943.

ON RECIPROCANTS AND DIFFERENTIAL INVARIANTS.

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I USE the term Reciprocant to denote a function of an arbitrary variable or variables and its differential coefficients, not connected with any differential equation; and Differential Invariant to denote a function of the coefficients of a differential equation and the differential coefficients of these coefficients. Halphen's differential invariants are thus reciprocants, but the term reciprocant is not made use of by him. I have entitled the present paper "On Reciprocants and Differential Invariants"; in the earlier part, (except that for preserving a chronological order, I briefly refer to Sir J. Cockle's Criticoids, which are differential invariants or rather seminvariants), I attend almost exclusively to Reciprocants, reproducing and explaining, and in some parts developing, the theories of Ampère, Halphen in his first two memoirs, and Sylvester.

I.

The notion of a reciprocant first presents itself in Ampère's "Mémoire sur les avantages qu'on peut retirer dans la théorie des courbes de la considération des paraboles osculatrices, avec des reflexions sur les fonctions différentielles dont la valeur ne change pas lors de la transformation des axes," *Journ. École Polyt.* t. VII. (1808), pp. 151—191 (sent to the Institute, Dec. 1803)*. We have (p. 167) for the radius of curvature the expression

$$-\frac{(1+y'^2)^{\frac{3}{2}}}{y''},$$

* A reciprocant, the Schwarzian derivative, occurs in Lagrange's memoir, "Sur la construction des cartes géographiques," *Nouv. Mém. de Berlin*, 1779, *Œuvres*, t. IV. p. 651, but scarcely *quâ* reciprocant, viz. the form in question $\frac{f'''}{f'}-\frac{3}{2}\left(\frac{f''}{f'}\right)^2$, presents itself in the equation

$$\frac{f'''(u+ti)}{f'(u+ti)}-\frac{3}{2}\left(\frac{f''(u+ti)}{f'(u+ti)}\right)^2=\frac{F'''(u-ti)}{F'(u-ti)}-\frac{3}{2}\left(\frac{F''(u-ti)}{F'(u-ti)}\right)^2,$$

and for the parameter of the osculating parabola the expression

$$-\frac{54y''^5}{\{y'''^2+3(y''^2-y'y''')^2\}^{\frac{3}{2}}};$$

and it is explicitly noticed that each of these expressions remains absolutely unaltered when the coordinates x, y are changed into any other (rectangular) coordinates whatever.

The functions here spoken of are thus orthogonal absolute reciprocants; the change which leaves them unaltered is that of x, y into X, Y, where

$$X = x\cos\theta + y\sin\theta + \alpha, \quad Y = -x\sin\theta + y\cos\theta + \beta.$$

For the more general change

$$X = ax + by + \alpha, \quad Y = cx + dy + \beta,$$

we find without difficulty

$$Y' = \frac{c+dy'}{a-by'}, \quad Y'' = \frac{(ad-bc)\,y''}{(a-by')^3},$$

and thus there is *not* an identity of form in the expressions

$$\frac{(1+Y'^2)^{\frac{3}{2}}}{Y''} \text{ and } \frac{(1+y'^2)^{\frac{3}{2}}}{y''}.$$

But if we disregard factors, or (what is the same thing) attend to the equations $y''=0$ and $Y''=0$, we see that for this more general change one of these equations implies the other; each of them is, in fact, the condition for an inflexion of the curve in (x, y) or (X, Y).

II.

The differential invariants, or rather seminvariants, called "Criticoids," were considered by Sir James Cockle in his paper "On Criticoids," *Phil. Mag.* t. XXXIX. (1870), pp. 201—221; these will be more particularly considered further on, but at present it is sufficient to remark that the functions in question are connected with a linear differential equation, viz. they are functions of the coefficients of a linear differential equation, such that effecting thereon a transformation of the dependent variable, say $y=f.Y$, where f is an arbitrary function of the independent variable x, the function is equal to the like function of the coefficients of the new equation. For instance, if by the transformation in question $y=f.Y$ we transform

$$\frac{d^2y}{dx^2}+2b\frac{dy}{dx}+cy=0 \text{ into } \frac{d^2Y}{dX^2}+2B\frac{dY}{dX}+CY=0,$$

then we have

$$B=b+\frac{f'}{f}, \quad C=c+2b\frac{f'}{f}+\frac{f''}{f},$$

and thence

$$C-B^2-B'=c-b^2-b',$$

where F is the conjugate function of f, an equation implying that each side thereof is equal to one and the same constant; $u-ti$ is of course not a function of $u+ti$, and thus we have here no property that the function $\frac{f'''}{f'}-\frac{3}{2}\left(\frac{f''}{f'}\right)^2$ remains unaltered when for the arguments x, y thereof we substitute X, Y, determinate functions of these arguments.

(where the accents denote differentiation in regard to x): and thus $c - b^2 - b'$ is an invariant of the form in question; say it is an α-seminvariant of the differential equation.

III.

We have four important Memoirs by Halphen, (1) "Thèse d'Analyse sur les invariants différentiels," Paris (1878), (2) "Sur les invariants différentiels des courbes gauches," *Jour. École Polyt.* Cah. XLVII. (1880), pp. 1—102; (3) "Mémoire sur la réduction des équations différentielles linéaires aux formes intégrables," *Mém. Sav. Étrang.* t. XXVIII. (1883), pp. 1—297, and (4) "Sur les invariants des équations différentielles linéaires du quatrième ordre," *Acta Math.* t. III. (1883), pp. 325—380.

In the Memoir, Halphen (1), the investigations are in the first instance presented in a geometrical form; the author considers for instance the inflexions of a plane curve, and so obtains the invariant y'', or using his notation

$$\frac{1}{1.2\ldots k}\frac{d^ky}{dx^k} = a_k,$$

say the invariant a_2 of the weight 2; similarly, the consideration of the sextactic points gives him the invariant $a_2^2a_5 - 3a_2a_3a_4 + 2a_3^3$ of the weight 9; and that of the points of nine-pointic contact a more complicated invariant of the weight 27: and with these he forms absolute invariants. But the formal analytical definition is given § 3, "Théorie des invariants différentiels jusqu'au huitième ordre exclusivement," viz. considering the general homographic transformation

$$X = \frac{\alpha x + \beta y + \gamma}{\alpha'' x + \beta'' y + \gamma''}, \quad Y = \frac{\alpha' x + \beta' y + \gamma'}{\alpha'' x + \beta'' y + \gamma''},$$

then if the equation

$$f\left(x,\ y,\ \frac{dy}{dx}, \cdots \frac{d^ny}{dx^n}\right) = 0$$

is unaltered in form by the change (x, y) into (X, Y), or what is the same thing if the function

$$f\left(x,\ y,\ \frac{dy}{dx}, \cdots \frac{d^ny}{dx^n}\right)$$

be unaltered save as to a factor, then such function is said to be a differential invariant: and the quotient of two functions having the same factor, which quotient is therefore unaltered, is said to be an absolute invariant. We may for these terms substitute reciprocant, and absolute reciprocant respectively. The theory is scarcely worked out from the definition, and indeed, as will presently appear, it is by no means easy to work out directly from the definition, even the foregoing sextactic invariant $a_2^2a_5 - 3a_2a_3a_4 + 2a_3^3$. I remark that, in what follows, instead of Halphen's $a_2, a_3, a_4, a_5, \ldots$ I write $a, b, c, d, \ldots$, viz. these are used to denote respectively

$$\tfrac{1}{2}\frac{d^2y}{dx^2},\quad \tfrac{1}{6}\frac{d^3y}{dx^3},\quad \tfrac{1}{24}\frac{d^4y}{dx^4},\quad \tfrac{1}{120}\frac{d^5y}{dx^5}, \ldots,$$

the first differential coefficient $\dfrac{dy}{dx}$ being denoted by t. The foregoing invariant, or say the sextactic reciprocant, is thus $= a^2d - 3abc + 2b^3$. (See *post*, XI.)

IV.

For the analytical theory, we have as above

$$X = \frac{\alpha x + \beta y + \gamma}{\alpha'' x + \beta'' y + \gamma''}, \quad Y = \frac{\alpha' x + \beta' y + \gamma'}{\alpha'' x + \beta'' y + \gamma''};$$

taking h, k for the increments of x, y respectively, we have

$$k = th + ah^2 + bh^3 + ch^4 + dh^5 + \ldots,$$

and, similarly, taking H, K for the increments of X, Y respectively, we have

$$K = TH + AH^2 + BH^3 + CH^4 + DH^5 + \ldots,$$

and we require the relations between the two sets of coefficients $(t, a, b, c, d, \ldots)$ and $(T, A, B, C, D, \ldots)$: I propose to develop these up to d, D, so as to obtain the theory for the form $a^2d - 3abc + 2b^3$.

Writing for shortness ξ, η, ζ for

$$\alpha x + \beta y + \gamma, \quad \alpha' x + \beta' y + \gamma', \quad \alpha'' x + \beta'' y + \gamma''$$

respectively, we have

$$X + H = \frac{\alpha\ (x+h) + \beta\ (y+k) + \gamma}{\alpha''(x+h) + \beta''(y+k) + \gamma''}, \quad = \frac{\xi + \alpha h + \beta k}{\zeta + \alpha'' h + \beta'' k},$$

and thence

$$H = \frac{\xi + \alpha h + \beta k}{\zeta + \alpha'' h + \beta'' k} - \frac{\xi}{\zeta},$$

$$= \frac{(\alpha\zeta - \alpha''\xi) h + (\beta\zeta - \beta''\xi) k}{\zeta(\zeta + \alpha'' h + \beta'' k)} = \frac{Ph + Qk}{\zeta(\zeta + \alpha'' h + \beta'' k)},$$

if

$$P = \alpha\zeta - \alpha''\xi, \quad Q = \beta\zeta - \beta''\xi,$$

or, for k substituting its value,

$$H = \frac{(P + Qt) h + Q (ah^2 + bh^3 + ch^4 + dh^5)}{\zeta \{\zeta + (\alpha'' + \beta'' t) h + \beta'' (ah^2 + bh^3 + ch^4 + dh^5)\}},$$

$$= \frac{L \{h + \lambda (ah^2 + bh^3 + ch^4 + dh^5)\}}{\zeta [1 + L'' \{h + \lambda'' (ah^2 + bh^3 + ch^4 + dh^5)\}]},$$

if

$$L = P + Qt, \quad \lambda = \frac{Q}{P + Qt},$$

$$L'' = \alpha'' + \beta'' t, \quad \lambda'' = \frac{\beta''}{\alpha'' + \beta'' t},$$

or, putting $\mu = \dfrac{\zeta}{L}$, say this is

$$\mu H = \frac{h + \lambda\,(ah^2 + bh^3 + ch^4 + dh^5)}{1 + L''\,\{h + \lambda''\,(ah^2 + bh^3 + ch^4 + dh^5)\}}.$$

Similarly, if

$$P' = \alpha'\zeta - \alpha''\eta,\quad Q' = \beta'\zeta - \beta''\eta,\quad L' = P' + Q't,\quad \lambda' = \frac{Q'}{P' + Q't},\ \text{and}\ \mu' = \frac{\zeta}{L'},$$

then

$$\mu' K = \frac{h + \lambda'\,(ah^2 + bh^3 + ch^4 + dh^5)}{1 + L''\,\{h + \lambda''\,(ah^2 + bh^3 + ch^4 + dh^5)\}};$$

we have thus found H and K each of them in terms of h, and the elimination of h leads to an equation between H and K expressible in the form

$$K = TH + AH^2 + BH^3 + CH^4 + DH^5.$$

Writing for convenience

$$ah^2 + bh^3 + ch^4 + dh^5 = h\Omega,$$

that is,

$$\Omega = ah + bh^2 + ch^3 + dh^4,$$

we have

$$\mu H = \frac{h\,(1 + \lambda\Omega)}{1 + L''h\,(1 + \lambda''\Omega)},$$

$$\mu' K = \frac{h\,(1 + \lambda'\Omega)}{1 + L''h\,(1 + \lambda''\Omega)},$$

and thence

$$K = \frac{\mu}{\mu'}\,H\,\frac{1 + \lambda'\Omega}{1 + \lambda\Omega},\quad = TH\,\frac{1 + \lambda'\Omega}{1 + \lambda\Omega},$$

since, when h is small or say $\Omega = 0$, we have $K = TH$; viz we thus have $T = \dfrac{\mu}{\mu'}$.

