912.

ON THE NOTION OF A PLANE CURVE OF A GIVEN ORDER.

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We have a complete geometrical notion of a curve of a given order, viz. a curve of the order n is a curve which is met by any line whatever in n points and no more; but starting with this definition, how do we know that there exists a curve of the order n? and, further, how do we know that it depends linearly on $\frac{1}{2}n(n+3)$ parameters, or, what is the same thing, that there is one and only one curve which can be drawn through $\frac{1}{2}n(n+3)$ given points?

The last-mentioned property does not by itself constitute a definition of a curve of the order n; thus $n=2$, we cannot define a curve of the second order as a curve which is uniquely determined by the condition of passing through 5 given points; for a cubic passing through 4 given points is a curve uniquely determined by the condition in question; but we may differentiate between these two solutions by adding the further condition that, when 3 of the 5 points are in a line, the curve of the second order shall include as part of itself this line. And we are thus led to the definition: A curve of the order n is a curve which is uniquely determined by the condition of passing through $\frac{1}{2}n(n+3)$ given points; and of being moreover such that, when $n+1$ of these points lie on a line, it includes as part of itself this line.

Starting from the foregoing definition, the first property is, I think, demonstrable, viz. the property that a curve of the order n is met by any line whatever in n points and no more. Thus $n=2$: start with a line chosen at pleasure, and on it take 2 points which are regarded as indeterminate points: to fix the ideas, let one of these be regarded as depending on a parameter λ and the other on a parameter μ, (so that when λ has a determinate value assigned to it, the first point becomes a determinate point, and similarly when μ has a determinate value assigned to it, the second point becomes a determinate point; and consequently, when λ and μ have determinate values, the 2 points are determinate points on the line). Take now any other 3 points not on the line; then, for the moment regarding the 2 points on the line as determinate points, we can through the 5 points draw a curve of the second order; this is the general curve of the second order through the 3 points; for it is a curve of the second order through the 3 points, and which, when the parameters λ and μ are regarded as undetermined and arbitrary, or choosable at pleasure, might be made to pass through any other two points whatever. But by hypothesis, the curve meets the line in question, that is, *any line*, in two points: and the conic cannot meet the line in more than two points; for in like manner, starting with the given line, and upon it 3 points (which may be considered as depending on the parameters λ, μ and ν respectively), and taking any two points not on the line, we have through the 5 points a conic; and this conic, regarding the parameters λ, μ, ν as undetermined and arbitrary, or choosable at pleasure, will be the general conic through the two points; for by a proper determination of the parameters, it might be made to pass through any other 3 points whatever. But, by hypothesis, the conic contains as part of itself the line; that is, the general conic through the 2 points contains as part of itself any line whatever, which is absurd.

So again for the cubic, $n=3$: here starting with a line taken at pleasure, we take on it 3 points, which may be regarded as depending on the parameters λ, μ, ν respectively, and we take any other 6 points not on the line. We have through the 9 points a cubic; and this is the general cubic through the 6 points, for it depends on the parameters λ, μ, ν, which might be determined so as to make the curve pass through any other 3 points whatever; and by hypothesis the cubic meets the line, that is, any line whatever, in 3 points. And it cannot meet it in more than 3 points: for starting with the same line and upon it 4 points depending on the parameters λ, μ, ν, ρ respectively, and taking any other 5 points not upon the line, we then have through the 9 points a cubic which will be the general cubic through the 5 points (for it depends on the 4 parameters λ, μ, ν, ρ). But by hypothesis, the cubic contains as part of itself the line, viz. the general cubic through the 5 points contains as part of itself any line whatever, which is absurd. The reasoning is quite general; and applying to a curve of the order n, the conclusion is that such a curve meets any line in n points and no more.