920.

ON ORTHOMORPHOSIS.

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1. THE equation $x_1 + iy_1 = \phi(x+iy)$, where ϕ is in general an imaginary function (that is, a function with imaginary coefficients), and where (x, y), (x_1, y_1) are rectangular coordinates, determines x_1, y_1, each of them as a function of (x, y), and hence, first eliminating y, we have a set of curves depending on the parameter x, and secondly eliminating x, we have a set of curves depending on the parameter y; we have thus two sets of curves, or say trajectories, which are orthomorphoses of the two sets of lines $x = \text{const.}$ and $y = \text{const.}$ respectively. They thus cut everywhere at right angles, and are moreover such that, giving to the parameters values at equal infinitesimal intervals (the same for x and y respectively), they form a double system of infinitesimal squares; say that we have two systems of square trajectories S and T. We may assume at pleasure a trajectory S, and also the consecutive trajectory S'; but it is at once seen geometrically that the entire system of trajectories S and T is then completely determined. For at any point t of S drawing a normal to meet S', and on S the element tt' equal to the normal distance at t, then at t' drawing a normal to meet S', and on S the element $t't''$ equal to the normal distance at t', and so on, we divide the strip between S and S' into infinitesimal squares: at the successive points of S' drawing normals, and on these measuring off distances equal to the successive elements of S', we construct a new curve S'', at the same time dividing the strip between S' and S'' into infinitesimal squares; and proceeding in this manner, we obtain the series of curves S, S', S'', S''', &c., and at the same time the series of curves T, T', T'', T''', &c., proceeding from the points t, t', t'', t''', &c., respectively, and forming with the first set of curves the double system of infinitesimal squares.

2. But to translate this into Analysis, and to obtain the equations of the two sets of curves, we proceed as follows*.

Suppose that the curve S is given in the form

$$x_1 = p, \quad y_1 = q,$$

where p, q are given real functions of a variable parameter θ: we obtain a series of square trajectories S and T, one of the first set being the given curve S, by forming from these equations the equation

$$x_1 + iy_1 = p + iq,$$

and writing therein $\theta = x + iy$. In fact, for the value $y = 0$, this equation becomes $x_1 = p$, $y_1 = q$, where p, q are now the same functions of x which they were originally of θ, and the elimination of x leads therefore to the equation of the given curve S. More generally, we may take $\theta = f(w)$, an arbitrary real function of w, and then assume $w = x + iy$: in this case, for $y = 0$, we have as before $x_1 = p$, $y_1 = q$, where p and q are now the same functions of $f(x)$ which they originally were of θ, and the elimination of x from these equations thus leads as before to the given curve S.

We have next to determine $f(w)$ in suchwise that the curve corresponding to an infinitesimal real value ϵ of y shall be the given consecutive curve S'. We assume that this curve S' is given by the equations

$$x_1 = p + \gamma P, \quad y_1 = q + \gamma Q,$$

where p, q are as before and P, Q are given real functions of θ; γ is a real infinitesimal. The expression for w as a function of θ is then determined by the condition

$$\gamma = C dw = \frac{(p'^2 + q'^2)\, d\theta}{p'Q - q'P},$$

where p', q' are the derived functions of p, q in regard to θ: C is a real constant, the value of which may be assumed at pleasure. Say we have

$$Cw = \int \frac{(p'^2 + q'^2)\, d\theta}{p'Q - q'P},$$

where the integral may be regarded as containing a real or imaginary constant of integration $x_0 + iy_0$; but this being so, we ultimately have

$$w + x_0 + iy_0 = x + iy, \quad \text{or} \quad w = (x - x_0) + i(y - y_0),$$

which is the same thing as $w = x + iy$.

3. We may, in the expressions for x_1, y_1, substitute for θ any other real value θ_1 whatever, say the equations thus became $x_1 = p_1 + \gamma P_1$, $y_1 = q_1 + \gamma Q_1$, values which give

$$x_1 + iy_1 = p_1 + iq_1 + \gamma(P_1 + iQ_1);$$

* The investigation is taken from the memoir, Beltrami, "Delle variabili complesse sopra una superficie qualunque," *Annali di Matem.*, t. I. (1867), pp. 329—366.

the proof consists in showing that there exists for θ_1 a value, differing infinitesimally from θ and such as to reduce this equation to

$$x_1 + iy_1 = p + iq + (p' + iq')\frac{i\gamma}{C}\frac{d\theta}{dw},$$

the value assumed by $p + iq$ on substituting for w the value $w + \frac{i\gamma}{C}$: for this being so, it is obvious that we have for S' the curve in the series of curves S which corresponds to the value $\frac{\gamma}{C}$ of y in $w = x + iy$. The value of θ_1 is

$$\theta_1 = \theta - \gamma\frac{p'P + q'Q}{p'^2 + q'^2},$$

for this gives

$$p_1 = p - \gamma p'\frac{p'P + q'Q}{p'^2 + q'^2}, \quad q_1 = q - \gamma q'\frac{p'P + q'Q}{p'^2 + q'^2};$$

and since γ is infinitesimal, we may in the terms γP_1 and γQ_1 of x_1 and y_1 write $\theta_1 = \theta$, and so reduce these terms to γP and γQ respectively: we thus have

$$x_1 = p_1 + \gamma P_1 = p - \gamma p'\frac{p'P + q'Q}{p'^2 + q'^2} + \gamma P, \quad = p - \gamma q'\frac{p'Q - q'P}{p'^2 + q'^2},$$

$$y_1 = q_1 + \gamma Q_1 = q - \gamma q'\frac{p'P + q'Q}{p'^2 + q'^2} + \gamma Q, \quad = q + \gamma p'\frac{p'Q - q'P}{p'^2 + q'^2},$$

or for $\frac{p'Q - q'P}{p'^2 + q'^2}$ substituting its value $\frac{1}{C}\frac{d\theta}{dw}$, these values are

$$x_1 = p - \frac{\gamma}{C}q'\frac{d\theta}{dw}, \quad y_1 = q + \frac{\gamma}{C}p'\frac{d\theta}{dw},$$

which give the before-mentioned equation

$$x_1 + iy_1 = p + iq + (p' + iq')\frac{i\gamma}{C}\frac{d\theta}{dw},$$

and the proof is thus completed.

The following two examples are given by Beltrami.