Developing, we have

$$K = TH\,\{1 + (\lambda' - \lambda)\,\Omega - \lambda\,(\lambda' - \lambda)\,\Omega^2 + \lambda^2\,(\lambda' - \lambda)\,\Omega^3 - \lambda^3\,(\lambda' - \lambda)\,\Omega^4\}.$$

We have

$$\begin{aligned}
\Omega &= ah + bh^2 + ch^3 + dh^4,\\
h &= \frac{1}{a}\,\Omega - \frac{b}{a^3}\,\Omega^2 + \frac{1}{a^5}\,(2b^2 - ac)\,\Omega^3 - \frac{1}{a^7}\,(5b^3 - 5abc + a^2d)\,\Omega^4\\
&= p\Omega + q\Omega^2 + r\Omega^3 - s\Omega^4,
\end{aligned}$$

suppose, and hence

$$\begin{aligned}
\mu H &= \frac{(p\Omega + q\Omega^2 + r\Omega^3 + s\Omega^4)\,(1 + \lambda\Omega)}{1 + L''\,(p\Omega + q\Omega^2 + r\Omega^3 + s\Omega^4)\,(1 + \lambda''\Omega)}\\
&= \frac{p\Omega + (q + \lambda p)\,\Omega^2 + (r + \lambda q)\,\Omega^3 + (s + \lambda r)\,\Omega^4}{1 + L''p\Omega + L''\,(q + \lambda''p)\,\Omega^2 + L''\,(r + \lambda''q)\,\Omega^3 + L''\,(s + \lambda''r)\,\Omega^4},
\end{aligned}$$

which gives H in terms of Ω, and conversely Ω in terms of H. We assume

$$\Omega = XH + YH^2 + ZH^3 + WH^4;$$

K is then given as above in terms of H, Ω, that is, in terms of H, and it is assumed that we have

$$K = TH + AH^2 + BH^3 + CH^4 + DH^5.$$

In the foregoing expression for K, substituting for Ω its value

$$XH + YH^2 + ZH^3 + WH^4,$$

and comparing coefficients, we find

$$\begin{aligned}
A &= (\lambda' - \lambda) \times \; X, \\
B &= (\lambda' - \lambda) \times \; Y - \lambda X^2, \\
C &= (\lambda' - \lambda) \times \; Z - \lambda . 2XY + \lambda^2 X^3, \\
D &= (\lambda' - \lambda) \times \; W - \lambda (2XZ + Y^2) + \lambda^2 . 3X^2 Y - \lambda^3 X^4;
\end{aligned}$$

and we have then, from the relation between H, Ω, to find the values of X, Y, Z, W. We have

$$\begin{aligned}
& p\,(XH + YH^2 + ZH^3 + WH^4) \\
& \qquad - \mu H \\
& + (q + \lambda p)\,(X^2H^2 + 2XYH^3 + (2XZ + Y^2)\,H^4) \\
& \qquad - \mu H . L'' p\,(XH + YH^2 + ZH^3) \\
& + (r + \lambda q)\,(X^3 H^3 + 3X^2 Y . H^4) \\
& \qquad - \mu H . L''(q + \lambda'' p)\,(X^2H^2 + 2XYH^3) \\
& + (s + \lambda r)\,(X^4 . H^5) \\
& \qquad - \mu H . L''(r + \lambda'' q)\,(X^3H^3) = 0.
\end{aligned}$$

Equating coefficients, we have

$$\begin{aligned}
& pX - \mu = 0, \\
& pY + (q + \lambda p)\,X^2 - \mu L'' pX = 0, \\
& pZ + (q + \lambda p)\,2XY + (r + \lambda q)\,X^3 - \mu L'' pY - \mu L''(q + \lambda'' p)\,X^2 = 0, \\
& pW + (q + \lambda p)(2XZ + Y^2) + (r + \lambda q)\,3X^2 Y + (s + \lambda r)\,X^4 \\
& \qquad - \mu L'' pZ - \mu L''(q + \lambda'' p)\,2XY - \mu L''(r + \lambda'' q)\,X^3 = 0,
\end{aligned}$$

and substituting for p, q, r, s their values, we find successively after reductions, which for W are somewhat troublesome, the values

$$\begin{aligned}
X &= \mu a, \\
Y &= \mu^2 \{b + aL'' - a^2\lambda\}, \\
Z &= \mu^3 \{c + 2bL'' + aL''^2 - 3ab\lambda + 2a^3\lambda^2 - 3a^2\lambda L'' + a^2\lambda'' L''\}, \\
W &= \mu^4 \{d + 3cL'' + 3bL''^2 + aL''^3 - (4ac + 2b^2)\,\lambda + 10a^2 b\lambda^2 - 5a^4\lambda^3 \\
& \qquad - 12ab\lambda L'' + 3ab\lambda'' L'' - 6a^2\lambda''^2 + 3a^2\lambda'' L''^2 + 10a^3\lambda^2 L'' - 4a^3\lambda\lambda'' L''\};
\end{aligned}$$

these values are to be substituted for X, Y, Z, W in the foregoing expressions of A, B, C, D.

I add the formulæ

$$\lambda' - \lambda = \frac{Q'}{P' + Q't} - \frac{Q}{P + Qt} = \frac{PQ' - P'Q}{(P + Qt)(P' + Q't)},$$

$$PQ' - P'Q = (\alpha\zeta - \alpha''\xi)(\beta'\zeta - \beta''\eta) - (\beta\zeta - \beta''\xi)(\alpha'\zeta - \alpha''\eta)$$

$$= \zeta \begin{vmatrix} \alpha, & \alpha', & \alpha'' \\ \beta, & \beta', & \beta'' \\ \xi, & \eta, & \zeta \end{vmatrix} = \zeta \begin{vmatrix} \alpha, & \alpha', & \alpha'' \\ \beta, & \beta', & \beta'' \\ \gamma, & \gamma', & \gamma'' \end{vmatrix} = (\alpha'' x + \beta'' y + \gamma'') \begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix},$$

$$\begin{aligned}P + Qt &= \alpha\zeta - \alpha''\xi + (\beta\zeta - \beta''\xi)\,t\\ &= \alpha\,(\alpha''x+\beta''y+\gamma'') - \alpha''\,(\alpha x+\beta y+\gamma) + t\,\{\beta\,(\alpha''x+\beta''y+\gamma'') - \beta''\,(\alpha x+\beta y+\gamma)\},\\ P' + Q't &= \alpha'\zeta - \alpha''\eta + (\beta'\zeta - \beta''\eta)\,t\\ &= \alpha'\,(\alpha''x+\beta''y+\gamma'') - \alpha''\,(\alpha'x+\beta'y+\gamma') + t\,\{\beta'\,(\alpha''x+\beta''y+\gamma'') - \beta''\,(\alpha'x+\beta'y+\gamma')\},\end{aligned}$$

that is,

$$P+Qt = \begin{vmatrix} \alpha, & \beta, & \gamma \\ \alpha'', & \beta'', & \gamma'' \\ t, & -1, & y-tx \end{vmatrix}, \quad P'+Q't = \begin{vmatrix} \alpha', & \beta', & \gamma' \\ \alpha'', & \beta'', & \gamma'' \\ t, & -1, & y-tx \end{vmatrix},$$

whence $\lambda' - \lambda$ is known.

Also

$$\mu = \frac{\zeta}{L} = \frac{\alpha''x+\beta''y+\gamma''}{P+Qt}, \quad \mu' = \frac{\zeta}{L'} = \frac{\alpha''x+\beta''y+\gamma''}{P'+Q't};$$

and

$$T = \frac{\mu}{\mu'} = \frac{L'}{L} = \frac{P'+Q't}{P+Qt}.$$

V.

We have

$$A = (\lambda' - \lambda)\,\mu a,$$

and thus a is a reciprocant, viz. $a = 0$ is the condition of an inflexion.

I remark that $ac - b^2$ is not a reciprocant; we have

$$AC - B^2 = (\lambda - \lambda')^2\,(XZ - Y^2),$$

and then

$$XZ - Y^2 = \mu^4\,\{(ac-b^2) - a^2\lambda\,(b - a^2\lambda) + a^3\,(\lambda'' - \lambda)\,L''\},$$

and thus $AC - B^2$ is not a multiple of $ac - b^2$, even in the particular case $L'' = 0$. I notice further that we have

$$\begin{aligned}(4AC - 5B^2) &= (\lambda' - \lambda)^2\,\{4XZ - 5Y^2 + 2\lambda X^2Y - \lambda^2X^4\},\\ &= (\lambda' - \lambda)^2\,\mu^4\,\{4ac - 5b^2 - 2abL'' - a^2L''^2 + 4a^3\lambda''L''\},\end{aligned}$$

viz. in the particular case $L'' = 0$, we have

$$4AC - 5B^2 = (\lambda' - \lambda)^2\,\mu^4\,(4ac - 5b^2),$$

and thus in the particular case $L'' = 0$ (or say, if $\alpha'' = 0$, $\beta'' = 0$, that is, if X, Y are mere linear functions of x, y) we have $4ac - 5b^2 = 0$ a reciprocant. But in the general case now in hand, it is not a reciprocant.

We have now to consider the sextactic form $a^2d - 3abc + 2b^3$. We have

$$A^2D - 3ABC + 2B^3 = (\lambda' - \lambda)^3\,\{X^2W - 3XYZ + 2Y^3 + \lambda X^2\,(XZ - Y^2)\},$$

and we proceed to calculate the expression in { }. Writing for shortness

$$\begin{aligned}X &= \mu a,\\ Y &= \mu^2\,(b + aL'' + Y_0),\\ Z &= \mu^3\,(c + 2bL'' + aL''^2 + Z_0),\\ W &= \mu^4\,(d + 3cL'' + 3bL''^2 + aL''^3 + W_0),\end{aligned}$$

and then, omitting a factor μ^6, the expression is

$$\begin{aligned} = \ & a^2d - 3abc + 2b^3 \\ & + a^2W_0 - 3a(b + aL'')Z_0 + \{(-3ac + 6b^2) + 6abL'' + 3a^2L''^2\}Y_0 \\ & + 6(b + aL'')Y_0^2 - 3aY_0Z_0 + 2Y_0^3 \\ & + \lambda a^2\{ac - b^2 + aZ_0 - 2(b + aL'')Y_0 - Y_0^2\}, \end{aligned}$$

where the terms after the first term are in fact $= 0$; and this being so, we have

$$A^2D - 3ABC + 2B^3 = (\lambda' - \lambda)^3\mu^6(a^2d - 3abc + 2b^3),$$

and thus $a^2d - 3abc + 2b^3$ is a reciprocant; but observe that the factor being here $(\lambda' - \lambda)^3\mu^6$, and in the equation $A = (\lambda' - \lambda)\mu a$ the factor being $(\lambda' - \lambda)\mu$, we cannot with the reciprocants a and $a^2d - 3abc + 2b^3$ form an absolute reciprocant.

To verify the evanescence of the above-mentioned terms, observe that, considering first the terms independent of L'', we have

$$\begin{aligned} a^2W_0 &= -4a^3c\lambda - 2a^2b^2\lambda + 10a^4b\lambda^2 - 5a^6\lambda^3, \\ -3abZ_0 &= +9a^2b^2\lambda - 6a^4b\lambda^2, \\ +(-3ac + 6b^2)Y_0 &= +3a^3c\lambda - 6a^2b^2\lambda, \\ -3aY_0Z_0 &= -9a^4b\lambda^2 + 6a^6\lambda^3, \\ +6bY_0^2 &= +6a^4b\lambda^2, \\ +2Y_0^3 &= -2a^6\lambda^3, \\ +\lambda a^2(ac - b^2) &= +a^3c\lambda - a^2b^2\lambda, \\ +\lambda a^3Z_0 &= -3a^4b\lambda^2 + 2a^6\lambda^3, \\ +\lambda a^2 . -2bY_0 &= +2a^4b\lambda^2, \\ +\lambda a^2 . -Y_0^2 &= -a^6\lambda^3; \end{aligned}$$

the sum of which is in fact $= 0$. And next for the terms containing L'', we have

$$\begin{aligned} & a^2W_0 = \\ & \quad -12a^3b\lambda L'' + 3a^3b\lambda''L'' - 6a^4\lambda L''^2 + 3a^4\lambda''L''^2 + 10a^5\lambda^2L'' - 4a^5\lambda\lambda''L'', \\ & -3abZ_0 = \\ & \quad +9a^3b\lambda L'' - 3a^3b\lambda''L'', \\ & -3a^2L''Z_0 = \\ & \quad +9a^3b\lambda L'' + 9a^4\lambda L''^2 - 3a^4\lambda''L''^2 - 6a^5\lambda^2L'', \\ & (6abL'' + 3a^2L''^2)Y_0 = \\ & \quad -6a^3b\lambda L'' - 3a^4\lambda L''^2, \\ & -3aY_0Z_0 = \\ & \quad -9a^5\lambda^2L'' + 3a^5\lambda\lambda''L'', \\ & +6(b + aL'')Y_0^2 = \\ & \quad +6a^5\lambda^2L'', \\ & +\lambda a^3Z_0 = \\ & \quad -3a^5\lambda^2L'' + a^5\lambda\lambda''L'', \\ & -2\lambda a^2(b + aL'')Y_0 = \\ & \quad +2a^5\lambda^2L'', \end{aligned}$$

the sum of which is also $= 0$.

Before going further, it is proper to remark that $a, b, c, \ldots$ as representing the second, third, &c., differential coefficients of y are considered as being of the orders $2, 3, 4, \ldots$, and the order of a reciprocant is taken to be the order of the highest letter contained therein: the degree means the degree in these letters $a, b, c, \ldots$, and the weight the weight in these letters, reckoning them as of the weights $2, 3, 4, \ldots$. A reciprocant is a homogeneous isobaric function of $a, b, c, \ldots$, not involving y or the first differential coefficient.

VI.

To the reciprocants thus obtained, say

$$U = a, \quad V = a^2d - 3abc + 2b^3,$$

Halphen adds another reciprocant Δ (that of nine-pointic contact) which he expresses in the form of a determinant.

Writing down in connexion therewith its developed expression, we have

$$\Delta = \begin{vmatrix} b, & c, & d, & e, & f \\ a, & b, & c, & d, & e \\ -a^2, & 0, & b^2, & 2bc, & 2bd + c^2 \\ 0, & a^2, & 2ab, & 2ac + b^2, & 2ad + 2bc \\ 0, & 0, & a^2, & 3ab, & 3ac + 3b^2 \end{vmatrix}, \quad = \begin{array}{lll} a^6 & df & +\ 1 \\ & e^2 & -\ 1 \\ \hline +a^5 & bcf & -\ 3 \\ & bde & +\ 3 \\ & c^2e & +\ 4 \\ & cd^2 & -\ 5 \\ \hline +a^4 & b^3f & +\ 2 \\ & b^2ce & -\ 5 \\ & b^2d^2 & -\ 1 \\ & bc^2d & +\ 14 \\ & c^4 & -\ 4 \\ \hline +a^3 & b^3cd & -\ 10 \\ & b^2c^3 & -\ 5 \\ \hline +a^2 & b^5d & +\ 4 \\ & b^4c^2 & +\ 15 \\ \hline +a^1 & b^6c & -\ 12 \\ +a^0 & b^8 & +\ 3, \end{array}$$

where observe that the whole term in the highest letter f is $a^4(a^2d - 3abc + 2b^3)f$, viz. the coefficient of f is the reciprocant $a^4(a^2d - 3abc + 2b^3)$, agreeing with a general theorem given by Halphen.

