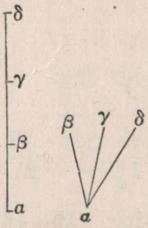
THE number of trees which can be formed with $n+1$ given knots $\alpha, \beta, \gamma, \ldots$ is $=(n+1)^{n-1}$; for instance $n=3$, the number of trees with the 4 given knots $\alpha, \beta, \gamma, \delta$ is $4^2=16$, for in the first form shown in the figure the $\alpha, \beta, \gamma, \delta$ may be arranged



$12+4=16$

in 12 different orders ($\alpha\beta\gamma\delta$ being regarded as equivalent to $\delta\gamma\beta\alpha$), and in the second form any one of the 4 knots $\alpha, \beta, \gamma, \delta$ may be in the place occupied by the α: the whole number is thus $12+4, =16$.

Considering for greater clearness a larger value of n, say $n=5$, I state the particular case of the theorem as follows:

No. of trees $(\alpha, \beta, \gamma, \delta, \epsilon, \zeta) =$ No. of terms of $(\alpha+\beta+\gamma+\delta+\epsilon+\zeta)^4\,\alpha\beta\gamma\delta\epsilon\zeta, = 6^4, = 1296$,

and it will be at once seen that the proof given for this particular case is applicable for any value whatever of n.

I use for any tree whatever the following notation: for instance, in the first of the forms shown in the figure, the branches are $\alpha\beta$, $\beta\gamma$, $\gamma\delta$; and the tree is said to be $\alpha\beta^2\gamma^2\delta$ (viz. the knots α, δ occur each once, but β, γ each twice); similarly in the second of the same forms, the branches are $\alpha\beta$, $\alpha\gamma$, $\alpha\delta$, and the tree is said to be $\alpha^3\beta\gamma\delta$ (viz. the knot α occurs three times, and the knots β, γ, δ each once). And so in other cases.

This being so, I write

$$(\alpha+\beta+\gamma+\delta+\epsilon+\zeta)^4\,\alpha\beta\gamma\delta\epsilon\zeta = \left.\begin{array}{rlr} 1 & \alpha^4 & 6 \\ +\ 4 & \alpha^3\beta & 30 \\ +\ 6 & \alpha^2\beta^2 & 15 \\ +\ 12 & \alpha^2\beta\gamma & 60 \\ +\ 24 & \alpha\beta\gamma\delta & 15 \end{array}\right\} \alpha\beta\gamma\delta\epsilon\zeta; \quad \begin{array}{r} 6 \\ 120 \\ 90 \\ 720 \\ \underline{360} \\ 1296, \end{array}$$

where the numbers of the left-hand column are the polynomial coefficients for the index 4; and the numbers of the right-hand column are the numbers of terms of the several types, 6 terms α^4, 30 terms $\alpha^3\beta$, 15 terms $\alpha^2\beta^2$, &c.: the products of the corresponding terms of the two columns give the outside column 6, 120, 90, &c.; and the sum of these numbers is of course 6^4, $=1296$.

It is to be shown that we have

1 tree $\alpha^4 . \alpha\beta\gamma\delta\epsilon\zeta$ $(=\alpha^5\beta\gamma\delta\epsilon\zeta)$; 4 trees $\alpha^3\beta . \alpha\beta\gamma\delta\epsilon\zeta$ $(=\alpha^4\beta^2\gamma\delta\epsilon\zeta)$, ..., 24 trees $\alpha\beta\gamma\delta . \alpha\beta\gamma\delta\epsilon\zeta$ $(=\alpha^2\beta^2\gamma^2\delta^2\epsilon\zeta)$:

for this being so, then by the mere interchange of letters, the numbers 1, 4, 6, ... of the left-hand column have to be multiplied by the numbers 6, 30, 15, ... of the right-hand column, and we have the numbers in the outside column, the sum of which is $=1296$ as above.

Start with the last term $\alpha\beta\gamma\delta . \alpha\beta\gamma\delta\epsilon\zeta$, $=\alpha^2\beta^2\gamma^2\delta^2\epsilon\zeta$. We have the trees

$$\epsilon\alpha\beta\gamma\delta\zeta \; (=\epsilon\alpha . \alpha\beta . \beta\gamma . \gamma\delta . \delta\zeta),$$

where the α, β, γ, δ may be written in any one of the 24 orders, and the number of such trees is thus $=24$. If we consider only the 12 orders ($\alpha\beta\gamma\delta$ being regarded as equivalent to $\delta\gamma\beta\alpha$), then the ϵ, ζ may be interchanged; and the number is thus 2×12, $=24$ as before.

Now for the δ of $\alpha\beta\gamma\delta$ substitute α, or consider the form $\alpha\beta\gamma\alpha . \alpha\beta\gamma\delta\epsilon\zeta$, $=\alpha^3\beta^2\gamma^2\delta\epsilon\zeta$. We see at once in the form $\epsilon\alpha . \alpha\beta . \beta\gamma . \gamma\delta . \delta\zeta$, which one it is of the two δ's that must be changed into α: in fact, changing the first δ, we should have $\epsilon\alpha . \alpha\beta . \beta\gamma . \gamma\alpha . \delta\zeta$ which contains a circuit $\alpha\beta\gamma$, and a detached branch $\delta\zeta$, and is thus *not* a tree: changing the second δ, we have $\epsilon\alpha . \alpha\beta . \beta\gamma . \gamma\delta . \alpha\zeta$ which is a tree $\alpha^3\beta^2\gamma^2\delta\epsilon\zeta$, $=\alpha\zeta . \alpha\epsilon . \alpha\beta . \beta\gamma . \gamma\delta$. And similarly for any other order of the $\alpha\beta\gamma\delta$, there is in each case only one of the δ's which can be changed into α; and thus from each of the 24 forms we obtain a tree $\alpha^3\beta^2\gamma^2\delta\epsilon\zeta$. But dividing the 24 forms into the $12+12$ forms corresponding to the interchange of the letters ϵ, ζ, then the first 12 forms, and the second 12 forms, give each of them the same trees $\alpha^3\beta^2\gamma^2\delta\epsilon\zeta$; and the number of these trees is thus $\frac{1}{2} . 24$, $=12$.