4. First, let the curves S, S' be given confocal ellipses,

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1, \quad \frac{x_1^2}{a^2 - 2cy} + \frac{y_1^2}{b^2 - 2cy} = 1.$$

Here

$$p = a\cos\theta, \quad q = b\sin\theta,$$

whence

$$p'^2 + q'^2 = a^2\sin^2\theta + b^2\cos^2\theta;$$

$$P = -\frac{c}{a}\cos\theta, \quad Q = -\frac{c}{b}\sin\theta,$$

whence

$$p'Q - q'P = \frac{c}{ab}(a^2\sin^2\theta + b^2\cos^2\theta),$$

and consequently

$$Cw = \int \frac{c}{ab}\,d\theta = \frac{c}{ab}\,\theta, \text{ or taking } C = \frac{c}{ab}, \text{ then } w = \theta,$$

and for the two sets of curves we have

$$x_1 + iy_1 = a\cos(x+iy) + ib\sin(x+iy),$$

which are two sets of confocal conics. In verification, write

$$\cos x = X, \quad \sin x = X', \quad \text{whence} \quad X^2 + X'^2 = 1,$$

$$\cos iy = Y, \quad \sin iy = iY', \quad \text{,,} \quad Y^2 - Y'^2 = 1,$$

and then

$$x_1 + iy_1 = a(XY - iX'Y') + ib(YX' + iY'X),$$

that is,

$$x_1 = X(aY - bY'),$$

$$y_1 = X'(bY - aY'),$$

and consequently

$$\frac{x_1^2}{X^2} - \frac{y_1^2}{X'^2} = a^2 - b^2, \text{ which are the curves } T,$$

$$\frac{x_1^2}{(aY-bY')^2} + \frac{y_1^2}{(bY-aY')^2} = 1, \quad \text{,,} \quad \text{,,} \quad \text{,,} \quad S;$$

where observe that

$$(aY-bY')^2 - (bY-aY')^2 = (a^2-b^2)(Y^2-Y'^2) = a^2 - b^2,$$

so that, putting $(aY-bY')^2 = a^2 + h$, we have

$$(bY-aY')^2 = a^2 + h - (a^2 - b^2), \quad = b^2 + h,$$

and thus the curves S are the confocal ellipses

$$\frac{x^2}{a^2+h} + \frac{y^2}{b^2+h} = 1, \quad h > -b^2;$$

and similarly, since $X^2 + X'^2 = 1$, putting $(a^2-b^2)X^2 = a^2 + k$, we have

$$(a^2-b^2)X'^2 = a^2 - b^2 - (a^2+k), \quad = -b^2 - k,$$

and thus the curves T are the confocal hyperbolas

$$\frac{x^2}{a^2+k} + \frac{y^2}{b^2+k} = 1, \quad -k < a^2 > b^2.$$

5. Next, let the curve S be a circle and the curve S' an interior non-concentric circle. It is convenient to take the circles, such that they have for their common chord the line $y_1 = 0$ and each meets this chord in the points $x_1 = \pm i$.

This being so, the equations of the two circles are

$$x_1^2 + y_1^2 - 2y_1 \operatorname{cosec}\mu = -1,$$

$$x_1^2 + y_1^2 - 2y_1 \operatorname{cosec}\mu\,(1 - \gamma\cos^2\mu) = -1,$$

and we have

$$p = \cot\mu\cos\theta, \quad q = \cot\mu\sin\theta + \operatorname{cosec}\mu,$$

$$P = -\cot\mu\cos\theta, \quad Q = -\cot\mu\sin\theta - \cos^2\mu\operatorname{cosec}\theta,$$

giving

$$p'^2 + q'^2 = \cot^2\mu,$$

$$p'Q - q'P = \cot^2\mu\,(1 + \cos\mu\sin\theta),$$

$$\gamma = Cdw = \frac{d\theta}{1 + \cos\mu\sin\theta}, \text{ or if } C = \frac{1}{\sin\mu}, \text{ then } dw = \frac{\sin\mu\, d\theta}{1 + \cos\mu\sin\theta}.$$

The integral of this may be written

$$\tan(\tfrac{1}{2}w - \tfrac{1}{2}iy_0) = \cot\tfrac{1}{2}\mu\tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\theta);$$

and if we here assume for the constant of integration y_0 a value such that

$$\tan\tfrac{1}{2}iy_0 = i\tan\tfrac{1}{2}\mu,$$

we find for w the equation

$$\tan\tfrac{1}{2}w = \cot\mu\, e^{i\theta} + i\operatorname{cosec}\mu,$$

we have $w = x + iy$, and

$$x_1 + iy_1 = p + iq = \cot\mu\, e^{i\theta} + i\operatorname{cosec}\mu,$$

hence

$$x_1 + iy_1 = \tan\tfrac{1}{2}(x + iy);$$

or writing, as we may do, $2x$, $2y$ in place of x, y respectively, say

$$x_1 + iy_1 = \tan(x + iy),$$

for the two sets of curves S and T.

Writing $\tan x = X$, $\tan iy = iY$, this gives

$$x_1 + iy_1 = \frac{X + iY}{1 - iXY} = \frac{(X + iY)(1 + iXY)}{1 + X^2Y^2},$$

that is,

$$x_1 = \frac{X(1 - Y^2)}{1 + X^2Y^2}, \quad y_1 = \frac{Y(1 + X^2)}{1 + X^2Y^2};$$

and therefore

$$x_1^2 + y_1^2 = \frac{X^2 + Y^2}{1 - X^2Y^2},$$

and thence

$$x_1^2 + y_1^2 - \left(Y + \frac{1}{Y}\right)y_1 = -1,$$

$$x_1^2 + y_1^2 - \left(X - \frac{1}{X}\right)x_1 = 1,$$

for the curves S and T respectively. The value $Y = \cot\frac{1}{2}\mu$ gives the original S-curve

$$x_1^2 + y_1^2 - 2y_1\operatorname{cosec}\mu = -1.$$

We have thus the well-known system of orthotomic circles, where the circles of the one set pass through the points $x_1 = 0$, $y_1 = \pm 1$, and those of the other set through the antipoints $x_1 = \pm i$, $y_1 = 0$.

6. A different solution is given in the Dissertation referred to below*: I reproduce this, giving the proof in a somewhat altered form. The equations of the curve S and of the consecutive curve S' are taken to be

$$\phi(x, y) = 0, \quad \phi(x, y) - 2u\,\psi(x, y) = 0;$$

in the investigation, we introduce the conjugate variables $z = x + iy$, $\bar{z} = x - iy$, and write ϕ, ψ to denote the values assumed by the functions $\phi(x, y)$, $\psi(x, y)$ when x, y are replaced therein by their values in terms of z, $\bar{z}$, viz. $x = \frac{1}{2}(z + \bar{z})$, $y = \dfrac{1}{2i}(z - \bar{z})$. But in the first instance we may, without attending to the meanings of z, $\bar{z}$, consider ϕ, ψ as denoting each of them a given function of the two coordinates z, $\bar{z}$.

Suppose now that the functions $f(z)$, $f_1(\bar{z})$ are determined by the equations

$$f(z) = \int \frac{1}{\psi} \frac{d\phi}{dz} dz, \quad f_1(\bar{z}) = \int \frac{1}{\psi} \frac{d\phi}{d\bar{z}} d\bar{z},$$

where in the first integral, after the calculation of $\dfrac{d\phi}{dz}$, $\bar{z}$ is expressed as a function of z by means of the assumed relation $\phi = 0$, and in the second integral after the calculation of $\dfrac{d\phi}{d\bar{z}}$, z is expressed as a function of $\bar{z}$ by means of the same relation $\phi = 0$; and where in each integral the constant is properly determined, as presently appearing.