Δ is of the degree 8 and the weight 24, and it is seen without difficulty that the factor is $=(\lambda' - \lambda)^8 \mu^{16}$. He remarks that $25\Delta^3 - 27V^8$ vanishes for $a = 0$ (viz. it becomes $256 \cdot 27b^{24} - 27 \cdot 256b^{24} = 0$), and not only so, but that it in fact contains the factor a^4. Assuming that this is so, viz. that the terms in a^0, a, a^2, a^3 all of them vanish, I have calculated the terms in a^4, and have thus obtained an incomplete expression for the quotient $(256\Delta^3 - 27V^8) \div a^4$, viz. this is

$$\begin{aligned} H = (256\Delta^3 - 27V^8) \div a^4 = \quad & 256a^{14}(df \ - e^2)^3 \\ & \vdots \\ +288b^{16} \quad & b^3f + 48 \\ & b^2ce + 120 \\ & b^2d^2 - 64 \\ & bc^2d + 9288 \\ & c^4 + 81; \end{aligned}$$

where I remark that the term in b^{16} presented itself in the form

$$\begin{aligned} & 256\,(54b^{19}f - 135b^{18}ce + 117b^{18}d^2 + 5346b^{17}c^2d + 9477b^{16}c^4) \\ - & \ 27\,(\qquad\qquad\qquad 1792b^{18}d^2 - 48384b^{17}c^2d + 90720b^{16}c^4). \end{aligned}$$

We have thus Halphen's reciprocant H of the weight 64.

The reciprocants thus far obtained are consequently

	deg.	weight	factor
$U = a$	1	2	$(\lambda' - \lambda)\,\mu$
$V = a^2d - 3abc + 2b^3$	3	9	$(\lambda' - \lambda)^3\,\mu^6$
Δ	8	24	$(\lambda' - \lambda)^8\,\mu^{16}$
$H = (256\Delta^3 - 27V^8) \div a^4$	20	64	$(\lambda' - \lambda)^{20}\,\mu^{44}$

VII.

Reciprocants of the same degree and weight have the same factor, and may thus be combined in the way of addition, viz. R and S being reciprocants of the same degree and the same weight, then (α, β being constants) we have $\alpha R + \beta S$ a reciprocant of the same degree and the same weight. The quotient of two such reciprocants has the factor unity, or say it is an absolute invariant. Thus Δ^3 and V^8 have each of them the degree 24 and weight 72, so that $\alpha\Delta^3 + \beta V^8$ is a reciprocant; in particular, $256\Delta^3 - 27V^8$ is a reciprocant having the factor a^4, and it thus gives rise to the foregoing reciprocant $H = (256\Delta^3 - 27V^8) \div a^4$.

Moreover $\Delta^3 \div V^8$ is an absolute reciprocant.

Any two reciprocants R, S of the same degree and same weight, or, what is the same thing, any absolute reciprocant $R \div S$ gives rise to a reciprocant $RS' - R'S$, where the accents denote differentiation in regard to x ($a'=3b$, $b'=4c$, $c'=5d,\ldots$); the order of the new reciprocant thus exceeds by unity the order of R or S, whichever of them is of the highest order.

From U, V, Δ, which are of the orders 2, 5, 7 respectively, it is thus possible to deduce a series of reciprocants T_8, T_9, $T_{10},\ldots$ of the orders 8, 9, 10,..., respectively: viz. we have from the absolute reciprocant $\Delta^3 V^{-8}$, first a reciprocant $3V\Delta' - 8V\Delta$, which, however, contains the factor U^3, and we have

$$U^3 T_8 = 3V\Delta' - 8V\Delta.$$

We then have the absolute reciprocant $U^4 T_8 . V^{-4}$, leading to

$$T_9 = UVT_8' \quad + 4(VU' - UV')\,T_8,$$

then the absolute reciprocant $U^4 T_9 . V^{-\frac{16}{3}}$, leading to

$$T_{10} = UVT_9' \quad + 4(VU' - \tfrac{4}{3}UV')\,T_9,$$

and so

$$T_{11} = UVT_{10}' \quad + 4(VU' - \tfrac{5}{3}UV')\,T_{10},$$

$$T_n = UVT'_{n-1} + 4\{VU' - \tfrac{1}{3}(n-6)\,UV'\}\,T_{n-1}.$$

Halphen considers that these are the only distinct reciprocants of the orders 2, 5, 7, 8, 9,..., n; but remarks that we can, with the reciprocants up to any given order, form algebraical combinations, in some cases containing as factor a power of the reciprocant U or V, so that, rejecting this factor, we have a new independent reciprocant, that is, a reciprocant not expressible as a rational and integral function of inferior reciprocants; an instance hereof is the foregoing reciprocant

$$H = (256\Delta^3 - 27V^8) \div a^4.$$

Other like forms are given by Halphen, viz. writing with him T for shortness in place of T_8, we have for T the following expression in the form of a determinant

$$T = \begin{vmatrix} 3b, & 2a, & a, & 0, & 0 & , & 0 \\ 4c, & 3b, & b, & a, & 2a^2 & , & 0 \\ 5d, & 4c, & c, & 2b, & 5ab & , & a^2 \\ 6e, & 5d, & d, & 3c, & 6ac+3b^2 & , & 3ab \\ 7f, & 6e, & e, & 4d, & 7ad+7bc & , & 4ac+2b^2 \\ 8g, & 7f, & f, & 6e, & 8ae+8bd+4c^2, & & 5ad+5bc \end{vmatrix},$$

and then

$$T_1 = (V^4T - \tfrac{1}{6}H) \div U,$$

$$G = (U^4T^2 + 9H) \div V^2,$$

$$\Theta = \{2U\Delta T' + T(8U'\Delta - 3U\Delta')\} \div V,$$

$$\Theta_1 = (\Theta + \tfrac{1}{4}G) \div V,$$

$$\Theta_2 = (\Theta_1 - \tfrac{45}{2}TV) \div U,$$

where Θ is given explicitly in the form of a determinant, viz. this is

$$\Theta_2 = -432 \begin{vmatrix} c, & d, & e, & f, & g, & h \\ b, & c, & d, & e, & f, & g \\ a, & b, & c, & d, & e, & f \\ a^2, & 2ab, & 2ac+b^2, & 2ad+2bc, & 2ae+2bd+c^2, & 2af+2be+2cd \\ 0, & a^2, & 2ab, & 2ac+b^2, & 2ad+2bc, & 2ae+2bd+c^2 \\ 0, & 0, & a^3, & 3a^2b, & 3a^2c+3ab^2, & 3a^2d+6abc+b^3 \end{vmatrix},$$

where $\Theta_2 = 0$ is the differential equation of the ninth order of the general cubic curve.

Halphen's $S, = UVT' + 4(VU' - UV')T$, is the above-mentioned reciprocant T_9; U is connected with Θ by the formula $2\Delta S - V^2\Theta = U^4T^2$.

Halphen considers incidentally (p. 56), what may be called a polar relation of the variables X, Y and x, y; viz. this is

$$X = \frac{dy}{dx},\quad Y = x\frac{dy}{dx} - y,$$

whence conversely

$$x = \frac{dY}{dX},\quad y = X\frac{dY}{dX} - Y.$$

We may here express Y and its differential coefficients in regard to X, say $Y, Y_1, Y_2, \ldots$, in terms of y and its differential coefficients in regard to x, say $y, y_1, y_2, \ldots$ (to avoid confusion with other formulæ, I purposely use these notations, abstaining from the introduction of the letters $a, b, c, \ldots$). The geometrical signification is that x, y, being point-coordinates, then X, Y are line-coordinates. We find, without difficulty,

$$Y_1 = x,\quad Y_2 = \frac{1}{y_2},\quad Y_3 = \frac{-y_3}{y_2^3},\quad Y_4 = \frac{-y_2y_4 - 3y_3^2}{y_2^5},\quad Y_5 = \frac{-y_2^2y_5 + 10y_2y_3y_4 + 15y_3^3}{y_2^7},\ \&c.$$

Hence, in particular,

$$9Y_2^2Y_5 - 45Y_2Y_3Y_4 + 40Y_3^3 = -\frac{1}{y_2^9}(9y_2^2y_5 - 45y_2y_3y_4 + 40y_3^3),$$

viz. the differential equation $9y_2^2y_5 - 45y_2y_3y_4 + 40y_3^3 = 0$ (in the former notation $a^2d - 3abc + 2b^3 = 0$) of the conic in point-coordinates gives, as it should do, the equation of like form $9Y_2^2Y_5 - 45Y_2Y_3Y_4 + 40Y_3^3 = 0$ of the conic in line-coordinates.

The geometrical applications throughout the memoir are very extensive and interesting.

VIII.

In Halphen's second memoir "Sur les invariants différentiels des courbes gauches" (1880), instead of a single dependent variable y a function of x, we have two dependent variables y, z, each of them a function of x.

The relations between the original variables x, y, z, and the new variables X, Y, Z, are of the general homographic form

$$X,\ Y,\ Z = \frac{\xi}{\omega},\ \frac{\eta}{\omega},\ \frac{\zeta}{\omega},$$

where

$$\begin{aligned}\xi &= \alpha x + \beta y + \gamma z + \delta\ ,\\ \eta &= \alpha' x + \beta' y + \gamma' z + \delta'\ ,\\ \zeta &= \alpha'' x + \beta'' y + \gamma'' z + \delta''\ ,\\ \omega &= \alpha''' x + \beta''' y + \gamma''' z + \delta''',\end{aligned}$$

but he does not from these formulæ deduce the expressions of $Y_{,}, Y_{,,}, \ldots, Z_{,}, Z_{,,}, \ldots$ in terms of $y', y'', \ldots, z', z'', \ldots$; the investigation is in some measure a geometrical one. The notation employed is

$$u = \tfrac{1}{12}(y''z''' - y'''z''),$$

viz. u is here the most simple reciprocant, the equation $u=0$ is obviously the condition of a plane curve:

$$a_n = \frac{1}{4.5.6\ldots n}\ \frac{y''z^{(n)} - y^{(n)}z''}{y''z''' - y'''z''},$$

$$b_n = \frac{-1}{3.4.5\ldots n}\ \frac{y'''z^{(n)} - y^{(n)}z'''}{y''z''' - y'''z''},$$

n greater than or equal to 4; viz. these two singly infinite series of symbols $a_4, a_5, a_6, \ldots, b_4, b_5, b_6, \ldots$, together with u, are used for the expression of the doubly infinite series $y^{(m)}z^{(n)} - y^{(n)}z^{(m)}$, by virtue of the identity

$$(y''z''' - y'''z'')(y^{(m)}z^{(n)} - y^{(n)}z^{(m)})$$
$$= (y''z^{(m)} - y^{(m)}z'')(y'''z^{(n)} - y^{(n)}z''') - (y''z^{(n)} - y^{(n)}z'')(y'''z^{(m)} - y^{(m)}z'''),$$

or, as this may be written,

$$\frac{1}{1.2\ldots m.1.2\ldots n}\ \frac{y^{(m)}z^{(n)} - y^{(n)}z^{(m)}}{u} = a_nb_m - a_mb_n.$$

The most simple reciprocant (after u) is

$$v = a_6 - 2b_5 - 3a_4a_5 + 3a_4b_4 + 2a_4^3,$$

or rather u^3v, which is an integral function of the differential coefficients: the signification of the equation $v=0$ is that the tangents of the curve belong to a linear complex. This is a property belonging to the tangents of a skew cubic, and the skew cubic thus satisfies the differential equation $u=0$. Another reciprocant is obtained

$$w = b_6 - a_4b_5 - 4a_5b_4 + 4a_4^2b_4 - 2a_4^2a_5 + 2b_4^2 + a_5^2 + a_4^4,$$

or rather u^4w, which is an integral function of the differential coefficients: and the skew cubic satisfies the second differential equation $w=0$. The formulæ enable the determination of the osculating skew cubic at any point of a skew curve.

The general theory is developed in a compact form in the theorems I. to VII., and it is shown that the investigation of all the reciprocants depends upon that of two reciprocants of the seventh order. The geometrical applications of the theory are very extensive and interesting.

IX.

Before going further, I remark that the homologic transformation

$$X = \alpha x + \beta y + \gamma, \quad Y = \alpha' x + \beta' y + \gamma',$$

(which is in point of generality intermediate between Halphen's homographic transformation

$$X = \frac{\alpha x + \beta y + \gamma}{\alpha'' x + \beta'' y + \gamma''}, \quad Y = \frac{\alpha' x + \beta' y + \gamma'}{\alpha'' x + \beta'' y + \gamma''},$$

and Sylvester's special transformation $X = y$, $Y = x$, and which includes as a particular case the rectangular transformation

$$X = x \cos\theta + y \sin\theta, \quad Y = -x \sin\theta + y \cos\theta),$$

does not appear to have been explicitly considered. Writing as in the general case $t, a, b, c, \ldots$, for $y', \frac{1}{2}y'', \frac{1}{6}y''', \ldots$, and $T, A, B, C, \ldots$, for $Y_,, \frac{1}{2}Y_{,,}, \frac{1}{6}Y_{,,,}, \ldots$; also taking h, k for the increments of x, y, and H, K for those of X, Y respectively, we have

$$H = \alpha h + \beta k, \quad k = th + ah^2 + bh^3 + ch^4 + \ldots,$$
$$K = \alpha' h + \beta' k,$$

that is,

$$H = (\alpha + \beta t) h + \beta (ah^2 + bh^3 + ch^4 + \ldots),$$
$$K = (\alpha' + \beta' t) h + \beta' (ah^2 + bh^3 + ch^4 + \ldots),$$

which, by the elimination of h, must lead to

$$K = TH + AH^2 + BH^3 + CH^4 + \ldots.$$

We have

$$T = \frac{\alpha' + \beta' t}{\alpha + \beta t};$$

and we write

$$\lambda = \frac{\beta}{\alpha + \beta t}, \quad \lambda' = \frac{\beta'}{\alpha' + \beta' t},$$

and thence

$$\lambda' - \lambda = \frac{\alpha\beta' - \alpha'\beta}{\alpha + \beta t \,.\, \alpha' + \beta' t};$$