And in like manner reducing the $\alpha\beta\gamma\delta$ to $\alpha^2\beta^2$, $\alpha^3\beta$ or α^4, we obtain in each case the number of trees equal to the proper sub-multiple of 24, viz. 6, 4, 1 in the three cases respectively (for the last case this is obvious, viz. there is 1 tree $\alpha^5\beta\gamma\delta\epsilon\zeta, = \alpha\beta\,.\,\alpha\gamma\,.\,\alpha\delta\,.\,\alpha\epsilon\,.\,\alpha\zeta$); and the subsidiary theorem is thus proved. Hence the original theorem is true: as already remarked, it is easy to see that the proof is perfectly general.

The theorem is one of a set as follows:

Let $(\lambda, \alpha, \beta, \gamma, \ldots)$ denote as above the trees with the given knots $\lambda, \alpha, \beta, \gamma, \ldots$; $(\lambda+\mu, \alpha, \beta, \gamma, \ldots)$ the pairs of trees with the given knots $\lambda, \mu, \alpha, \beta, \gamma, \ldots$, the knots λ, μ belonging always to different trees; $(\lambda+\mu+\nu, \alpha, \beta, \gamma, \ldots)$ the triads of trees with the given knots $\lambda, \mu, \nu, \alpha, \beta, \gamma, \ldots$, the knots λ, μ, ν always belonging to different trees; and so on: then if $i+1$ be the number of the knots $\lambda, \mu, \nu, \ldots$, and n the number of the knots $\alpha, \beta, \gamma, \ldots$, the number of trees or pairs, or triads, &c., of trees is $=(i+1)(i+n+1)^{n-1}$. In particular, if $i=0$, then n being the number of knots $\alpha, \beta, \gamma, \ldots$, and therefore $n+1$ the whole number of knots $\lambda, \alpha, \beta, \gamma, \ldots$, the number of trees is $=(n+1)^{n-1}$ as before.

As a simple example, consider the pairs $(\lambda+\mu, \alpha, \beta)$: here $i=1$, $n=2$, and we have $(i+1)(i+n+1)^{n-1} = 2\,.\,4, = 8$: in fact, the pairs of trees are

$$(\lambda\alpha, \alpha\beta, \mu), (\lambda\beta, \beta\alpha, \mu), (\lambda\alpha, \lambda\beta, \mu),$$
$$(\mu\alpha, \alpha\beta, \lambda), (\mu\beta, \beta\alpha, \lambda), (\mu\alpha, \mu\beta, \lambda);\ (\lambda\alpha, \mu\beta), (\lambda\beta, \mu\alpha).$$

We may arrange the trees $(\alpha, \beta, \gamma, \delta, \epsilon)$ as follows:

$$\begin{array}{llllll}
(\alpha, \beta, \gamma, \delta, \epsilon) = & \alpha\beta & (\beta, \gamma, \delta, \epsilon); & 125 = & 4\times 1\,.\,4^2 = & 64 \\
& +\alpha\beta\,.\,\alpha\gamma & (\beta+\gamma, \delta, \epsilon) & & +6\times 2\,.\,4^1 & 48 \\
& +\alpha\beta\,.\,\alpha\gamma\,.\,\alpha\delta & (\beta+\gamma+\delta, \epsilon) & & +4\times 3\,.\,4^0 & 12 \\
& +\alpha\beta\,.\,\alpha\gamma\,.\,\alpha\delta\,.\,\alpha\epsilon & & & +1 & 1 \\
\hline
& & & & & 125,
\end{array}$$

viz. to obtain the trees $(\alpha, \beta, \gamma, \delta, \epsilon)$, we join on the branch $\alpha\beta$ to any tree $(\beta, \gamma, \delta, \epsilon)$: the branches $\alpha\beta$, $\alpha\gamma$ to any pair of trees $(\beta+\gamma, \delta, \epsilon)$; the branches $\alpha\beta$, $\alpha\gamma$, $\alpha\delta$ to any triad of trees $(\beta+\gamma+\delta, \epsilon)$; and take lastly the tree $\alpha\beta\,.\,\alpha\gamma\,.\,\alpha\delta\,.\,\alpha\epsilon$: the knots $\beta, \gamma, \delta, \epsilon$ being then interchanged in every possible manner. The whole number of trees 125 is thus obtained as $=64+48+12+1$; the theorem is of course perfectly general.

The foregoing theory in effect presents itself in a paper by Borchardt, "Ueber eine der Interpolation entsprechende Darstellung der Eliminations-Resultante," *Crelle*, t. LVII. (1860), pp. 111—121, viz. Borchardt there considers a certain determinant, composed of the elements 10, 12, ..., $1n$, 20, 21, 23, ..., $2n$, ..., $n0$, $n1$, ..., $nn-1$, and represented by means of the trees $(0, 1, 2, \ldots, n)$; the branches of the tree being the aforesaid elements, and the tree being regarded as equal to the product of the several branches: the number of terms of the determinant is thus $=(n+1)^{n-1}$ as above.