We have

$$f'(z)\,dz + f_1'(\bar{z})\,d\bar{z} = \frac{1}{\psi}\left(\frac{d\phi}{dz} dz + \frac{d\phi}{d\bar{z}} d\bar{z}\right),$$

where the expression on the right-hand side is $= 0$, if only $\phi = 0$; that is, ϕ being $= 0$, we have $f'(z)\,dz + f_1'(\bar{z})\,d\bar{z} = 0$; and consequently $f(z) + f_1(\bar{z}) = \text{const.}$, viz. this last equation, existing as a consequence of the equation $\phi = 0$, can be nothing else than a different form of the equation $\phi = 0$. We may, by a proper determination of the constants of integration of the two integrals, make the constant of the last-mentioned equation to be $= 0$; we thus have the relation $f(z) + f_1(\bar{z}) = 0$, equivalent to the relation $\phi = 0$.

Writing now $z + \delta z$, $\bar{z} + \delta\bar{z}$ in place of z, $\bar{z}$ respectively, where δz, $\delta\bar{z}$ are arbitrary infinitesimal increments, we have

$$f(z + \delta z) + f_1(\bar{z} + \delta\bar{z}) = 0,$$

equivalent to

$$\phi(z + \delta z, \ \bar{z} + \delta\bar{z}) = 0,$$

* Meyer, *Ueber die von gerade Linien und von Kegelschnitten gebildeten Schaaren von Isothermen, sowie über einige von speciellen Curven dritter Ordnung gebildete Schaaren von Isothermen:* Inaugural Dissertation der Universität zu Göttingen; 4°, Zürich, 1879. I quote the title in full as it explains the object of the paper; the Isothermals referred to are, of course, systems of orthotomic curves S and T: Plates I. to V. exhibit cases where the curves of at least one of the systems are conics, and Plates VI. to XIV. cases where the curves of at least one of the systems are cubics; the discussion of the several systems is very full and interesting.

that is,

$$f(z)+f'(z)\,\delta z+f_1(\bar z)+f_1'(\bar z)\,\delta\bar z=0,$$

equivalent to

$$\phi+\frac{d\phi}{dz}\,\delta z+\frac{d\phi}{d\bar z}\,\delta\bar z=0\,;$$

and if we here assume

$$\delta z=-u\div f'(z),$$

$$\delta\bar z=-u\div f_1'(\bar z),$$

then we have

$$f(z)+f_1(\bar z)-2u=0,$$

equivalent to

$$\phi-u\left\{\frac{1}{f'(z)}\,\frac{d\phi}{dz}+\frac{1}{f_1'(\bar z)}\,\frac{d\phi}{d\bar z}\right\}=0,$$

that is, in virtue of

$$f'(z)=\frac{1}{\psi}\,\frac{d\phi}{dz}\ \text{ and }\ f_1'(\bar z)=\frac{1}{\psi}\,\frac{d\phi}{d\bar z},$$

equivalent to

$$\phi-2u\psi=0.$$

There is the difficulty that the last-mentioned values of $f'(z)$, $f_1'(\bar z)$ do not subsist absolutely, but only when $\phi=0$; the answer to this is that the equation $\phi-2u\psi=0$, where u is infinitesimal, is in effect the equation $\phi=0$. Supposing that z, $\bar z$, instead of being connected by the equation $\phi=0$, are connected by the equation $\phi-2u\psi=0$, then the value of

$$\frac{1}{f'z}\,\frac{d\phi}{dz}+\frac{1}{f_1'(\bar z)}\,\frac{d\phi}{d\bar z},$$

instead of being actually equal to 2ϕ, will be equal to 2ϕ plus an infinitesimal value containing the factor u, and consequently on substituting this value in the expression

$$\phi-u\left\{\frac{1}{f'(z)}\,\frac{d\phi}{dz}+\frac{1}{f_1'(\bar z)}\,\frac{d\phi}{d\bar z}\right\},$$

and neglecting powers of u, we obtain as above the expression $\phi-2u\psi$.

If we now write $z=x+iy$, $\bar z=x-iy$, and assume that the equation $\phi(x, y)=0$ is a real equation, then clearly $f(z)$, $f_1(\bar z)$ as above defined will be conjugate functions of z, $\bar z$ respectively: and u being no longer infinitesimal, we shall have

$$f\,(z)=u+iv,$$

an equation implying the conjugate relation

$$f_1(\bar z)=u-iv.$$

Considering this equation $f(z)=u+iv$ as thus denoting the two equations, and successively eliminating the parameters v and u, we obtain first a system of curves S depending on the parameter u, and secondly a system of curves T depending on the parameter v, which are two systems of orthotomic curves. And by what precedes, it appears that the curve S belonging to the value $u=0$ of the parameter u, and the consecutive curve S' belonging to a real infinitesimal value $u=u$, are the given curves

$$\phi(x, y)=0 \text{ and } \phi(x, y)-2u\psi(x, y)=0,$$

respectively.

7. For instance, let the curves S and S' be the circles

$$x^2+y^2-1=0, \quad x^2+y^2-1-2ux=0;$$

here

$$\phi=z\bar{z}-1, \quad \psi=\tfrac{1}{2}(z+\bar{z}),$$

$$\frac{1}{\psi}\frac{d\phi}{dz}=\frac{2\bar{z}}{z+\bar{z}}=\frac{2\frac{1}{z}}{z+\frac{1}{z}}=\frac{2}{z^2+1};$$

whence, adding $\frac{1}{2}\pi$ for the constant of integration,

$$\int\frac{1}{\psi}\frac{d\phi}{dz}dz=2\tan^{-1}z+\tfrac{1}{2}\pi.$$

The two sets of curves are thus given by

$$u+iv=\tan^{-1}z=\tan^{-1}(x+iy), \quad 2\tan^{-1}(x+iy)=u-\tfrac{1}{2}\pi+iv,$$

that is,

$$x+iy=\tan(\tfrac{1}{2}u-\tfrac{1}{4}\pi+iv);$$

viz. here, if

$$\tan(\tfrac{1}{2}u-\tfrac{1}{2}\pi)=\alpha, \quad \tan\tfrac{1}{2}iv=i\beta,$$

$$x+iy=\frac{\alpha+i\beta}{1-i\alpha\beta}=\frac{\alpha(1-\beta^2)+i\beta(1+\alpha^2)}{1+\alpha^2\beta^2},$$

that is,

$$x=\frac{\alpha(1-\beta^2)}{1+\alpha^2\beta^2}, \quad y=\frac{\beta(1+\alpha^2)}{1+\alpha^2\beta^2},$$

giving

$$x^2+y^2=\frac{\alpha^2+\beta^2}{1+\alpha^2\beta^2},$$

and thence

$$x^2+y^2-1-\left(\alpha-\frac{1}{\alpha}\right)x=0, \quad x^2+y^2+1-\left(\beta-\frac{1}{\beta}\right)y=0,$$

for the two systems of curves. The first of these, substituting for α its value, becomes

$$x^2+y^2-1-2x\tan u=0,$$

and thus the curves, for $u=0$ and u infinitesimal, are

$$x^2+y^2-1=0 \quad\text{and}\quad x^2+y^2-1-2ux=0,$$

as they should be.