moreover

$$\mu = \frac{1}{\alpha + \beta t}, \quad \mu' = \frac{1}{\alpha' + \beta' t},$$

and therefore

$$T = \frac{\mu}{\mu'}.$$

Also

$$\Omega = ah + bh^2 + ch^3 + \dots;$$

whence, if

$$h = p\Omega + q\Omega^2 + r\Omega^3 + s\Omega^4 + \dots,$$

then

$$p = \frac{1}{a},$$

$$q = -\frac{1}{a^3} b,$$

$$r = -\frac{1}{a^5}(ac - 2b^2),$$

$$s = -\frac{1}{a^7}(a^2 d - 5abc + 5b^3),$$

$$\vdots$$

and we have

$$H = (\alpha + \beta t) h + \beta h\Omega, \ = \left(\frac{1}{\mu} + \beta\Omega\right)(p\Omega + q\Omega^2 + r\Omega^3 + \dots),$$

giving

$$\Omega = XH + YH^2 + ZH^3 + WH^4 + \dots,$$

where, substituting for p, q, r, ..., their values,

$$X = \mu a,$$

$$Y = \mu^2 . b - a^2\lambda,$$

$$Z = \mu^3 . c - 3ab\lambda + 2a^3\lambda^2,$$

$$W = \mu^4 . d - (4ac + 2b^2)\lambda + 10a^2b\lambda^2 - 5a^4\lambda^3,$$

$$\vdots$$

and then

$$K = HT\frac{1 + \lambda'\Omega}{1 + \lambda\Omega}, \ = HT\frac{1 + \lambda'(XH + YH^2 + \dots)}{1 + \lambda(XH + YH^2 + \dots)},$$

giving

$$K = TH + AH^2 + BH^3 + CH^4 + \dots,$$

where

$$A = \lambda' - \lambda . X,$$

$$B = \lambda' - \lambda . Y - \lambda X^2,$$

$$C = \lambda' - \lambda . Z - \lambda 2XY + \lambda^2 X^3,$$

$$D = \lambda' - \lambda . W - \lambda(2XZ + Y^2)\lambda^2 . 3X^2Y - \lambda^3 X^4,$$

whence, substituting for X, Y, Z, W, ..., their values, we have $T = \frac{\mu}{\mu'}$, *ut suprà*, and

$$A = (\lambda' - \lambda)\mu . a,$$

$$B = (\lambda' - \lambda)\mu^2 . b - 2a^2\lambda,$$

$$C = (\lambda' - \lambda)\mu^3 . c - 5ab\lambda + 5a^3\lambda^2,$$

$$D = (\lambda' - \lambda)\mu^4 . d + (-6ac - 3b^2)\lambda + 21a^2b\lambda^2 - 14a^4\lambda^3,$$

$$\vdots$$

It will be observed that μ' enters only into the equation $T=\frac{\mu}{\mu'}$: there are thus no reciprocants containing t, i.e. there is here nothing analogous to Sylvester's impure reciprocants. But we have as reciprocants, all his pure reciprocants, for instance

$$4AC-5B^2=(\lambda'-\lambda)^2(4ac-5b^2), \text{ \&c.}$$

Putting

$$\alpha=0, \quad \beta=1, \quad \alpha'=1, \quad \beta'=0,$$

then

$$\lambda=\frac{1}{t}, \quad \lambda'=0, \quad \mu=\frac{1}{t}, \quad \mu'=1;$$

then $T=\frac{1}{t}$, viz. we have here no arbitrary constant entering into this equation only, and there are thus reciprocants containing t. We have, in fact, the formulæ of Sylvester's theory.

In the rectangular case,

$$\alpha=\cos\theta, \quad \beta=\sin\theta, \quad \alpha'=-\sin\theta, \quad \beta'=\cos\theta,$$

we have

$$T=\frac{\cos\theta+t\sin\theta}{-\sin\theta+t\cos\theta},$$

where θ enters into the other equations, and there are thus impure reciprocants containing t, and reciprocants pure and impure which are not reciprocants in the general homologic case; but I do not go into the question of these orthogonal reciprocants.

X.

We have next Sylvester:—"Lectures on the theory of Reciprocants" (reported by J. Hammond), *Amer. Math. Journ.*, t. VIII. (1886), pp. 196—260.

The lectures were delivered at Oxford, *Inaugural Lecture*, Dec. 1885, starting from the Schwarzian function*

$$\frac{y'''}{y'}-\tfrac{3}{2}\left(\frac{y''}{y'}\right)^2,$$

which acquires only a factor by the interchange of x and y,

$$\left[\frac{y'''}{y'}-\tfrac{3}{2}\left(\frac{y''}{y'}\right)^2=-x_1^2\left\{\frac{x_{,,,}}{x_,}-\tfrac{3}{2}\left(\frac{x_{,,}}{x_,}\right)^2\right\}\right].$$

In lecture 2, pp. 203 *et seq.*, Sylvester considers the general theory of the functions which remain unaltered, except as to a factor, by the interchange of x, y: viz. writing

$$t, a, b, c, \ldots \text{ for } y', y'', y''', y'''', \ldots,$$

or again

$$t, a_0, a_1, a_2, a_3, \ldots \text{ for } y', \tfrac{1}{2}y'', \tfrac{1}{6}y''', \tfrac{1}{24}y'''', \ldots,$$

* Schwarzian $=\frac{y'''}{y'}-\frac{3}{2}\left(\frac{y''}{y'}\right)^2=6\frac{b}{t}-\frac{3}{2}\left(\frac{2a}{t}\right)^2=\frac{6\,(bt-a^2)}{t^2}$, where the reciprocant is $=bt-a^2$.

and so

$$T,\ \alpha,\ \beta,\ \gamma, \ldots \text{ for } x_{,},\ x_{,,},\ x_{,,,},\ x_{,,,,}, \ldots,$$

or again

$$T,\ \alpha_0,\ \alpha_1,\ \alpha_2,\ \alpha_3, \ldots \text{ for } x_{,},\ \tfrac{1}{2}x_{,,},\ \tfrac{1}{6}x_{,,,},\ \tfrac{1}{24}x_{,,,,}, \ldots$$

he gives the equations

$$\begin{array}{llll} \alpha = -a & \div t^3, \text{ or } & \alpha_0 = -a_0 & \div t^3, \\ \beta = -bt + 3a^2 & \div t^5, & \alpha_1 = -a_1 t + 2a_0^2 & \div t^5, \\ \gamma = -ct^2 + 10abt - 15a^3 & \div t^7, & \alpha_2 = -a_2 t^2 + 5a_0 a_1 t - 5a_0^3 & \div t^7, \\ \vdots & & \vdots & \end{array}$$

and obtains with $(a, b, c, \ldots)$ reciprocants such as a, $2bt - 3a^2$, &c., viz. we have

$$a = -t^3 . \alpha,$$

$$2bt - 3a^2 = -t^6 . 2\beta T - 3\alpha^2,$$

and further on, like forms with $(a_0, a_1, a_2, \ldots)$.

It is to be remarked that it is preferable to deal with the quantities $(a_0, a_1, a_2, \ldots)$ which represent $(\frac{1}{2}y'', \frac{1}{6}y''', \frac{1}{24}y'''', \ldots)$ rather than with $(a, b, c, \ldots)$ which represent $(y', y'', y''', \ldots)$. I do this in the sequel, *changing the notation*, and writing $(t, a, b, c, \ldots)$ to denote $(y', \frac{1}{2}y'', \frac{1}{6}y''', \frac{1}{24}y'''', \ldots)$, viz. my $(t, a, b, c, \ldots)$ are not Sylvester's $(t, a, b, c, \ldots)$, but are his $(t, a_0, a_1, a_2, \ldots)$.

A reciprocant with Sylvester is thus a function of y and its differential coefficients in regard to x, which except as to a factor remain unaltered when x, y are interchanged, or say, when x, y are changed into X, Y, where $X = y$, $Y = x$. This is a much less general change than Halphen's, and thus every reciprocant (Halphen) is a reciprocant (Sylvester), but not conversely. The reciprocants (Sylvester) are far more numerous. I remark that incidentally Sylvester considers reciprocants, which remain unaltered save as to a factor, when x, y are changed into X, Y, where

$$X = \alpha x + \beta y + \gamma, \quad Y = \alpha' x + \beta' y + \gamma',$$

which again is a less general change than Halphen's—it has been in what precedes alluded to as the particular case $L'' = 0$, that is, $\alpha'' = 0$, $\beta'' = 0$. It may be remarked that the proper orthogonal substitution corresponding to Sylvester's substitution is not $X = y$, $Y = x$, but $X = y$, $Y = -x$, viz. we have here the determinant $\alpha\beta' - \alpha'\beta = +1$.

I develop Sylvester's theory of reciprocants; but I wish first to point out the resemblance in form between this theory and that of seminvariants. In the theory of seminvariants, from the set of quantities $(a, b, c, \ldots)$ in connexion with an arbitrary quantity θ, we deduce a new set $(a', b', c', \ldots)$, where

$$\begin{aligned} a' &= a, \\ b' &= b + a\theta, \\ c' &= c + 2b\theta + a\theta^2, \\ d' &= d + 3c\theta + 3b\theta^2 + a\theta^3, \\ &\vdots \end{aligned}$$

or, what is the same thing, if $\theta' = -\theta$, then

$$\begin{aligned} a &= a', \\ b &= b' + a'\theta', \\ c &= c' + 2b'\theta' + a'\theta'^2, \\ d &= d' + 3c'\theta' + 3b'\theta'^2 + a'\theta'^3, \\ &\vdots \end{aligned}$$

and this being so there exist functions which are the same for unaccented and the accented letters respectively; for instance, $a', = a$:

$$\begin{aligned} a'c' &= \quad ac + 2ab\theta + a^2\theta^2, \\ -b'^2 &= -b^2 - 2ab\theta - a^2\theta^2, \end{aligned}$$

that is, $a'c' - b'^2 = ac - b^2$; and similarly

$$a'^2d' - 3a'b'c' + 2b'^3 = a^2d - 3abc + 2b^3, \text{ \&c.};$$

these functions a, $ac - b^2$, $a^2d - 3abc + 2b^3$, &c., are called seminvariants.

Similarly, in Sylvester's theory of reciprocants, starting from y, a given function of x, we have $(t, a, b, c, \ldots)$ denoting $(y', \frac{1}{2}y'', \frac{1}{6}y''', \frac{1}{24}y'''')$; and conversely $(t', a', b', c', \ldots)$ denoting $(x_{,}, \frac{1}{2}x_{,,}, \frac{1}{6}x_{,,,}, \frac{1}{24}x_{,,,,}, \ldots)$; taking h, k for the increments of x and y respectively, we have

$$\begin{aligned} k &= th + ah^2 + bh^3 + ch^4 + dh^5 + \ldots, \\ h &= t'k + a'k^2 + b'k^3 + c'k^4 + d'k^5 + \ldots, \end{aligned}$$

which equations determine the relations between $(t, a, b, c, d, \ldots)$ and $(t', a', b', c', d', \ldots)$, viz. we have $tt' = 1$, and then

$$\begin{aligned} a' &= -a && \div t^3, \\ b' &= -bt + 2a^2 && \div t^5, \\ c' &= -ct^2 + 5abt - 5a^3 && \div t^7, \\ d' &= -dt^3 + (6ac + 3b^2)\,t^2 - 21a^2bt + 14a^4 && \div t^9, \\ e' &= -et^4 + (7ad + 7bc)\,t^3 - (28a^2c + 28ab^2)\,t^2 + 84a^3bt - 42a^5 && \div t^{11}, \\ &\vdots \end{aligned}$$

and conversely,

$$\begin{aligned} a &= -a' && \div t'^3, \\ b &= -b't' + 2a'^2 && \div t'^5, \\ c &= -c't'^2 + 5a'b't' - 5a'^3 && \div t'^7, \\ &\vdots \end{aligned}$$

and we thence deduce functions which have equal or opposite values for the accented and unaccented letters respectively; these functions may or may not contain t, t'. Thus we have

$$a't'^{-\frac{3}{2}} = -at^{-\frac{3}{2}},$$
$$(b't' - a'^2)\, t'^{-3} = -(bt - a^2)\, t^{-3},$$
$$(2c't' - 5a'b')\, t'^{-\frac{7}{2}} = -(2ct - 5ab)\, t^{-\frac{7}{2}},$$
$$(4a'c' - 5b'^2)\, t'^{-4} = (4ac - 5b^2)\, t^{-4},$$
$$\{(1 + t'^2)\, b' - 2a'^2 t'\}\, t'^{-3} = \{(1 + t^2)\, b - 2a^2 t\}\, t^{-3},$$
&c.

These functions of the unaccented letters, or say the same functions omitting the exterior power of t, are called Reciprocants; thus we have the reciprocants a, $bt - a^2$ $2ct - 5ab$, $4ac - 5b^2$, $(1 + t^2)\, b - 2a^2 t$, &c. Observe that, when the exterior powers of t are omitted, then the values are equal or opposite to those with the accented letters save as to a power of t, viz. the forms are

$$a = -a't^3,$$
$$bt - a^2 = -(b't' - a'^2)\, t^6,$$
$$2ct - 5ab = -(2c't' - 5a'b')\, t^7,$$
$$4ac - 5b^2 = (4a'c' - 5b'^2)\, t^8,$$
$$\{(1 + t^2)\, b - 2a^2 t\} = -\{(1 + t'^2)\, b' - 2a'^2 t'\}\, t^6,$$
&c.

A reciprocant is said to be odd or even according as the sign on the right-hand side is $-$ or $+$; the index of t on the right-hand side is said to be the weight of the reciprocant. Reciprocants may be combined in the way of addition if and only if they are each of them of the same weight and parity, viz. this being so, then if λ, λ', ..., are mere numbers, $\lambda R + \lambda' R' + \ldots$ will be a reciprocant of the same weight and parity with each of the reciprocants R, R', ...

Reciprocants may in every case be combined in the way of multiplication; viz. two or more reciprocants may be multiplied together giving a reciprocant the parity of which is the sum of the parities, and its weight the sum of the weights of the component reciprocants. Thus $bt - a^2$ and $2ct - 5ab$ being odd reciprocants of the weights 6 and 7 respectively, we have $(bt - a^2)(2ct - 5ab)$ an even reciprocant of the weight 13.

