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ON THE DEVELOPMENT OF $(1+n^2x)^{\frac{m}{n}}$.

[From the *Messenger of Mathematics*, vol. XXII. (1893), pp. 186—190.]

It is a known theorem that, if $\frac{m}{n}$ be any fraction in its least terms, the coefficients of the development of $(1+n^2x)^{\frac{m}{n}}$ are all of them integers, or, what is the same thing, that

$$\frac{m \,.\, m-n \ldots m-(r-1)\,n}{1 \,.\, 2 \ldots r}\, n^r$$

is an integer. The greater part, but not the whole, of this result comes out very simply from Mr Segar's very elegant theorem, *Messenger*, vol. XXII. (1893), p. 59, "the product of the differences of any r unequal numbers is divisible by $(r-1)!!$" or, as it may be stated, if $\alpha, \beta, \gamma, \ldots$ are any r unequal numbers, then $\zeta^{\frac{1}{2}}(\alpha, \beta, \gamma, \ldots)$ is divisible by $\zeta^{\frac{1}{2}}(0, 1, 2, \ldots, r-1)$.

In fact, writing $r+1$ for r and considering the numbers

$$m+n,\ n,\ 2n,\ 3n,\ \ldots\ (r-1)\,n;$$

then neglecting signs

$$\begin{aligned}\zeta^{\frac{1}{2}}(\alpha, \beta, \gamma, \ldots) \text{ is } = \ & m \,.\, m-n \ldots m-(r-1)\,n,\\ & \times 1n \,.\, 2n \ldots (r-1)\,n,\\ & \times 1n \,.\, 2n \ldots (r-2)\,n,\\ & \vdots\\ & \times 1n \,.\, 2n,\\ & \times 1n,\end{aligned}$$

which is

$$= m \,.\, m-n \ldots m-(r-1)\,n \times n^{\frac{1}{2}r \,.\, r-1} \times \zeta^{\frac{1}{2}}(0, 1, 2, \ldots, r-1),$$

and similarly

$$\zeta^{\frac{1}{2}}(0,\ 1,\ 2,\ \ldots,\ r) = 1\,.\,2\,.\,3\ldots r \times \zeta^{\frac{1}{2}}(0,\ 1,\ 2,\ \ldots,\ r-1);$$

so that, omitting the common factor $\zeta^{\frac{1}{2}}(0,\ 1,\ 2,\ \ldots,\ r-1)$, we have

$$m\,.\,m-n\ldots m-(r-1)\,n\,.\,n^{\frac{1}{2}r\,.\,r-1} \text{ divisible by } 1\,.\,2\,.\,3\ldots r.$$

It thus appears that the fraction

$$\frac{m\,.\,m-n\ldots m-(r-1)\,n}{1\ .\ \ 2\ \ \ldots\ \ \ r},$$

when reduced to its least terms, will contain in the denominator only products of powers of the prime factors of n; and it remains to show that multiplying this by n^r it will become integral, or what is the same thing that

$$\frac{n^r}{1\,.\,2\,.\ldots r}$$

in its least terms will not contain in the denominator any prime factor of n.

Considering in succession the prime numbers 2, 3, 5, ..., first the number 2, we see that in the product $1\,.\,2\,.\,3\,.\ldots r$, the number of terms divisible by 2 is $=\left(\frac{r}{2}\right)$, the number of terms divisible by 4 is $=\left(\frac{r}{4}\right)$, that by 8 is $=\left(\frac{r}{8}\right)$, and so on, where $\left(\frac{r}{2}\right)$ denotes the integer part of $\frac{r}{2}$, and so in other cases. Hence the product contains the factor 2, with the exponent $\left(\frac{r}{2}\right)+\left(\frac{r}{4}\right)+\left(\frac{r}{8}\right)+\ldots$, which exponent is less than

$$\frac{r}{2}+\frac{r}{4}+\frac{r}{8}+\ldots\ ad\ inf.$$

is less than r, say it is less than (r). Similarly for the number 3, the product contains the factor 3, with the exponent

$$\left(\frac{r}{3}\right)+\left(\frac{r}{9}\right)+\left(\frac{r}{27}\right)+\ldots,$$

which exponent is less than

$$\frac{r}{3}+\frac{r}{9}+\frac{r}{27}+\ldots\ ad\ inf.$$

is less than $\frac{1}{2}r$, say it is at most $=(\frac{1}{2}r)$; and so it contains the factor 5 with an exponent which is less than $\frac{1}{4}r$, say it is at most $=(\frac{1}{4}r)$, and generally the prime factor p with an exponent which is less than $\frac{1}{p-1}r$: say it is at most $=\left(\frac{1}{p-1}r\right)$.