8. Reverting to the first-mentioned solution, the equation of the circle $x_1^2+y_1^2=1$ is satisfied by $x_1=\cos\theta$, $y_1=\sin\theta$, and we have therefore $x_1+iy_1=e^{i\theta}$, and hence by what precedes, writing for convenience $X+iY$ instead of $x+iy$, we have

$$x_1+iy_1=e^{i\phi(X+iY)}$$

(ϕ being any real function) for a set of curves S and T, or say S_1 and T_1, wherein one of the curves S_1 is the circle $x_1^2+y_1^2=1$; in fact, writing $Y=0$, we have $x_1+iy_1=e^{i\phi(X)}$, and thence $x_1-iy_1=e^{-i\phi(X)}$, and consequently $x_1^2+y_1^2-1=0$ for one of the S_1 curves.

But similarly, if ψ be any real function, then we have $x+iy=e^{i\psi(X+iY)}$ for a set of curves S and T, wherein one of the S curves is the circle $x^2+y^2-1=0$, and thus the two equations $x_1+iy_1=e^{i\phi(X+iY)}$, $x+iy=e^{i\psi(X+iY)}$ establish a correspondence between the two circumferences $x_1^2+y_1^2-1=0$ and $x^2+y^2-1=0$. To eliminate the X, Y, we may write

$$i\psi(X+iY)=\log(x+iy),$$

or say

$$-\psi(X+iY)=i\log(x+iy);$$

then $\phi(X+iY)$ is a real function of $-\psi(X+iY)$, say it is $=f\{-\psi(X+iY)\}$, and it is thus $=f\{i\log(x+iy)\}$, or we have

$$x_1+iy_1=e^{if\{i\log(x+iy)\}}:$$

as the formula for the correspondence in question. In verification, observe that changing the sign of i, we have

$$x_1-iy_1=e^{-if\{-i\log(x-iy)\}};$$

here if $x^2+y^2-1=0$, that is, $(x+iy)(x-iy)=1$, we have

$$\log(x-iy)=-\log(x+iy),$$

and the last equation is

$$x_1-iy_1=e^{-if\{i\log(x+iy)\}},$$

so that we have $x_1^2+y_1^2-1=0$; and thus the two circumferences $x^2+y^2-1=0$ and $x_1^2+y_1^2-1=0$ correspond to each other.

9. We may write down *à priori* a formula which must be equivalent to the foregoing, viz. if $\phi(x+iy)$ be a real or imaginary function of $x+iy$, and $\bar{\phi}$ be the conjugate function (obtained by changing the sign of i in the coefficients), the formula is

$$x_1+iy_1=\frac{\phi(x+iy)}{\bar{\phi}\left(\dfrac{1}{x+iy}\right)};$$

in fact, here changing the sign of i, we have

$$x_1-iy_1=\frac{\bar{\phi}(x-iy)}{\phi\left(\dfrac{1}{x-iy}\right)},$$

which if $x^2+y^2-1=0$, that is, $(x+iy)(x-iy)=1$, becomes

$$x_1-iy_1=\frac{\bar{\phi}\left(\dfrac{1}{x+iy}\right)}{\phi(x+iy)},$$

and the two equations give $x_1^2+y_1^2-1=0$; the two circumferences thus correspond to each other. It is to be noticed that, if the numerator function contain a factor $(x+iy)^m$, say if $\phi(x+iy)=(x+iy)^m\,\psi(x+iy)$, then the denominator will be

$$(x+iy)^{-m}\,\bar{\psi}\left(\frac{1}{x+iy}\right),$$

and the formula thus becomes

$$x_1+iy_1=\frac{(x+iy)^{2m}\,\psi(x+iy)}{\bar{\psi}\left(\dfrac{1}{x+iy}\right)},$$

which is thus not really more general than the original form. I recur further on to this transformation between the two circumferences.

10. The curve S may be taken to be a closed curve enclosing a singly connected area F (say such a curve is a Contour), and the curve S' a like curve lying wholly inside S, and such that the normal distance between the two curves is everywhere infinitesimal. It is not in general the case, but it may happen that the successive curves S'', S''', &c. will be all of them like curves (each enclosed in the preceding and enclosing the succeeding one), of continually diminishing area, and continually approximating to a point: the curves T will then all of them pass through this point and may be called Radials. We may say that the area F is then squarewise contractible, or contracted, into a point.

11. In particular, a circle is squarewise contractible into a point, viz. if the point be the centre, then the curves S are concentric circles, and the curves T are radii; as is known and will appear, the circle is also contractible into any interior eccentric point whatever. It may be remarked that (the squares being infinitesimal) the numbers of the contours and radials are each of them infinite, but further the number of contours is infinitely great in comparison with that of the radials. Thus in the case just referred to, if to construct a figure we imagine the circumference of the circle divided into any large number n of equal parts, the radials will be the n radii drawn to the points of division: taking the radius to be $=1$, and writing for shortness $\alpha=\dfrac{2\pi}{n}$, the sides of the successive squares along any radius will be α, $\alpha(1-\alpha)$, $\alpha(1-\alpha)^2$, ..., and we require an infinite number of these to make up the entire radius; $1=\alpha\{1+(1-\alpha)+(1-\alpha)^2+\ldots\}$; viz. the number of radials being any large number n whatever, the number of contours will be actually infinite.

12. We may compare the circle as thus contracted into its centre, or rather the quadrant of a circle, with an infinite strip of finite breadth divided into squares by two sets of parallel orthotomic lines: here if x_1, y_1 refer to the positive quadrant of the circle $x_1^2+y_1^2=1$, and x, y to the infinite strip $x=0$ to $-\infty$, $y=0$ to $\tfrac{1}{2}\pi$, we have $x+iy=\log(x_1+iy_1)$, that is,

$$x=\log\sqrt{(x_1^2+y_1^2)},\quad y=\tan^{-1}\frac{y_1}{x_1};$$

the successive concentric quadrants correspond to equal lines parallel to the axis of y, and the successive radii to infinite lines parallel to the axis of x.

13. Considering the contour S as given, then for a consecutive interior contour S' assumed at pleasure, the area F does not contract into a point: but we have the theorem that the consecutive contour S' can be found, and that in one way only, such that the area F shall contract into a given interior point. To do this is, in effect, to make the given area to correspond say to the circle $x_1^2 + y_1^2 - 1 = 0$, the contour S and the successive interior contours to the circumference of the circle and to the circumferences of the successive interior concentric circles, and finally the given interior point of F to the centre $x_1 = 0$, $y_1 = 0$ of the circle: and thus the theorem is identical with Riemann's theorem "It is possible and that in one way only to make a given singly connected area F correspond to a circle, in such wise that a given interior point of F shall correspond to the centre of the circle, and that a given boundary point of F shall correspond to a given point on the circumference of the circle." The last clause as to the given boundary point of F is necessary in order to make the correspondence a completely definite one, for without the clause it would obviously be allowable to give to the circle an arbitrary rotation about its centre.

14. The proof depends on the following lemma*.

For the given area F, it is possible to find, and that in one way only, a real function ξ of the coordinates (x, y) satisfying the following conditions:

(1) ξ is throughout the area finite and continuous, except only that in the neighbourhood of a given point thereof, taken to be the point $x = 0$, $y = 0$, it is $= \log \sqrt{(x^2 + y^2)}$.

(2) At the boundary of the area ξ is $= 0$.