A reciprocant is pure or impure according as it does not or does contain t. The foregoing equations for a', b', c', ..., may be obtained each from the next preceding one by operating upon it with

$$\frac{1}{\lambda} t^{-1} (2a\partial_t + 3b\partial_a + 4c\partial_b + 5d\partial_c + \ldots),$$

where λ is a positive integer, $= 3 +$ weight of the letter operated upon ($3 + 3$, $= 6$ in operating on d' to obtain e', $3 + 4 = 7$ in operating on e' to obtain f', and so on).

For example,

$$e' = \tfrac{1}{6}t^{-1}(2a\partial_t + 3b\partial_a + 4c\partial_b + 5d\partial_c + 6e\partial_d)\,d'$$
$$= t^{-1}(\tfrac{1}{3}a\partial_t + \tfrac{1}{2}b\partial_a + \tfrac{2}{3}c\partial_b + \tfrac{5}{6}d\partial_c + e\partial_d)\,d',$$

viz. operating herewith on

$$d', = \{-dt^3 + (6ac+3b^2)\,t^2 - 21a^2bt + 14a^4\} \div t^9,$$

we obtain

t^4,	t^3 ,	t^2 ,	t ,	t^0	$\div t^{11}$,
	$2ad$,	$-14a^2c - 7ab^2$,	$56a^3b$,	$-42a^5$,	
	$3bc$,	$-21ab^2$,	$28a^3b$,		
	$4bc$,	$-14a^2c$,			
	$5ad$,				
$-e$,					
$-e$,	$7ad+7bc$,	$-28a^2c-28ab^2$,	$84a^3b$,	$-42a^5$,	

the value given above.

The proof is at once obtained: except as to a numerical factor, the operation is in fact equivalent to the differentiation of a function of y, and its derived coefficients y', y'', ..., in regard to x.

Any pure reciprocant is reduced to zero by the operator

$$V = 4\,.\,\tfrac{1}{2}a^2\partial_b + 5ab\partial_c + 6(ac+\tfrac{1}{2}b^2)\,\partial_d + 7(ad+bc)\,\partial_e + 8(ae+bd+\tfrac{1}{2}c^2)\,\partial_f + \ldots,$$

or say V is an annihilator of any pure reciprocant: compare herewith the theorem $\Delta_1 = a\partial_b + 2b\partial_c + 3c\partial_d + \ldots$ is an annihilator of any seminvariant; and conversely, any homogeneous isobaric function of $a, b, c, \ldots$ reduced to zero by V is a pure reciprocant. For instance, for the pure reciprocant $5ac-4b^2$, we have

$$(2a^2\partial_b + 5ab\partial_c)(4ac-5b^2) = -20a^2b + 20a^2b, = 0.$$

Any rational and integral homogeneous reciprocant, operated upon with

$$G, = (3bt-3a^2)\,\partial_a + (4ct-4ab)\,\partial_b + (5dt-5ac)\,\partial_c + \ldots,$$

produces a like reciprocant. Thus operating upon a with G, we obtain $3bt-3a^2$, or say $bt-a^2$, and operating hereon again with G, a new reciprocant, and so on.

The series of reciprocants thus derived from a is

$$a,$$
$$bt - a^2,$$
$$2ct - 5ab,$$
$$\left.\begin{array}{r} 2dt^2 - 6ac \\ -3b^2 \end{array}\right|\, t + 7a^2b,$$

$$\begin{array}{r|l} 2et^2 - \ \ 7ad & t^2 + \ \ \ 8a^2c \\ -\ \ 7bc & \ \ \ +\ \ 11ab^2, \end{array}$$

$$\begin{array}{r|l|l} 14ft^3 - 56ae & t^2 + 103a^2d & t - \ \ 88a^3c \\ -56bd & \ \ \ +199abc & -121a^2b^2, \\ -28c^2 & \ \ \ +\ \ 33b^3 & \end{array}$$

&c.

These are, in fact, connected with the terms of

$$\left(\frac{1}{\sqrt{t}}\,\frac{d}{dx}\right)^i \log t, \quad = -\left(\frac{1}{\sqrt{t'}}\,\frac{d}{dy}\right)^i \log t',$$

where observe that $\partial_x t = 2a$, $\partial_x a = 3b$, $\partial_x b = 4c$, &c. Thus

$$\left(\frac{1}{\sqrt{t}}\,\frac{d}{dx}\right)\log t = \frac{a}{t^{\frac{3}{2}}}; \quad \left(\frac{1}{\sqrt{t}}\,\frac{d}{dx}\right)^2 \log t = \frac{1}{\sqrt{t}}\,\frac{d}{dx}\,\frac{a}{t^{\frac{3}{2}}} = \frac{3\,(bt - a^2)}{t^3}, \text{ \&c.}$$

A like generator for pure reciprocants is

$$H, = 4\,(ac - b^2)\,\partial_b + 5\,(ad - bc)\,\partial_c + 6\,(ae - bd)\,\partial_d + \ldots;$$

thus operating herewith upon $(4ac - 5b^2)$, we obtain

$$(4ac - b^2)\,(-10b) + 5\,(ad - bc)\,4a, \quad = 20a^2d - 60abc + 40b^3,$$

viz. we have thus the pure reciprocant $a^2d - 3abc + 2b^3$.

In repeating this process, we may reduce by means of powers and products of earlier reciprocants, and we can also in many cases throw out powers of the reciprocant a. Thus forming the expression for $H\,(a^2d - 3abc + 2b^3)$, this is given by column 1 of the annexed form: multiplying by 25, and adding $24\,(4ac - 5b^2)^2$, so as to eliminate the term in b^4, we obtain the expression in col. 4: or throwing out the numerical factor 3, and also the factor a, we have the

	1, H,	2, $25H$,	3, $24H^2$,	4, $\div 3$,	5,	
a^3e	$+\ 6$	$+\ 150$		$+\ 150$	$+\ 50$	a^2e
a^2bd	$-\ 21$	$-\ 525$		$-\ 525$	$-\ 175$	abd
a^2c^2	$-\ 12$	$-\ 300$	$+\ 384$	$+\ 84$	$+\ 28$	ac^2
ab^2c	$+\ 51$	$+\ 1275$	$-\ 960$	$+\ 315$	$+\ 105$	b^2c
b^4	$-\ 24$	$-\ 600$	$+\ 600$			

reciprocant $+\ 50a^2e -$ &c. in col. 5, and the outside right-hand column of literal terms.

We thus obtain the series of pure reciprocants

$a+1$	$ac+4$	a^2d+1	a^2e+50	a^3f+10	a^2g+14	a^3h+7	a^3i+420	
	b^2-5	$abc-3$	$abd-175$	a^2be-40	$abf-63$	a^2bg-35	$a^2bh-2310$	
		b^3+2	ac^2+28	a^2cd-12	$ace-1350$	$a^2cf-539$	$a^2cg-25648$	$+1176$
			b^2c+105	ab^2d+65	ad^2+1470	$a^2de+605$	$a^2df+9240$	-8085
				abc^2+16	$b^2e+1782$	ab^2f+735	$a^2e^2+21780$	$+7040$
				b^3c-39	$bcd-4158$	$abce+306$	$ab^2g+36680$	-1470
					c^3+2310	abd^2-2135	$abcf+85386$	$+18963$
						$ac^2d+1001$	$abde-191730$	-16940
						$b^3e-1485$	$ac^2e-59220$	-27160
						$b^2cd+3465$	$acd^2+120540$	$+26460$
						bc^3-1925	$b^3f-126945$	-9555
							$b^2ce+252126$	$+28098$
							$b^2d^2+169260$	$+12740$
							$bc^2d-419034$	-52822
							$c^4+129360$	$+21560$

I remark that a pure reciprocant not included in the foregoing series is

$a^2ce+800$
a^2d^2-875
$ab^2e-1000$
$abcd+2450$
ac^3-1344
b^2c^2-35

XI.

Halphen assumes that $a^2d-3abc+2b^3$ is the sextactic reciprocant, or, what is the same thing, that the general conic satisfies the differential equation of the fifth order $a^2d-3abc+2b^3$; the proof is as follows.

The general equation of the conic may be written

$$y+Ax+B=\sqrt{k}\sqrt{(x-\alpha\,.\,x-\beta)},$$

which depends on the five parameters A, B, k, α, β.

We have

$$y' + A = \tfrac{1}{2}\sqrt{k}\,\frac{2x-\alpha-\beta}{\sqrt{(x-\alpha\,.\,x-\beta)}},$$

$$y'' \;= \tfrac{1}{4}\sqrt{k}\left\{\frac{4}{\sqrt{(x-\alpha\,.\,x-\beta)}} - \frac{(2x-\alpha-\beta)^2}{(x-\alpha\,.\,x-\beta)^{\frac{3}{2}}}\right\},$$

$$= \frac{-\tfrac{1}{4}\sqrt{k}\,(\alpha-\beta)^2}{\{x-\alpha\,.\,x-\beta\}^{\frac{3}{2}}},$$

or say

$$y''^{-\frac{2}{3}} = (\tfrac{1}{2})^{-\frac{4}{3}}\,k^{-\frac{1}{3}}\,(\alpha-\beta)^{-\frac{4}{3}}\,x-\alpha\,.\,x-\beta\,;$$

and we have thus the differential equation $\left(\dfrac{d}{dx}\right)^3 y''^{-\frac{2}{3}} = 0$. This gives

$$\left(\frac{d}{dx}\right)^2 y''^{-\frac{5}{3}}y''' = 0,$$

$$\frac{d}{dx}\quad y''^{-\frac{5}{3}}y'''' - \tfrac{5}{3}y''^{-\frac{8}{3}}y'''^2 = 0,$$

$$y''^{-\frac{5}{3}}y^{\text{v}} \quad - \tfrac{5}{3}y''^{-\frac{8}{3}}y'''y''''$$

$$- \tfrac{10}{3}y''^{-\frac{8}{3}}y'''y'''' + \tfrac{40}{9}y''^{-\frac{11}{3}}y'''^3 = 0,$$

that is,

$$9y''^{-\frac{5}{3}}y^{\text{v}} - 45y''^{-\frac{8}{3}}y'''y'''' + 40y''^{-\frac{11}{3}}y'''^3 = 0,$$

or say

$$9y''^2y^{\text{v}} \quad - 45y''y'''y'''' \quad + 40y'''^3 \quad = 0.$$

But y'', y''', y'''', $y^{\text{v}} = 2a,\ 6b,\ 24c,\ 120d$, and the equation thus is $4320\,(a^2d - 3abc + 2b^3) = 0$, viz. it is $a^2d - 3abc + 2b^3 = 0$.

XII.

Sylvester's Lectures are published in the *American Mathematical Journal* as follows: Lectures 1 to 10, t. VIII. (1886), pp. 196—260; lectures 11 to 32, t. IX. (1887), viz. 11 to 16, pp. 1—37, 17 to 24, pp. 113—161, and 25 to 32, pp. 297—352; and lectures 33 and 34 (34 by Mr Hammond), t. X. (1888), pp. 1—16. In the footnote p. 7 to lecture 12, writing $y_1, y_2, y_3, y_4, \ldots$, for the derived functions of y, he adopts definitely the notation $t, a, b, c, \ldots$, to mean the reduced functions, $y_1,\ \dfrac{1}{1\,.\,2}y_2,\ \dfrac{1}{1\,.\,2\,.\,3}y_3,\ \dfrac{1}{1\,.\,2\,.\,3\,.\,4}y_4, \ldots$. In lecture 13, p. 20, he introduces the term Principiant—"instead of the cumbrous terms Projective Reciprocants or Differential Invariants, it is better to use the single word Principiants to denominate that crowning class or order of Reciprocants which remain to a factor *près*, unaltered for any homographic substitutions impressed on the variables"—that is, Halphen's Differential Invariant = Principiant. And in lectures 22 *et seq.*, Sylvester develops an important theory in regard to Principiants, connecting them with reciprocants and seminvariants, viz. he considers a series of seminvariants $N, A_0, A_1, A_2, \ldots$, and a series of reciprocants $M, A, B, C, \ldots$, such that the seminvariants formed with either sets of capitals are identical with each other (for instance $A_0A_2 - A_1^2 = AC - B^2$), and that any such seminvariant is a Principiant.

In what follows,

instead of $N, A_0, A_1, A_2, \ldots$, I write $N, A, B, C, \ldots$,

and instead of $M, A, B, C, \ldots$, I write $\mathfrak{M}, \mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots$,

so that $N, A, B, C, \ldots$, are seminvariants and $\mathfrak{M}, \mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots$, are reciprocants.