This is

$$1\,.\,2\,.\,3\ldots r = \frac{1}{K}\,2^{(r)}\,3^{(\frac{1}{2}r)}\,5^{(\frac{1}{4}r)}\ldots,$$

where K is a whole number. Hence if $n = 2^\alpha 3^\beta 5^\gamma \ldots$, we have

$$\frac{n^r}{1.2.3\ldots r} = K 2^{r\alpha-(r)} . 3^{r\beta-(\frac{1}{2}r)} . 5^{r\gamma-(\frac{1}{4}r)} \ldots,$$

and here for every prime number 2, 3, 5, ... which is a factor of n, that is, for which the corresponding exponent α, β, γ, ... is not $=0$, the exponents $r\alpha-(r)$, $r\beta-(\frac{1}{2}r)$, $r\gamma-(\frac{1}{4}r)$, ... are all of them positive; and thus the fraction in its least terms does not contain in the denominator any prime factor of n; this is the theorem which was to be proved.

Mr Segar's theorem may without loss of generality be stated as follows: if β, γ, ... are any $r-1$ unequal positive integers (which for convenience may be taken in order of increasing magnitude), then $\zeta^{\frac{1}{2}}(0, \beta, \gamma, \ldots)$ is divisible by $\zeta^{\frac{1}{2}}(0, 1, 2, \ldots, r-1)$. A proof, in principle the same as his, is as follows:

We have the determinant

$$\left|\begin{matrix} 1, & a^\beta, & a^\gamma, \ldots \\ & b & \\ & c & \\ & \vdots & \end{matrix}\right| \text{ divisible by } \left|\begin{matrix} 1, & a, & a^2, \ldots \\ & b & \\ & c & \\ & \vdots & \end{matrix}\right|,$$

viz. the quotient is a rational and integral function of a, b, c, ... with coefficients which are positive integers; hence putting $a=b=c$, ... $=1$, the quotient will be a positive integer number. Considering the numerator determinant, and for a, b, c, ... writing therein $1+a$, $1+b$, $1+c$, ... respectively, where a, b, c, ... are ultimately to be put each $=0$, the value is

$$= \left|\begin{matrix} 1, & 1+\beta_1 a+\beta_2 a^2 \ldots, & 1+\gamma_1 a+\gamma_2 a^2+\ldots, \ldots \\ & b & \\ & c & \\ & \vdots & \end{matrix}\right|,$$

where β_1, β_2, ... denote the binomial coefficients

$$\frac{\beta}{1}, \frac{\beta . \beta-1}{1.2}, \&c.:$$

attending only to the lowest powers of a, b, c, ... which enter into the formula, this is

$$= \left|\begin{matrix} 1, & & \\ 1, & \beta_1, & \beta_2 \\ 1, & \gamma_1, & \gamma_2 \\ \vdots & & \end{matrix}\right| \left|\begin{matrix} 1, & a, & a^2, \ldots \\ 1, & b, & b^2, \\ 1, & c, & c^2, \\ \vdots & & \end{matrix}\right|,$$

or what is the same thing it is

$$= M \begin{vmatrix} 1, & & \dots \\ 1, & \beta, & \beta^2, \\ 1, & \gamma, & \gamma^2, \\ \vdots & & \end{vmatrix} \div \begin{vmatrix} 1, & a, & a^2 \\ 1, & b, & b^2 \\ 1, & c, & c^2 \end{vmatrix}, = M\zeta^{\frac{1}{2}}(0, \beta, \gamma, \dots) \begin{vmatrix} 1, & a, & a^2, \dots \\ 1, & b, & b^2, \\ 1, & c, & c^2, \\ \vdots & & \end{vmatrix},$$

where M is a mere number: it will be recollected that in this form, $a, b, c, \dots$ are not the original $a, b, c, \dots$. Putting herein $\beta, \gamma, \dots = 1, 2, \dots$, the denominator determinant is

$$= M\zeta^{\frac{1}{2}}(0, 1, 2, \dots) \begin{vmatrix} 1, & a, & a^2, \dots \\ 1, & b, & b^2, \\ 1, & c, & c^2, \\ \vdots & & \end{vmatrix},$$

and hence the quotient, which as already seen is an integer number, is equal to $\zeta^{\frac{1}{2}}(0, \beta, \gamma, \dots) \div \zeta^{\frac{1}{2}}(0, 1, 2, \dots)$, the theorem in question.

The original theorem as to the form of $(1+n^2x)^{\frac{m}{n}}$ is a particular case of Eisenstein's very general theorem that, in the development of any algebraical function of x, it is always possible by substituting for x a proper multiple of x, to make all the coefficients integers. It may be remarked that this would not be so if we had only

$$m \,.\, m-n \,.\, \dots \, m-(r-1)\,n \,.\, n^{\frac{1}{2}r\,.\,r-1}$$

divisible by $1\,.\,2\,\dots\,r$; for then, writing Nx for x, the form of the coefficient would have been

$$\frac{KN^r}{n^r \,.\, n^{\frac{1}{2}r\,.\,r-1}}, = \frac{KN^r}{n^{\frac{1}{2}r\,.\,r+1}},$$

and there would be no value (however great) of N by which the denominator factor $n^{\frac{1}{2}r\,.\,r+1}$ could be got rid of.