(3) Throughout the area ξ satisfies the partial differential equation

$$\frac{d^2\xi}{dx^2} + \frac{d^2\xi}{dy^2} = 0.$$

In fact, if ξ be thus determined, it follows from (3) that we have

$$-\frac{d\xi}{dy}\,dx + \frac{d\xi}{dx}\,dy$$

an exact differential, whence putting this $= d\eta$, or determining η by the quadrature

$$\eta = \int\left(-\frac{d\xi}{dy}\,dx + \frac{d\xi}{dx}\,dy\right),$$

we have

$$\frac{d\eta}{dx} = -\frac{d\xi}{dy}, \quad \frac{d\eta}{dy} = \frac{d\xi}{dx},$$

and thence $\xi + i\eta =$ a function of $x + iy$.

* This proof is taken from the paper, Christoffel, "Sul problema delle temperature stazionarie et la rappresentazione di una data superficie," *Annali di Matem.*, t. I. (1867), pp. 89—103.

Writing then $x_1 + iy_1 = e^{\xi+i\eta} = \psi(x+iy)$ suppose, we have an orthomorphosis of the area into the circle $x_1^2 + y_1^2 - 1 = 0$, viz. for any point (x, y) of the boundary $\xi = 0$, and consequently $x_1 + iy_1 = e^{i\eta}$, thence $x_1 - iy_1 = e^{-i\eta}$, and therefore $x_1^2 + y_1^2 - 1 = 0$, or the point corresponds to a point on the circumference of the circle. Also, for a point x, y in the neighbourhood of $x = 0$, $y = 0$, we have

$$x_1 + iy_1 = e^{\log \surd(x^2+y^2) + i\eta}, \quad = \surd(x^2+y^2)\, e^{i\eta},$$

that is, for $x = 0$, $y = 0$ we have $x_1 = 0$, $y_1 = 0$, or the given point of F corresponds to the centre of the circle.

Some further considerations as to the continuity of η would be requisite in order to show that to the series of circles $x_1^2 + y_1^2 = c^2$, as c continuously increases from 0 to 1, there correspond closed curves surrounding the assumed origin and each other, and passing continuously to the boundary of F, and thus to complete the proof that we thus obtain an orthomorphosis of F into the circle $x_1^2 + y_1^2 = 1$; but I abstain from a further discussion of the question.

15. Reverting to the lemma, this in the first place asserts the existence of a function ξ satisfying the prescribed conditions, and next that the function is completely determinate, or say that there is but one such function. As to the latter point, suppose that there is a second function ξ_1 satisfying the same conditions, then throughout the area $\xi - \xi_1$ is finite and continuous; by general principles in the theory of functions, this implies that the function has throughout the area a constant value, and since this value at the boundary is $= 0$, it must be always $= 0$, that is, $\xi - \xi_1 = 0$, or $\xi_1 = \xi$. As to the former point, it is to be remarked that we obtain $\dfrac{d^2\xi}{dx^2} + \dfrac{d^2\xi}{dy^2} = 0$ as the general condition for a minimum value of the double integral $\displaystyle\iint\left\{\left(\frac{d\xi}{dx}\right)^2 + \left(\frac{d\xi}{dy}\right)^2\right\} dx\,dy$; if then we assume that there exists a function $\xi = 0$ at the boundary of the area, and finite and continuous throughout the area, except only that in the neighbourhood of a given point thereof, say $x = 0$, $y = 0$, it becomes $= \log \surd(x^2 + y^2)$, and that for all the values which satisfy these conditions it is such that the above-mentioned double integral taken over the given area shall be a minimum, it follows that there exists a function ξ satisfying the conditions of the lemma.

16. As a simple illustration, suppose that the area F is that of the circle $x^2 + y^2 = 1$, and that the excepted point is the point $x = 0$, $y = 0$, the centre of the circle. We have here a function ξ satisfying the conditions of the lemma, viz. this is $\xi = \log \surd(x^2 + y^2)$; the resulting value of η is $\eta = \tan^{-1}\dfrac{y}{x}$, and we have then $\xi + i\eta$ a function of $x + iy$, viz. this is $= \log(x + iy)$. Hence $e^{\xi+i\eta} = x + iy$, and the resulting orthomorphosis of the two circles is the identical one

$$x_1 + iy_1 = x + iy,$$

viz. we have

$$x_1 = x, \quad y_1 = y.$$

17. The contraction of a circle to a given eccentric point has, in fact, been exhibited in the foregoing example in No. 5, but I resume the question from a different point of view. Writing for shortness $z_1 = x_1 + iy_1$, $z = x + iy$; using also a bar for the conjugate function, $\bar{z} = x - iy$, and so in other cases $\alpha = a + \mathrm{a}i$, $\bar{\alpha} = a - \mathrm{a}i$, $\beta = b + \mathrm{b}i$, &c., the general formula of (7) may be written in the form

$$z_1 = \frac{z-\alpha}{1-\bar{\alpha}z} \cdot \frac{z-\beta}{1-\bar{\beta}z} \ldots,$$

viz. we thus have a transformation between the circumferences $x^2 + y^2 - 1 = 0$, $x_1^2 + y_1^2 - 1 = 0$. In fact, repeating the verification, the change of sign of i gives

$$\bar{z}_1 = \frac{\bar{z}-\bar{\alpha}}{1-\alpha\bar{z}} \cdot \frac{\bar{z}-\bar{\beta}}{1-\beta\bar{z}} \ldots.$$

Here the product of the α-factors is

$$\frac{z\bar{z} - (\alpha\bar{z} + \bar{\alpha}z) + \alpha\bar{\alpha}}{1 - (\alpha\bar{z} + \bar{\alpha}z) + \alpha\bar{\alpha}z\bar{z}},$$

which, if $z\bar{z} = 1$, becomes $= 1$; the same property exists as to the β-factors, &c., and hence putting $z\bar{z} = 1$, we have $z_1\bar{z}_1 = 1$, that is, the two circumferences correspond to each other. The correspondence would have subsisted if a factor $e^{i\lambda}$ had been introduced into the expression of z_1, but the effect is merely to rotate the circle $x_1^2 + y_1^2 - 1 = 0$ about its centre, and there is no real gain of generality.

18. The most simple case is

$$z_1 = \frac{z-\alpha}{1-\bar{\alpha}z}.$$

I assume here that $\alpha_1 = a + \mathrm{a}i$, is an interior point of the circle $x^2 + y^2 - 1 = 0$ (viz. $a^2 + \mathrm{a}^2 < 1$); hence $z = \alpha$ gives $z_1 = 0$, viz. to the point $x = a$, $y = \mathrm{a}$, within the circle $x^2 + y^2 - 1 = 0$, there corresponds the centre of the circle $x_1^2 + y_1^2 - 1 = 0$; and we have in this equation the theory of the squarewise contraction of the circle to the eccentric point $x = a$, $y = \mathrm{a}$. There is no loss of generality in assuming $\mathrm{a} = 0$; doing this the formula becomes

$$z_1 = \frac{z-a}{1-az},$$

where $a^2 < 1$, or if to fix the ideas a be taken to be positive, then $a < 1$.