The expressions of these functions up to E, $\mathfrak{E}$, are

$N=$	$A=$	$B=$	$C=$	$8D=$	$6E=$
$ac + 1$	$a^2d + 1$	$a^3e + 1$	$a^4f + 1$	$a^5g + 8$	$a^6h + 6$
$b^2 - 1$	$abc - 3$	$a^2bd - 4$	$a^3be - 5$	$a^4bf - 48$	$a^5bg - 42$
	$b^3 + 2$	$c^2 - 2$	$cd - 5$	$ce - 48$	$cf - 42$
		$ab^2c + 10$	$a^2b^2d + 15$	$d^2 - 25$	$de - 45$
		$b^4 - 5$	$bc^2 + 15$	$a^3b^2e + 168$	$a^4b^2f + 168$
			$ab^3c - 35$	$bcd + 342$	$bce + 345$
			$a^0b^5 + 14$	$c^3 + 56$	$bd^2 + 180$
				$a^2b^3d - 452$	$c^2d + 174$
				$b^2c^2 - 681$	$a^3b^3e - 510$
				$ab^4c + 1020$	$b^2cd - 1578$
				$a^0b^6 - 340$	$bc^3 - 522$
					$a^2b^4d + 1299$
					$b^3c^2 + 2622$
					$ab^5c - 2877$
					$a^0b^7 + 822$
± 1	± 3	± 11	± 45	± 1594	± 5616

In terms of the fundamental seminvariants as indicated by their initial and final terms

$$ac \infty b^2 = ac - b^2, \quad a^2d \infty b^3 = a^2d - 3abc + 2b^3, \text{ \&c.},$$

the expressions of these functions are

$$N = (ac \infty b^2),$$

$$A = (a^2d \infty b^3),$$

$$B = (a^3e \infty a^2c^2) \quad - \quad 5\,(a^2c^2 \infty b^4),$$

$$C = (a^4f \infty a^2bc^2) \quad - \quad 7\,(a^3cd \infty b^5),$$

$$8D = 8\,(a^5g \infty a^4d^2) \; - 168\,(a^4ce \infty a^3c^3) \; - 113\,(a^4d^2 \infty a^2b^2c^2) + 340\,(a^3c^3 \infty b^6),$$

$$2E = 2\,(a^6h \infty a^4bd^2) - \; 32\,(a^5cf \infty a^3bc^2) - \; 37\,(a^5de \infty a^2b^3c^2) + 137\,(a^2cd^2 \infty b^7).$$

$4\mathfrak{M}=$	$\mathfrak{A}=$	$2\mathfrak{B}=$	$4\mathfrak{C}=$	$8\mathfrak{D}=$	$48\mathfrak{E}=$
$ac+4$	a^2d+1	a^3e+2	a^4f+4	a^5g+8	a^6h+48
b^2-5	$abc-3$	a^2bd-7	a^3be-16	a^4bf-36	$a^5bg-240$
	b^3+2	c^2-4	$cd-20$	$ce-48$	$cf-336$
		$a\,b^2c+17$	a^2b^2d+45	d^2-25	$de-360$
		a^0b^4-8	bc^2+52	a^3b^2e+114	a^4b^2f+840
			$a\,b^3c-103$	$bcd+282$	$bce+2184$
			a^0b^5+38	c^3+56	bd^2+1140
				a^2b^3d-295	$c^2d+1392$
				b^2c^2-513	$a^3b^3e-2400$
				$a\,b^4c+657$	$b^2cd-8880$
				a^0b^6-200	bc^3-3504
					$a^2b^4d+5955$
					$b^3c^2+13836$
					$a\,b^5c-13065$
					a^0b^7+3390
$+4-5$	± 3	± 19	± 139	± 1117	± 28785

In terms of the fundamental reciprocants of the table, *ante* p. 387, say these are

$$4ac-5b^2=P_2,\quad a^2d-3abc+2b^3=P_3,\quad 50a^2e-\ldots+105b^2c=P_4,\quad 10a^3f-\ldots-39b^3c=P_5,$$

$$14a^2g-\ldots+2310b^2c=P_6,\quad 7a^3h-\ldots-1925bc^3=P_7,\text{*}$$

the expressions of these functions are

$$4\mathfrak{M}=P_2,$$
$$\mathfrak{A}=P_3,$$
$$50\mathfrak{B}=aP_4-8P_2^2,$$
$$40\mathfrak{C}=4aP_5-38P_2P_3,$$
$$2800\mathfrak{D}=200a^3P_6+1266aP_2P_4-302750P_3^3-9128P_2^3,$$
$$1680\mathfrak{E}=240a^3P_7+2940aP_2P_5-3156aP_3P_4+2373P_3P_2^2.$$

Sylvester remarks that a Principiant, e.g. $a^2d-3abc+2b^3$, is at once a reciprocant, and in the theory of seminvariants (where $a, b, c, \ldots$, are the coefficients of the theory) a seminvariant; and conversely that any reciprocant which is also a seminvariant is a principiant: *quà* seminvariant, the principiant is annihilated by

$$\Omega=a\partial_b+2b\partial_c+3c\partial_d+4d\partial_e+5e\partial_f+\ldots,$$

* The eighth column $+420a^3i\ldots+129360c^4$, is *not* the proper value of P_8; the proper value is a linear combination of the eighth and ninth columns, eighth column $+6$ ninth column, viz. $P_8=420a^3i\ldots-102102bc^2d$: see as to this my paper "Tables of Pure Reciprocants to the weight 8," *Amer. Math. Jour.*, t. xv. (1893), pp. 75—77, [933].

and *quà* reciprocant is annihilated by

$$V = 2a^2\partial_b + 5ab\partial_c + (6ac + 3b^2)\,\partial_d + (7ad + 7bc)\,\partial_e + (8ae + 8bd + 4c^2)\,\partial_f + \ldots;$$

and any function which is annihilated by each of these operators is a principiant.

We form the foregoing series of functions $N = ac - b^2$, $A, B, C, \ldots$, as follows, viz. if

$$G' = (4ac - 5b^2)\,\partial_b + (5ad - 7bc)\,\partial_c + (6ae - 9bd)\,\partial_d + (7af - 11be)\,\partial_e + \ldots,$$

then

$$\begin{aligned} 5A &= G'N, \\ 6B &= G'A, \\ 7C &= G'B + NA, \\ 8D &= G'C + 2NB, \\ 9E &= G'D + 3NC; \end{aligned}$$

and similarly we form the foregoing series of functions $\mathfrak{M}, \mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots$, as follows, viz. if

$$\mathfrak{G}' = 4\,(ac - b^2)\,\partial_b + 5\,(ad - bc)\,\partial_c + 3\,(ae - bd)\,\partial_d + 7\,(af - be)\,\partial_e + \ldots,$$

($\mathfrak{G}'$ is Sylvester's G), then

$$\begin{aligned} 5\mathfrak{A} &= \mathfrak{G}'\mathfrak{M}, \\ 6\mathfrak{B} &= \mathfrak{G}'\mathfrak{A}, \\ 7\mathfrak{C} &= \mathfrak{G}'\mathfrak{B} - \mathfrak{M}\mathfrak{A}, \\ 8\mathfrak{D} &= \mathfrak{G}'\mathfrak{C} - 2\mathfrak{M}\mathfrak{B}, \\ 9\mathfrak{E} &= \mathfrak{G}'\mathfrak{D} - 3\mathfrak{M}\mathfrak{C}. \end{aligned}$$

As already mentioned, $N, A, B, C, \ldots$, are seminvariants, $\mathfrak{M}, \mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots$, are reciprocants. Putting for shortness $\frac{1}{2}b = \theta$, the two sets of functions are connected by

$$\begin{aligned} A &= \mathfrak{A}, & &\text{or conversely } & \mathfrak{A} &= A, \\ B &= \mathfrak{B} - \theta\mathfrak{A}, & & & \mathfrak{B} &= B + \theta A, \\ C &= \mathfrak{C} - 2\theta\mathfrak{B} + \theta^2\mathfrak{A}, & & & \mathfrak{C} &= C + 2\theta B + \theta^2 A, \\ D &= \mathfrak{D} - 3\theta\mathfrak{C} + 3\theta^2\mathfrak{B} - \theta^3\mathfrak{A}, & & & \mathfrak{D} &= D + 3\theta C + 3\theta^2 B + \theta^3 A, \\ &\vdots & & & &\vdots \end{aligned}$$

so that, one of the sets being calculated, the other set can be at once deduced therefrom. These equations give

$$AC - B^2 = \mathfrak{A}\mathfrak{C} - \mathfrak{B}^2, \quad A^2D - 3ABC + 2B^3 = \mathfrak{A}^2\mathfrak{D} - 3\mathfrak{A}\mathfrak{B}\mathfrak{C} + 2\mathfrak{B}^3, \text{ \&c.},$$

viz. as mentioned above, any seminvariant in the letters $A, B, C, \ldots$, is equal to the same seminvariant in the letters $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots$; or, what is the same thing, any principiant has the same expression in the letters $A, B, C, \ldots$, and in the letters $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \ldots$ respectively.

We have thus the entire series of Principiants,

$$AC - B^2, \quad A^2D - 3ABC + 2B^3, \quad AE - 4BD + 3C^2, \text{ \&c.}$$

We may express Halphen's reciprocants in terms of the capitals A, B, C,.... We have

$$U = a, \quad V = A, \quad \Delta = AC - B^2.$$

To obtain formulæ for the higher reciprocants, we require the derived functions A', B', C', ..., where the accent denotes differentiation in regard to x.

We have

$$\partial_x = 3b\partial_a + 4c\partial_b + 5d\partial_c + \ldots,$$

(whence in particular $a' = 3b$, $b' = 4b$, $c' = 5c$, ..., as is obvious). Hence, writing

$$G' = a(3b\partial_a + 4c\partial_b + 5d\partial_c + \ldots) - b(3a\partial_a + 5b\partial_b + 7c\partial_c + \ldots),$$

this may be written

$$G' = a\partial_x - bw,$$

if w be the weight of the homobaric function operated upon, reckoning the weights of $a, b, c, \ldots$, as 3, 5, 7, ... respectively, and consequently the weights of A, B, C, ..., as 15, 20, 25, ... respectively. We thus have

$$\begin{aligned} 5A &= (a\partial_x - 10b)N, \\ 6B &= (a\partial_x - 15b)A, \\ 7C &= (a\partial_x - 20b)B + NA, \\ 8D &= (a\partial_x - 25b)C + 2NB, \\ 9E &= (a\partial_x - 30b)D + 3NC, \\ &\vdots \end{aligned}$$

of which the first gives only $A = a^2d - 3abc + 2b^3$. The other equations give the required formulæ

$$\begin{aligned} aA' &= 6B + 15bA, \\ aB' &= 7C + 20bB - NA, \\ aC' &= 8D + 25bC - 2NB, \\ aD' &= 9E + 30bD - 3NC, \\ &\vdots \end{aligned}$$

Halphen's H is given by $U^4H = 256\Delta^3 - 27V^8$, viz. we thus have

$$a^4H = 256(AC - B^2)^3 - 27A^8.$$

His T is defined by the equation $U^3T = 3V\Delta' - 8V'\Delta$, that is,

$$a^3T = 3A(AC - B^2)^2 - 8A'(AC - B^2),$$

which is

$$\begin{aligned} &= 3A(AC' - 2BB' + CA') - 8A'(AC - B^2), \\ &= 3A^2C' - 6ABB' + (-5AC + 8B^2)A', \end{aligned}$$

or substituting for A', B', C', their values we find

$$a^4T = 24(A^2D - 3ABC + 2B^3),$$

which expression for the reciprocant T was given by Sylvester.

Halphen's T_1 is $\frac{1}{U}(V^4T-\frac{1}{6}H)$, viz. we thus have

$$6a^5T_1=24A^4(A^2D-3ABC+2B^3)-256(AC-B^2)^3+27A^8.$$

His G is given by $V^2G=U^4T^2+9H$, viz. we have

$$A^2a^4G=576(A^2D-3ABC+2B^3)^2+2604(AC-B^2)^3-243A^8,$$

where the whole divides by A^2; throwing out this factor, we find

$$a^4G=576(A^2D^2-6ABCD+4AC^3+4B^3D-3B^2C^2)-243A^6.$$

From the expression for T, I find

$$a^5T'=216A^2E-288ABD-504AC^2+576B^2C+1152(A^2D-3ABC+2B^3),$$

and I thence deduce for Halphen's Θ,

$$=\frac{1}{V}\{2U\Delta T'+T(8U'\Delta-3U\Delta')\},$$

the formula

$$a^4\Theta=432(AC-B^2)(AE-4BD+3C^2)-576(A^2D^2-6ABCD+4AC^3+4B^3D-3B^2C^2),$$

or, what is the same thing,

$$=-144(AC-B^2)(AE-4BD+3C^2)+576A(ACE-AD^2-B^2E+2BCD-C^3).$$

Also $\Theta_1=\frac{1}{V}(\Theta+\frac{1}{4}G)$, that is, $a^4\Theta_1=\frac{1}{A}(a^4\Theta+\frac{1}{4}a^4G)$,

$$=\frac{432}{A}\{(AC-B^2)(AE-4BD+3C^2)-(A^2D^2-6ABCD+4AC^3+4B^3D-3B^2C^2)\},$$

or finally,

$$a^4\Theta_1=432(ACE-AD^2-B^2E+2BCD-C^3)-\tfrac{243}{4}A^5.$$

Again,

$$\Theta_2=\frac{1}{U}(\Theta_1-\tfrac{45}{2}TV)=\frac{1}{a}(\Theta_1-\tfrac{45}{2}TA),$$

whence

$$a^5\Theta_2=a^4\Theta_1-\tfrac{45}{2}a^4TA,$$

or substituting,

$$a^5\Theta_2=432(ACE-AD^2-B^2E+2BCD-C^3)-540A(A^2D-3ABC+2B^3)-\tfrac{243}{4}A^5.$$

Writing

$$\begin{aligned}&AC-B^2=\mathbf{C},\\&A^2D-3ABC+2B^3=\mathbf{D},\\&A^2D^2-6ABCD+4AC^3+4B^3D-3B^2C^2=\square,\\&AE-4BD+3C^2=\mathbf{I},\\&ACE-AD^2-B^2E+2BCD-C^3=\mathbf{J},\end{aligned}$$

the foregoing results become $U=a,\ V=A,\ \Delta=\mathbf{C}$,

$$\begin{aligned}
a^4H \text{ (Halphen) } &= 256\mathbf{C}^3 - 27A^8,\\
a^4T &= 24\mathbf{D},\\
6a^5T_1 &= 24A^4\mathbf{D} - 256\mathbf{C}^3 + 27A^8,\\
a^4G &= 576\square - 243A^6,\\
a^4\Theta &= 144\mathbf{CI} + 576A\mathbf{J},\\
a^4\Theta_1 &= 432\mathbf{J} - \tfrac{243}{4}A^5,\\
a^5\Theta_2 &= 432\mathbf{J} - 540A\mathbf{D} - \tfrac{243}{4}A^5.
\end{aligned}$$

XIII.