19. To compare with a former investigation, writing $\xi + i\eta = \log(x_1 + iy_1)$, we have

$$\xi + i\eta = \log\frac{x-a+iy}{1-ax-aiy} = \log\frac{\sqrt{\{(x-a)^2+y^2\}}}{\sqrt{\{(1-ax)^2+a^2y^2\}}} + i\tan^{-1}\left\{\frac{y}{x-a} + \frac{ay}{1-ax}\right\},$$

and consequently

$$\xi = \log\frac{\sqrt{\{(x-a)^2+y^2\}}}{\sqrt{\{(1-ax)^2+a^2y^2\}}},$$

which is $= 0$ at the boundary $x^2 + y^2 - 1 = 0$, and is finite and continuous throughout the area except in the neighbourhood of the point $x = a$, $y = 0$, for which it is

$= \log \sqrt{\{(x-a)^2+y^2\}}$; moreover, ξ satisfies the partial differential $\dfrac{d^2\xi}{dx^2}+\dfrac{d^2\xi}{dy^2}=0$, and starting from the given value of ξ we should obtain the foregoing value of η, and thence $x_1+iy_1=e^{\xi+i\eta}$, that is,

$$x_1+iy_1=\frac{x-a+iy}{1-ax-aiy}, \text{ or } z_1=\frac{z-a}{1-az},$$

as the equation for the orthomorphosis of the circle $x_1{}^2+y_1{}^2-1=0$ into the circle $x^2+y^2-1=0$, the centre $x_1=0$, $y_1=0$ corresponding to the eccentric point $x=a$, $y=0$.

20. In further development of the solution, writing

$$z_1\bar{z}_1=\frac{z\bar{z}-a(z+\bar{z})+a^2}{1-a(z+\bar{z})+a^2z\bar{z}},$$

that is,

$$x_1{}^2+y_1{}^2=\frac{x^2+y^2+a^2-2ax}{a^2(x^2+y^2)-2ax+1},$$

and then assuming $x_1{}^2+y_1{}^2=\lambda_1{}^2$, we have as the contour corresponding thereto

$$x^2+y^2+a^2-2ax=\lambda^2\{a^2(x^2+y^2)-2ax+1\},$$

that is,

$$(x^2+y^2)(1-a^2\lambda^2)-2a(1-\lambda^2)x+a^2-\lambda^2=0,$$

or say

$$x^2+y^2-\frac{2a(1-\lambda^2)}{1-a^2\lambda^2}x+\frac{a^2-\lambda^2}{1-a^2\lambda^2}=0,$$

or, what is the same thing,

$$\left\{x-\frac{a(1-\lambda^2)}{1-a^2\lambda^2}\right\}^2+y^2=\frac{\lambda^2(1-a^2)^2}{(1-a^2\lambda^2)^2}.$$

The equation may also be written

$$\left\{x-\tfrac{1}{2}\left(\frac{1}{a}+a\right)\right\}\left\{x-\tfrac{1}{2}\left(\frac{1}{a}+a\right)+\left(\frac{1}{a}-a\right)\frac{1+a^2\lambda^2}{1-a^2\lambda^2}\right\}+y^2+\tfrac{1}{4}\left(\frac{1}{a}-a\right)^2=0,$$

showing that the contour circles all pass through two imaginary points

$$x=\tfrac{1}{2}\left(\frac{1}{a}+a\right),\quad y=\pm\tfrac{1}{2}i\left(\frac{1}{a}-a\right).$$

21. Writing next

$$x_1+iy_1=\frac{(x-a+iy)(1-ax-aiy)}{(1-ax)^2+a^2y^2};$$

that is,

$$x_1=\frac{-a+x(1+a^2)-a(x^2+y^2)}{(1-ax)^2+a^2y^2},\quad y_1=\frac{y(1-a^2)}{(1-ax)^2+a^2y^2}:$$

then to the radius $x_1-\theta y_1=0$ corresponds the radial

$$-a+x(1+a^2)-a(x^2+y^2)-\theta y(1-a^2)=0,$$

that is,

$$(x-a)\left(x-\frac{1}{a}\right)+y^2+\theta y\left(\frac{1}{a}-a\right)=0,$$

which is a circle passing through the two real points $y=0$, $x=a$, or $\frac{1}{a}$, being the antipoints of the before-mentioned pair of points.

22. Hence for the contraction of the circle $x^2+y^2-1=0$ to the interior point $x=a$, $y=0$, calling this point A, and taking A' the image hereof $\left(\text{coordinates } x=\frac{1}{a},\ y=0\right)$, we see that the contour circles are the circles all passing through the antipoints of A, A', and having thus for their common chord the line bisecting AA' at right angles, and that the radial circles are the circles all passing through the two points A, A'.

In what precedes, the distance AA' is $=\frac{1}{a}-a$, or say the half distance is $=\frac{1}{2}\left(\frac{1}{a}-a\right)$, but altering the scale so as to make this half distance $=1$, and moreover taking the origin at the middle point of AA', the equations of the contour circles and the radial circles assume the forms

$$x^2+y^2-2\alpha x+1=0,$$

and

$$x^2+y^2-2\beta y-1=0,$$

respectively.

23. It is better, interchanging x, y and altering the constants, to write these in the forms

$$x^2+y^2-\left(b+\frac{1}{b}\right)y+1=0,$$

$$x^2+y^2-\left(a-\frac{1}{a}\right)x-1=0,$$

viz. as in effect appearing in No. 5, these are the equations derived from $x+iy=\tan(u+iv)$ where $\tan u=a$, $\tan iv=ib$.

24. Passing now to the form

$$z_1=\frac{z-\alpha}{1-\bar{\alpha}z}\,\frac{z-\beta}{1-\bar{\beta}z},$$

to obtain the most simple instance, I write $a=0$, $b=\sqrt{2}$, so that the form is

$$z_1=\frac{z(z-\sqrt{2})}{1-z\sqrt{2}},$$

we thus obtain

$$z_1\bar{z}_1=\frac{z\bar{z}\,\{z\bar{z}-\sqrt{2}\,(z+\bar{z})+2\}}{1-\sqrt{2}\,(z+\bar{z})+2z\bar{z}},$$

that is,

$$x_1^2 + y_1^2 = \frac{(x^2+y^2)\{x^2+y^2-2\sqrt{2}\,x+2\}}{2(x^2+y^2)-2x\sqrt{2}+1},$$

and thence

$$x_1^2 + y_1^2 - 1 = \frac{(x^2+y^2-1)\{x^2+y^2-2\sqrt{2}\,x+1\}}{(x\sqrt{2}-1)^2+2y^2},$$

so that the circumference $x_1^2 + y_1^2 - 1 = 0$ corresponds to the two circumferences $x^2+y^2-1=0$, $x^2+y^2-2\sqrt{2}\,x+1=0$. The second of these is $(x-\sqrt{2})^2+y^2-1=0$; hence, putting $x+\frac{1}{\sqrt{2}}$ instead of x, the two circles are

$$\left(x+\frac{1}{\sqrt{2}}\right)^2+y^2-1=0, \quad \left(x-\frac{1}{\sqrt{2}}\right)^2+y^2-1=0,$$

(the former of these being the circle originally represented by $x^2+y^2-1=0$), and the last equation becomes

$$x_1^2+y_1^2-1=\frac{\left\{\left(x+\frac{1}{\sqrt{2}}\right)^2+y^2-1\right\}\left\{\left(x-\frac{1}{\sqrt{2}}\right)^2+y^2-1\right\}}{2(x^2+y^2)}.$$