The letters $t, a, b, \ldots$, of a reciprocant represent (it will be remembered) mere numerical multiples of the derived functions of a variable y, in regard to the independent variable x, and thus equating any reciprocant to zero, we have a differential equation in y. For instance, if the orthogonal reciprocant $c(1+t^2)-5abt+5a^3$ be put $=0$, $(t, a, b, c = y_1, \frac{1}{2}y_2, \frac{1}{6}y_3, \frac{1}{24}y_4)$, this is the differential equation

$$y_4(1+y_1^2)-10y_1y_2y_3+15y_2^3=0$$

of the order 4. The integral hereof is expressible by the two equations

$$x=\int\frac{dt}{\sqrt{(\kappa U+\lambda V)}}+\mu,\quad y=\int\frac{tdt}{\sqrt{(\kappa U+\lambda V)}}+\nu,$$

where U, V are real functions of a parameter t, viz. $U+iV=(1+it)^6$, $U-iV=(1-it)^6$, $\{i=\sqrt{(-1)}$ as usual$\}$, so that $U=1-15t^2+15t^4-t^6$, $V=6t-20t^3+6t^5$; and $\kappa, \lambda, \mu, \nu$ are the four constants of integration. In verification hereof, we have

$$dx=\frac{dt}{\sqrt{(\kappa U+\lambda V)}},\quad dy=\frac{tdt}{\sqrt{(\kappa U+\lambda V)}},$$

whence

$$\frac{dy}{dx}=y_1,\ =t,$$

hence

$$y_2=\frac{dt}{dx}=\sqrt{(\kappa U+\lambda V)},\text{ or say } y_2^2=\kappa U+\lambda V,$$

and thence, using an accent to denote differentiation in regard to t, we have

$$2y_3=\kappa U' + \lambda V',$$

$$2\frac{y_4}{y_2}=\kappa U''+\lambda V'',$$

or, eliminating the κ and λ,

$$\begin{vmatrix} y_2^3, & U, & V \\ 2y_2y_3, & U', & V' \\ 2y_4, & U'', & V'' \end{vmatrix}=0.$$

Substituting for U, V either of the above-mentioned forms, there is in each case a factor in t which divides out; rejecting this factor, the equation is found to be

$$2y_4(1+t^2)-20ty_2y_3+30y_2^3=0,$$

that is, substituting for t its value $=y_1$, and throwing out the factor 2, we have the differential equation

$$y_4(1+y_1^2)-10y_1y_2y_3+15y_2^3=0.$$

XIV.

But when the reciprocant equated to zero is a Principiant, we have the far more important results obtained in Halphen's Memoir (1): and which are, in Sylvester's lectures, exhibited in a more complete form by expressing Halphen's reciprocants in terms of the foregoing functions A, B, C,.... I recall that a, b, c,... are numerical multiples of differential coefficients of the orders 2, 3, 4,... respectively; A, B, C,... contain d, e, f,... respectively, that is, differential coefficients of the orders 5, 6, 7,... respectively, so that according as the highest capital letter is A, B, C,... respectively, the order of the differential equation is 5, 6, 7,... respectively.

But the equation, Principiant $=0$, may be interpreted in a different manner, viz. if instead of regarding the differential equation as an equation for the determination of y, we regard therein y as a given function of x, {that is, (x, y) as the coordinates of a point on a given curve}, then the differential equation serves for the determination of those points on the curve for which a given condition is satisfied, or say it is the condition in order to the existence of a singular point of determinate character.

Halphen's lowest reciprocants are: as already mentioned:

$$\begin{aligned} U &= a, \\ V &= a^2d-3abc+2b^3, \ = A, \\ \Delta & \qquad\qquad\qquad\quad , \ = AC-B^2. \end{aligned}$$

$U=0$, that is, $a=0$, is a differential equation of the second order, viz. it is the differential equation of a line: otherwise it is the condition for a point of inflexion.

$V=0$, that is, $A=0$, is a differential equation of the fifth order, viz. it is the differential of a conic: otherwise it is the condition for a sextactic point.

$\Delta=0$, that is, $AC-B^2=0$, is a differential equation of the seventh order, it is the condition for what Halphen calls a point of coincidence, viz. this is a point on a given curve, such that for it the cubic of nine-pointic intersection becomes a nodal cubic having the point for node and through it one branch of eight-pointic intersection and one branch of simple intersection. Observe that this is more than the condition that the cubic of nine-pointic intersection shall be a nodal cubic, and accordingly $\Delta=0$ is not the differential equation of a nodal cubic. In fact, a nodal cubic depends on 8 parameters and has therefore a differential equation of the order 8; this will be obtained further on. But $\Delta=0$ is the differential equation of the order 7 of a curve such that every point thereof is a coincident point.

The differential equation of the order 7,

$$2^4 . 7^3 . \{(\lambda-2)(\lambda+1)(2\lambda-1)\}^2 . \Delta^3 - 3^3 . 5^2 (\lambda^2-\lambda+1)^3 V^8 = 0,$$

has the integral $\alpha = \beta^\lambda \gamma^{1-\lambda}$, where α, β, γ represent arbitrary linear functions $lx+my+n$: we may without loss of generality take the linear functions to be of the form $x+my+n$, introducing in this case another constant C, and writing the integral equation in the form $\alpha = C\beta^\lambda\gamma^{1-\lambda}$; the number of arbitrary constants is thus $=7$.

If $\lambda = 2$, -1 or $\frac{1}{2}$, the integral equation is $\alpha = \beta^2\gamma^{-1}$, $\alpha = \beta^{-1}\gamma^2$ or $\alpha = \beta^{\frac{1}{2}}\gamma^{\frac{1}{2}}$; or say it is $\beta^2 = \alpha\gamma$, $\gamma^2 = \alpha\beta$ or $\alpha^2 = \beta\gamma$, viz. each of these represents a conic. The differential equation is here $V = 0$, viz. we have the foregoing result that $V = 0$, that is, $A = 0$, is the differential equation of a conic.

If $\lambda = \infty$, 0 or 1, the differential equation is

$$2^6 . 7^3 . \Delta^3 - 3^3 . 5^2 . V^8 = 0;$$

the integral equation is here of the form $\frac{\beta}{\gamma} = \log\frac{\alpha}{\gamma}$; in fact we have $\frac{\alpha}{\gamma} = \left(\frac{\beta}{\gamma}\right)^\lambda$, or for α, β, and γ writing $\lambda\alpha$, $\beta+\lambda\gamma$, and $\lambda\gamma$ respectively, this is $\frac{\alpha}{\gamma} = \left(1+\frac{\beta}{\lambda\gamma}\right)^\lambda$, which when λ is indefinitely large becomes $\frac{\alpha}{\gamma} = \exp.\frac{\beta}{\gamma}$, that is, $\frac{\beta}{\gamma} = \log\frac{\alpha}{\gamma}$. And similarly for $\lambda = 0$ and $\lambda = 1$.

If $\lambda^2 - \lambda + 1 = 0$, that is, if $\lambda = -\omega$, where ω is an imaginary cube root of unity, the differential equation is $\Delta = 0$, that is, $AC - B^2 = 0$ (of the order 7 as in the general case). The integral equation is $\alpha = \beta^{-\omega}\gamma^{-\omega^2}$, viz. this is the equation of the curve every point of which is a coincident point.

If $\lambda = 3$, the differential equation is $2^8\Delta^3 - 3^3V^8 = 0$, or say $256\Delta^3 - 27V^8 = 0$; Halphen's H is a function such that $U^4H = 256\Delta^3 - 27V^8$, so that this is

$$H_3 = U^{-4}(256\Delta^3 - 27V^8) = 0.$$

The integral equation is $\alpha = \beta^3\gamma^{-2}$, that is, $\alpha\gamma^2 = \beta^3$, or the curve is a cuspidal cubic, depending upon 7 constants; and it thus appears that the differential equation of a cuspidal cubic is the equation of the order 7

$$a^{-4}\{256(AC-B^2)^3 - 27A^8\} = 0.$$

Write

$$\Theta = 64(A^2D^2 - 6ABCD + 4AC^3 + 4B^3D - 3B^2C^2) + 144A^2(A^2D - 3ABC + 2B^3) + 81A^6$$
$$= 64\square + 144A^2\mathbf{D} + 81A^6,$$

$$\Phi = 8(A^2D - 3ABC + 2B^3) + 9A^4 = 8\mathbf{D} + 9A^4,$$

$$\Psi = 3072(AC-B^2)^3 \qquad = 3072\mathbf{C}^3,$$

where identically

$$\Phi^2 + \tfrac{1}{12}\Psi = A^2\Theta.$$

Halphen and Sylvester find that, for a cubic curve the invariants of which are S and T ($S = -l + l^4$, $T = 1 - 20l^3 - 8l^6$ for the canonical form $x^3 + y^3 + z^3 + 6lxyz = 0$), we have

$$4\theta^2 S = \Theta^2 - 4\Phi\Psi,$$

$$\theta^3 T = \Theta^3 - 6\Theta\Phi\Psi + 6A^2\Psi^2,$$

(θ an arbitrary multiplier); and we thence have

$$\frac{(\Theta^2 - 4\Phi\Psi)^3}{(\Theta^3 - 6\Theta\Phi\Psi + 6A^2\Psi^2)^2} = \frac{64S^3}{T^2},$$

that is,

$$T^2 (\Theta^2 - 4\Phi\Psi)^3 - 64S^3 (\Theta^3 - 6\Theta\Phi\Psi + 6A^2\Psi^2)^2 = 0,$$

or if, as with Halphen $h = \frac{T^2}{64S^3}$, then

$$(\Theta^3 - 6\Theta\Phi\Psi + 6A^2\Psi^2)^2 - h(\Theta^2 - 4\Phi\Psi)^3 = 0,$$

which is the differential equation of the order 8 of a cubic curve having the absolute invariant $S^3 \div T^2$.

If for shortness

$$P = \Theta^2 - 4\Phi\Psi,$$

$$R = \Theta^3 - 6\Theta\Phi\Psi + 6A^2\Psi^2,$$

$$Q = R^2 - P^3,$$

then the foregoing equation $\frac{P^3}{R^2} = \frac{64S^3}{T^2}$ gives

$$\frac{Q}{R^2} = \frac{T^2 - 64S^3}{T^2} = \frac{h-1}{h},$$

and thus for a nodal cubic we have $Q = 0$; we thus have

$P = 0$, for the differential equation of a cubic for which $S = 0$,

$R = 0$, " " " " " $T = 0$,

$Q = 0$, " " " nodal cubic,

these being each of them of the order 8. But the equation $Q = 0$ is reducible to a more simple form. We have

$$Q = R^2 - P^3 = \begin{array}{ll} \Theta^6 & -\ \Theta^6 \\ -12\Theta^4\Phi\Psi & +12\Theta^4\Phi\Psi \\ +12\Theta^3A^2\Psi^2 & \\ +36\Theta^2\Phi^2\Psi^2 & -48\Theta^2\Phi^2\Psi^2 \\ -72\Theta\Phi A^2\Psi^3 & +64\Phi^3\Psi^3 \\ +36A^4\Psi^4 & \end{array} = \Psi^2 \left\{ \begin{array}{l} 12A^2\Theta^3 \\ -12\Theta^2\Phi^2 \\ -72A^2\Theta\Phi\Psi \\ +64\Phi^3\Psi \\ +36A^4\Psi^2 \end{array} \right\},$$

or, since in the expression in { } the first and second terms are $12\Theta^2(A^2\Theta - \Phi^2) = \Theta^2\Psi$, this is

$$Q = \Psi^3 \begin{Bmatrix} \Theta^2 \\ -72A^2\Theta\Phi \\ +64\Phi^3 \\ +36A^4\Psi \end{Bmatrix} = \Psi^3 \begin{Bmatrix} \Theta^2 \\ -72A^2\Theta\Phi \\ +64\Phi^3 \\ +432A^4(A^2\Theta - \Phi^2) \end{Bmatrix},$$

or finally

$$Q = \Psi^3 \{(\Theta - 36A^2\Phi + 216A^6)^2 - 64(\Phi - 9A^4)^3\}.$$

We thus have for the differential equation of the eighth order of the nodal cubic

$$(\Theta - 36A^2\Phi + 216A^6)^2 - 64(\Phi - 9A^4)^3 = 0,$$

or, since

$$\Theta = 64\square + 144A^2\mathbf{D} + 81A^6, \text{ and } \Phi = 8\mathbf{D} + 9A^4,$$

the differential equation of the eighth order for the nodal cubic finally is

$$(64\square - 144A^2\mathbf{D} + 27A^6)^2 + 32768\mathbf{D}^3 = 0.$$

Halphen has $\Theta_2 = 0$ for the differential equation of a cubic curve. Putting, as before,

$$\mathbf{I} = AE - 4BD + 3C^2,$$

$$\mathbf{J} = ACE - AD^2 - B^2E + 2BCD - C^3,$$

(and therefore $\mathbf{IC} - A\mathbf{J} = \square$, $\mathbf{I}^3 - 27\mathbf{J}^2 =$ Quartic Disct.), then, by what precedes, the differential equation of the order 9 of a cubic curve is

$$a^{-5}\{432\mathbf{J} - 540A\mathbf{D} - \tfrac{243}{4}A^5\} = 0.$$

Recapitulating.

Order. The differential equation for a cubic curve:—

8. for which invt. $S = 0$, is

$$P = \Theta^2 - 12288\mathbf{C}^3(8\mathbf{D} + 9A^4) = 0,$$

8. for which invt. $T = 0$, is

$$R = \Theta^3 - 18432\mathbf{C}^3(8\mathbf{D} + 9A^4) + 55623104\mathbf{C}^9A^2 = 0,$$

8. nodal cubic

$$Q \div (3072\Psi)^3 = (64\square - 144A^2\mathbf{D} + 27A^6)^2 + 32768\mathbf{D}^3 = 0,$$

8. for invt. $64S^3 \div T^2 = h$, is $(h-1)R^2 - hQ = 0$,

9. for general cubic is $1728\mathbf{J} - 2160A\mathbf{D} - 243A^5 = 0$,

where for shortness Θ is retained to signify its value

$$= 64\square + 144A^2\mathbf{D} + 81A^6.$$

Halphen by a polar transformation finds that these same differential equations, changing therein the sign of **D**, apply to curves of the third class: in particular, the last equation, changing therein the sign of **D**, applies to the general curve of the third class, or sextic curve with nine cusps.

XV.

A very important notion in the theory, as well of Seminvariants as of Reciprocants, is that of MacMahon's *Multilinear Operator*, see his paper "Theory of a Multilinear Partial Differential Operator, with applications to the theories of Invariants and Reciprocants," *Proc. Lond. Math. Soc.*, t. XVIII. (1886), pp. 61—88: this operator plays so important a part in the theories to which it relates, that I venture to reproduce the definition of it in what appears to me a simplified and more easily intelligible form, and to recapitulate some of the leading properties.