25. Writing here $x_1^2+y_1^2-c^2=0$, we have

$$\{x^2+y^2-\tfrac{1}{2}+x\sqrt{2}\}\{x^2+y^2-\tfrac{1}{2}-x\sqrt{2}\}+2(x^2+y^2)(1-c^2)=0,$$

that is,

$$(x^2+y^2)^2-(2c^2+1)x^2-(2c^2-1)y^2+\tfrac{1}{4}=0,$$

a bicircular quartic which (or rather a branch whereof) is the contour corresponding to the circle $x_1^2+y_1^2-c^2=0$. For $c=1$, the curve breaks up into the two circles

$$\left(x+\frac{1}{\sqrt{2}}\right)^2+y^2-1=0, \quad \left(x-\frac{1}{\sqrt{2}}\right)^2+y^2-1=0;$$

and for $c=0$, it breaks up into

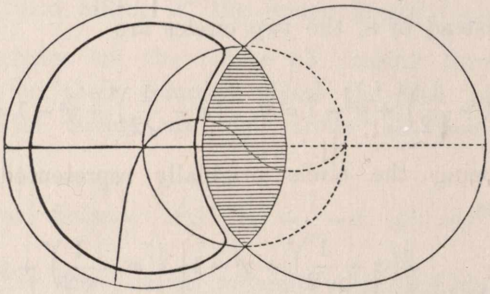
$$\left(x+\frac{1}{\sqrt{2}}\right)^2+y^2=0, \quad \left(x-\frac{1}{\sqrt{2}}\right)^2+y^2=0,$$

that is, into the two points $y=0$, $x=\pm\frac{1}{\sqrt{2}}$. We have to consider the curves for values of c between these two values $c=1$ and $c=0$.

26. For any such value of c, the curve consists of two symmetrical ovals, situate within the two circles respectively: we are concerned only with the oval lying within the circle $\left(x+\frac{1}{\sqrt{2}}\right)^2+y^2-1=0$; this is shown in the figure for a value of c, slightly inferior to 1, viz. it is an indented oval close approximating to the unshaded portion of the circle, say this is the contour S' consecutive to the circle which is the contour S.

As c diminishes and becomes ultimately $=0$, this oval continually diminishes, after a time losing the indentation, and approximating more and more to the form of a circle having for its centre the point $x = -\frac{1}{\sqrt{2}}$, $y = 0$, which is the centre of the circle, and ultimately reducing itself to this point.

It will be noticed that the consecutive contour S' as shown in the figure is not at an infinitesimal distance from every part of the circle which is the contour S; for a part of the circle, the consecutive contour or contour at an infinitesimal distance



is a part of the symmetrically situated oval, lying outside of the circle. And thus in the present case we do *not* obtain a contraction of the circle into its centre: what we do obtain is a contraction of the unshaded portion of the circle into the centre; this is as it should be, for the *only* contraction of the circle into its centre is by the series of concentric circles.

27. For the radials, writing as before $x + \frac{1}{\sqrt{2}}$ in place of x, that is, $z + \frac{1}{\sqrt{2}}$ in place of z, we have

$$z_1 = x_1 + iy_1 = \frac{\left(z + \frac{1}{\sqrt{2}}\right)\left(z - \frac{1}{\sqrt{2}}\right)}{-z}, \; = -z + \frac{\frac{1}{2}}{z} = -x - iy + \frac{\frac{1}{2}(x - iy)}{x^2 + y^2}.$$

Hence

$$x_1 = -x + \frac{\frac{1}{2}x}{x^2 + y^2}, \; = \frac{-x(x^2 + y^2 - \frac{1}{2})}{x^2 + y^2},$$

$$y_1 = -y - \frac{\frac{1}{2}y}{x^2 + y^2}, \; = \frac{-y(x^2 + y^2 + \frac{1}{2})}{x^2 + y^2},$$

and thus to the radius $x_1 - \theta y_1 = 0$ there corresponds the curve

$$x(x^2 + y^2 - \tfrac{1}{2}) - \theta y(x^2 + y^2 + \tfrac{1}{2}) = 0,$$

or say

$$(x - \theta y)(x^2 + y^2) - \tfrac{1}{2}(x + \theta y) = 0,$$

a circular cubic. In particular, for the line $x_1 = 0$, we have $\theta = 0$, and the curve is $x(x^2 + y^2 - \frac{1}{2}) = 0$, or say $x^2 + y^2 - \frac{1}{2} = 0$, the half circumference of which is shown in the figure: for the line $y_1 = 0$, we have $\theta = \infty$, and the curve is $y(x^2 + y^2 + \frac{1}{2}) = 0$ that is, $y = 0$.

The curve passes through the origin $x = 0$, $y = 0$, and the equation of the tangent there is $x+\theta y=0$. Moreover it passes through the centre of the circle, $x=\frac{1}{\sqrt{2}}$, $iy=0$, and the equation of the tangent there is easily found to be

$$x+\frac{1}{\sqrt{2}}-\theta y=0.$$

We may find where the curve cuts the circle $\left(x+\frac{1}{\sqrt{2}}\right)^2+y^2-1=0$, viz. this is $x^2+y^2=-x\sqrt{2}+\frac{1}{2}$, and substituting in the equation of the curve, we find

$$x^2-\theta xy+\frac{\theta}{\sqrt{2}}y=0,$$

or say

$$\left(x-\frac{1}{\sqrt{2}}\right)\left(x-\theta y+\frac{1}{\sqrt{2}}\right)+\tfrac{1}{2}=0,$$

a hyperbola which by its intersections with the circle determines the points in question. There are two real intersections, but of these only one is in the arc bounding the unshaded portion of the circle: the other intersection serves to determine by symmetry the intersection of the circular cubic with the other boundary of the unshaded portion: see the figure which exhibits the path of the circular cubic through the unshaded portion, and into and beyond the shaded portion or lens. I remark that the two systems of curves are considered by Meyer and shown in his Plate XIII.

28. The formula

$$z_1=\frac{z-\alpha}{1-\alpha z}$$

has been considered for the case of an interior point α; the case of an exterior point α might be considered in like manner, but it is obvious that we pass from one to the other, by a transformation $\frac{1}{z_1}$ for z_1; and it thus easily appears that, if α be an exterior point, we obtain a transformation between the area of the circle $x_1^2+y_1^2-1=0$ and the infinite area exterior to the circle $x^2+y^2-1=0$.

Similarly, in the formula

$$z_1=\frac{z-\alpha}{1-\bar{\alpha}z}\,\frac{z-\beta}{1-\bar{\beta}z},$$

the case considered ($\alpha=a=0$, $\beta=b=\sqrt{2}$) has been that of an interior point α, and an exterior point β. I omit the case of two exterior points, since this can be by the transformation $\frac{1}{z_1}$ for z_1 reduced to that of two interior points, but it is proper to consider this last case.

29. Considering then the case

$$z_1=\frac{z-\alpha\,.\,z-\beta}{1-\bar{\alpha}z\,.\,1-\bar{\beta}z},$$

where α, β are interior points, the most simple case is when $\alpha=a$, $\beta=-a$, positive and <1. Here

$$z_1=\frac{z^2-a^2}{1-a^2z^2},$$

and we have

$$x_1{}^2+y_1{}^2=\frac{(x^2+y^2)^2-2a^2(x^2-y^2)+a^4}{1-2a^2(x^2-y^2)+a^4(x^2+y^2)^2},$$

or say

$$x_1{}^2+y_1{}^2-1=\frac{(1-a^4)\{(x^2+y^2)^2-1\}}{1-2a^2(x^2-y^2)+a^4(x^2+y^2)^2},$$

which last form shows that to the circumference $x_1{}^2+y_1{}^2-1=0$, there corresponds the circumference $x^2+y^2-1=0$ (and besides this, only the imaginary curve $x^2+y^2+1=0$). Writing $x_1{}^2+y_1{}^2=c^2$, we have the contour

$$(c^2a^4-1)(x^2+y^2)^2-2a^2(c^2-1)(x^2-y^2)+c^2-a^4=0,$$

a bicircular quartic. For $c=1$, we have as already mentioned the circle $x^2+y^2-1=0$; for c less than 1, a curve lying within this circle, diminishing with c, and after a time acquiring on each side of the axis of x an indentation or assuming an hour-glass form; for the value $c^2=a^4$, the equation becomes

$$(a^4+1)(x^2+y^2)^2-2a^2(x^2-y^2)=0,$$

and the curve is a figure of eight, the two loops enclosing the points $y=0$, $x=a$ and $x=-a$ respectively; and as c^2 diminishes to zero, the curve consists of two detached ovals lying within the two loops of the figure of eight, and ultimately reducing themselves to the two points respectively. There is no difficulty in finding the equation of the curves corresponding to a radius $x_1-\theta y_1=0$, but the configuration of these curves is at once seen from that of the former system. We may in the present case say that the circle is squarewise contracted to the figure of eight; and then further that each loop of the figure of eight is squarewise contracted to a point; but we do *not* have a squarewise contraction of the circle to a point.

30. A closed curve or contour may be squarewise contracted not into a point but into a finite line: we see this in the case of a system of confocal ellipses, which gives the contraction of an ellipse into the thin ellipse which is the finite line joining the two foci. There is a like contraction of the circle; this is, in fact, given by the formula due to Schwarz, "Ueber einige Abbildungsaufgaben," *Crelle*, t. LXX. (1869), pp. 105—120 (see p. 115) for the orthomorphosis of the ellipse into a circle: this is

$$x_1+iy_1=\operatorname{sn}\left\{\frac{2K}{\pi}\sin^{-1}(x+iy)\right\},$$

where, if $a^2-b^2=1$ and $a=\cos\dfrac{i\pi K'}{4K}$, or, what is the same thing, $ib=\sin\dfrac{i\pi K'}{4K}$, then the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{a^2-1}=1$ is transformed into the circle $x_1{}^2+y_1{}^2-\dfrac{1}{K}=0$. The contours and trajectories for the ellipse are the confocal ellipses and hyperbolas respectively;

and for the circle, they are two sets of bicircular quartics, such that the portions within the circle have a configuration resembling that of the confocal ellipses and hyperbolas within the ellipse. To investigate the formulæ, it is convenient to introduce the function

$$X+iY, \quad =\sin^{-1}(x+iy);$$

we then have

$$x+iy=\sin(X+iY),$$

or say

$$x=\sin X\cos iY, \quad iy=\cos X\sin iY,$$

so that to any given value of Y there corresponds the ellipse

$$\frac{x^2}{\cos^2 iY}+\frac{y^2}{-\sin^2 iY}=1,$$

and then

$$x_1+iy_1=\operatorname{sn}\frac{2K}{\pi}(X+iY);$$

or if

$$s=\operatorname{sn}\frac{2KX}{\pi}, \qquad is_1=\operatorname{sn}\frac{2KiY}{\pi},$$

$$c=\operatorname{cn}\frac{2KX}{\pi}=\sqrt{(1-s^2)}, \qquad c_1=\operatorname{cn}\frac{2KiY}{\pi}=\sqrt{(1+s_1^2)},$$

$$d=\operatorname{dn}\frac{2KX}{\pi}=\sqrt{(1-k^2s^2)}, \quad d_1=\operatorname{dn}\frac{2KiY}{\pi}=\sqrt{(1+k^2s_1^2)},$$

then

$$x_1=\frac{sc_1d_1}{1+k^2s^2s_1^2}, \quad y_1=\frac{s_1cd}{1+k^2s^2s_1^2},$$

giving

$$x_1^2+y_1^2=\frac{s^2+s_1^2}{1+k^2s^2s_1^2},$$

and therefore $x_1^2+y_1^2-\frac{1}{k}=0$ if $1-ks_1^2=0$, that is, if $s_1=\frac{1}{\sqrt{k}}$, or since

$$is_1=\operatorname{sn}\frac{2KiY}{\pi}, \quad \text{if} \quad \frac{2KiY}{\pi}=\tfrac{1}{2}iK', \quad \text{or} \quad Y=\frac{\pi K'}{4K};$$

hence defining a as above, it appears that the elliptic periphery $\frac{x^2}{a^2}+\frac{y^2}{a^2-1}=1$ corresponds to the circumference $x_1^2+y_1^2-\frac{1}{k}=0$. By the introduction of $X+iY$ as above, the circle and the ellipse are each compared with a rectangle; the reduction of the circle to the rectangle, as given by the foregoing equation $x_1+iy_1=\operatorname{sn}\frac{2K}{\pi}(X+iY)$, or what is substantially the same thing by an equation $x_1+iy_1=\operatorname{sn}(X+iY)$, is more fully discussed in my paper, Cayley: "On the Binodal Quartic and the graphical representation of the Elliptic Functions," *Camb. Phil. Trans.*, t. XIV. (1889), pp. 484—494, [891], and in a paper "On some problems of orthomorphosis," *Crelle*, t. CVII. (1891), pp. 262—277, [921].

31. The whole theory, and in particular Riemann's theorem before referred to,

are intimately connected with Cauchy's theorem, "If a function $f(z)$ is holomorphic over a simply connected plane area, and if t denote any point within the area, then

$$f(t) = \frac{1}{2\pi i} \int \frac{f(z)}{z-t} dt,$$

where z denotes $x+iy$, and the integral is taken in the positive sense along the boundary of the area." See Briot and Bouquet, *Théorie des fonctions elliptiques* (Paris, 1875), p. 136.

Here in order to obtain by means of the theorem the value of the function $f(z)$ for a given point $t\,(=a+ib)$ within the area, we require to know the values of $f(z)$ for the several points of the boundary: viz. if z refers to a point P on the boundary, and if we represent the value $f(z)$ by a point P_1 in a second figure, then these points P_1 form a closed curve or boundary in this second figure, and we require to know not only the form of this boundary, but also the several points P_1 thereof which correspond to the several points P of the first-mentioned boundary, or say we require to know the correspondence of the two boundaries: this being known, we have by the theorem the value of $f(t)$, that is, the point a_1+ib_1 within the second area, which corresponds to the point $t=a+ib$ within the first area. The (1, 1) correspondence of the two areas is of course implied in the assertion that $f(t)$ has a determinate value, determined by means of the given values of $f(z)$ along the boundary.