I take with him the letters to be $(a_0, a_1, a_2, a_3, \ldots) = (a, b, c, d, \ldots)$, viz. the first letter is a_0 or a, the second is a_1 or b, and so on, but when we are not concerned with a term of indefinite rank, I use always $(a, b, c, d, \ldots)$ and the like in regard to any other series of letters. This being so, the definition of the *Multilinear Operator* of four elements, which is here alone in question, is

$$(\mu,\ \nu : m,\ n), = \quad \begin{array}{r|c|c|c|c|c|c|}
 & n= & & & & & \\
 & 0 & 1 & 2 & 3 & \ldots & n \\
\hline
(\mu\quad\ )\,A & \partial_a & \partial_b & \partial_c & \partial_d & \ldots & \partial_{a_n} \\
+\,(\mu+\ \nu)\,B & \partial_b & \partial_c & \partial_d & \partial_e & \ldots & \partial_{a_{n+1}} \\
+\,(\mu+2\nu)\,C & \partial_c & \partial_d & \partial_e & \partial_f & \ldots & \partial_{a_{n+2}} \\
+\,(\mu+3\nu)\,D & \partial_d & \partial_e & \partial_f & \partial_g & \ldots & \partial_{a_{n+3}} \\
\vdots & & & & & &
\end{array},$$

where

	$m=$ 1	2	3	…	m
$A=$	a	$\tfrac{1}{2}a^2$	$\tfrac{1}{3}a^3$	…	$\dfrac{1}{m}a^m$
$B=$	b	ab	a^2b	…	$a^{m-1}b$
$C=$	c	$ac+\tfrac{1}{2}b^2$	a^2c+ab^2	…	$a^{m-1}c+\tfrac{1}{2}(m-1)\,a^{m-2}b^2$
$D=$	d	$ad+bc$	$a^2d+2abc+\tfrac{1}{3}b^3$	…	$a^{m-1}d+(m-1)\,a^{m-2}\,bc+\tfrac{1}{6}(m-1)(m-2)\,a^{m-3}b^3$,
⋮					

where the expressions for $B, C, D, \ldots$, are obtained successively from that of $A, = \dfrac{1}{m}a^m$, by Arbogast's rule of derivation, viz. we operate on the last letter of a term, and when there is a last but one letter which in alphabetical order immediately precedes the last letter, then also upon the last but one letter, and whenever the term thus obtained ends in a power, we divide by the index of the power. Thus the term $\tfrac{1}{2}(m-1)\,a^{m-2}b^2$ in the third line gives in the fourth line

$$\begin{aligned}&\tfrac{1}{2}(m-1)\,a^{m-2}\,.\,2bc+\tfrac{1}{2}(m-1)(m-2)\,a^{m-3}b\,.\,b^2\,.\,\tfrac{1}{3},\\ &=(m-1)\,a^{m-2}bc+\tfrac{1}{6}(m-1)(m-2)\,a^{m-3}b^3.\end{aligned}$$

Going a step further, we form in this manner the expression

$$E = a^{m-1}e + (m-1)\,a^{m-2}(bd + \tfrac{1}{2}c^2) + \tfrac{1}{6}(m-1)(m-2)\{a^{m-3}.\,3b^2c + (m-3)\,a^{m-4}b\,.\,b^3.\,\tfrac{1}{4}\},$$

that is,

$$\begin{aligned} E = \;& a^{m-1}e \\ &+ a^{m-2}\;\{(m-1)\,bd + \tfrac{1}{2}(m-1)\,c^2\} \\ &+ a^{m-3}.\,\tfrac{1}{2}\,(m-1)(m-2)\,b^2c \\ &+ a^{m-4}.\,\tfrac{1}{24}(m-1)(m-2)(m-3)\,b^4, \end{aligned}$$

which is sufficient to explain the rule.

It may be noticed that, the operator being linear in μ, ν, we have

$$(\mu,\ \nu;\ m,\ n) = \mu\,(1,\ 0;\ m,\ n) + \nu\,(0,\ 1;\ m,\ n).$$

The Alternant of two multilinear operators.

If P, Q are the two operators, we have as usual

$$P\,.\,Q = PQ + P * Q,$$

where $P.Q$ denotes the successive operation first with Q and then with P upon any operand, PQ is the mere algebraical product of the operators, and $P * Q$ is the operator obtained by the operation of P upon Q.

Similarly,

$$Q\,.\,P = QP + Q * P,$$

and since QP is the same thing as PQ, we obtain

$$P\,.\,Q - Q\,.\,P = (P * Q) - (Q * P),$$

either of which equal expressions, or say rather the second of them, is called the alternant of P, Q and is written $[P.Q]$: viz. as the definition of the alternant, we have

$$[P\,.\,Q] = (P * Q) - (Q * P).$$

We have the remarkable theorem, that the alternant of any two operators $(\mu',\ \nu';\ m',\ n')$, $(\mu,\ \nu;\ m,\ n)$ is an operator $(\mu_1,\ \nu_1;\ m_1,\ n_1)$; where

$$\begin{aligned} \mu_1 &= (m'+m-1)\left\{\frac{\mu'}{m'}(\mu + n'\nu) - \frac{\mu}{m}(\mu' + n\nu')\right\}, \\ \nu_1 &= (n'-n)\,\nu'\nu + \frac{m-1}{m'}\,\mu'\nu - \frac{m'-1}{m}\,\mu\nu', \\ m_1 &= m'+m-1, \\ n_1 &= n'+n, \end{aligned}$$

or since m_1, n_1 have such simple expressions, say it is an operator $(\mu_1,\ \nu_1;\ m'+m-1,\ n'+n)$, where μ_1, ν_1 have the values just written down.

We see at once how these values $m_1 = m' + m - 1$, $n_1 = n' + n$, arise: $Q = (\mu, \nu; m, n)$ contains the letters $(a, b, c, d, \ldots)$ in the degree m, and it contains differential symbols ∂ which, operating on any function of these letters, diminish the degree by 1; similarly $P = (\mu', \nu'; m', n')$ contains the letters $(a, b, c, d, \ldots)$ in the degree m', and it contains the differential symbols ∂ which, operating on any function of the letters, diminish the degree by 1. Hence $P * Q$ contains the letters in the degree $(m' - 1) + m$, and similarly $Q * P$ contains them in the same degree $(m - 1) + m'$: thus in the alternant the degree is $m' + m - 1$. Again, in Q the weights of the successive functions $A, B, C, \ldots$ are 0, 1, 2,... and these are combined with differential symbols $\partial_{a_n}, \partial_{a_{n+1}}, \ldots$ which operating on any function of the letters diminish the weight by $n, n+1, \ldots$ respectively: that is, the terms $A\partial_{a_n}, B\partial_{a_{n+1}}, \ldots$ each diminish the weight by n. So in $P = (\mu', \nu'; m', n')$ the weights of the successive terms $A, B, C, \ldots$ are 0, 1, 2, ... and these are combined with differential symbols $\partial_{a_{n'}}, \partial_{a_{n'+1}}, \ldots$ which operating on any function of the letters diminish the weights by $n', n'+1, \ldots$ respectively. We may say that Q is a sum of terms such as $\Theta_k \partial_{a_{n+k}}$, where the subscript k of Θ denotes the weight of Θ; operating hereon with P, the corresponding term of $P * Q$ is of the form $\Theta_{k-n'} \partial_{a_{n+k}}$, or (what is the same thing) the general form of the terms of $P * Q$ is $\Theta_k \partial_{a_{n'+n+k}}$; and in like manner this is the general form of the terms of $Q * P$; and it is thus also the general form of the terms of the alternant $[P . Q]$. It thus appears that, admitting the alternant to be an operator $(\mu_1, \nu_1; m_1, n_1)$, the value of n_1 is $n' + n$.

As an instance of the theorem, we may take

$$[(1,\ 0,\ 1,\ 1),\ (1,\ 0,\ 2,\ 1)] = (1,\ 0,\ 2,\ 2),$$

we have here

$$(1,\ 0;\ 1,\ 1) * (1,\ 0;\ 2,\ 1) - (1,\ 0;\ 2,\ 1) * (1,\ 0;\ 1,\ 1),$$

$$\begin{array}{r|l|l|r} \multicolumn{1}{r}{} & \multicolumn{1}{l}{*} & \multicolumn{1}{l}{-} & \multicolumn{1}{r}{*} \\ a\partial_b & \tfrac{1}{2}a^2\partial_b & \tfrac{1}{2}a^2\partial_b & a\partial_b \\ + b\partial_c & + ab\partial_c & + ab\partial_c & + b\partial_c \\ + c\partial_d & + (ac + \tfrac{1}{2}b^2)\,\partial_d & + (ac + \tfrac{1}{2}b^2)\,\partial_d & + c\partial_d \\ + d\partial_e & + (ad + bc)\,\partial_e & + (ad + bc)\,\partial_e & + d\partial_e \\ + e\partial_f & + (ae + bd + \tfrac{1}{2}c^2)\,\partial_f & + (ae + bd + \tfrac{1}{2}c^2)\,\partial_f & + e\partial_f \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

$$\begin{array}{rl|ll} = & a\,(a\partial_c + b\partial_d + c\partial_e + d\partial_f + \ldots) - & \tfrac{1}{2}a^2\partial_c & = \tfrac{1}{2}a^2\partial_c \\ & + b\,(\qquad a\partial_d + b\partial_e + c\partial_f + \ldots) & + ab\partial_d & + ab\partial_d \\ & + c\,(\qquad\qquad a\partial_e + b\partial_f + \ldots) & + (ac + \tfrac{1}{2}b^2)\,\partial_e & + (ac + \tfrac{1}{2}b^2)\,\partial_e \\ & + d\,(\qquad\qquad\qquad a\partial_f + \ldots) & + (ad + bc)\,\partial_f & + (ad + bc)\,\partial_f, \\ & \vdots & \vdots & \vdots \end{array}$$

(= 1, 0, 2, 2), as it should be.

For the general proof, writing for a moment

$$C_0 = \mu A_0, \quad C_1 = (\mu + \nu) A_1, \quad C_2 = (\mu + 2\nu) A_2, \ \&c.,$$

and similarly

$$C_0' = \mu' A_0', \quad C_1' = (\mu' + \nu') A_1', \quad C_2' = (\mu' + 2\nu') A_2', \ \&c.,$$

where the accented symbols refer to the values $(\mu', \nu'; m', n')$, we have

$$[(\mu', \nu'; m', n').(\mu, \nu; m, n)],$$

$$\begin{array}{rllll} = & C_0'\partial_{a_{n'}} & * \quad C_0\partial_{a_n} & - C_0\partial_{a_n} & * \quad C_0'\partial_{a_{n'}} \\ & + C_1'\partial_{a_{n'+1}} & + C_1\partial_{a_{n+1}} & + C_1\partial_{a_{n+1}} & + C_1'\partial_{a_{n'+1}} \\ & + C_2'\partial_{a_{n'+2}} & + C_2\partial_{a_{n+2}} & + C_2\partial_{a_{n+2}} & + C_2\partial_{a_{n'+2}} \\ & \vdots & \vdots & \vdots & \vdots \\ & & + C_{n'}\partial_{a_{n+n'}} & & + C_n\partial_{a_{n'+n}} \\ & & + C_{n'+1}\partial_{a_{n+n'+1}} & & + C_{n+1}\partial_{a_{n'+n+1}}; \end{array}$$

or observing that in the series $C_0, C_1, C_2, \ldots$, the first term that contains $a_{n'}$ is $C_{n'}$, and the like as regards the series $C_0', C_1', C_2', \ldots$, this is

$$\begin{aligned} = \ & \{(C_0'\partial_{a_{n'}}) C_{n'} - (C_0\partial_{a_n}) C_n'\} \partial_{a_{n+n'}} \\ + & \{(C_0'\partial_{a_{n'}} + C_1'\partial_{a_{n'+1}}) C_{n'+1} - (C_0\partial_{a_n} + C_1\partial_{a_{n+1}}) C'_{n+1}\} \partial_{a_{n+n'+1}} \\ + & \{(C_0'\partial_{a_{n'}} + C_1'\partial_{a_{n'+1}} + C_2'\partial_{a_{n'+2}}) C_{n'+2} - (C_0\partial_{a_n} + C_1\partial_{a_{n+1}} + C_2\partial_{a_{n+2}}) C'_{n+2}\} \partial_{a_{n+n'+2}} \\ & \vdots \end{aligned}$$

and it is thus of the form

$$C_0''\partial_{a_{n+n'}} + C_1''\partial_{a_{n+n'+1}} + C_2''\partial_{a_{n+n'+2}} + \&c.,$$

i.e. the value of n_1 is $= n + n'$.

I confine myself to the comparison of the first and second coefficients: substituting for the C's their expressions in terms of the A's, we ought to have

$$\mu' A_0'\partial_{a_{n'}} (\mu + n'\nu) A_{n'} - \mu A_0 \partial_{a_n} (\mu' + n\nu') A_n' = \mu_1 A_0'',$$

$$(\mu' A_0'\partial_{a_{n'}} + (\mu' + \nu') A_1'\partial_{a_{n'+1}}) \{\mu + (n'+1)\nu\} A_{n'+1}$$

$$- (\mu A_0\partial_{a_n} + (\mu + \nu) A_1\partial_{a_{n+1}}) \{\mu' + (n+1)\nu'\} A'_{n+1} = (\mu_1 + \nu_1) A_1''.$$

Now assuming $m_1 = m' + m - 1$, and attending to the values

$$\begin{aligned} & A_0 = \frac{1}{m} a_0^m, \quad A_0' = \frac{1}{m'} a_0^{m'}, \quad A_0'' = \frac{1}{m' + m - 1} a_0^{m'+m-1}, \\ & \qquad\qquad\qquad\quad \vdots \qquad\qquad\qquad\quad \vdots \\ & A_1 = a_0^{m-1} a_1, \\ & A_2 = a_0^{m-1} a_2 + \tfrac{1}{2} a_0^{m-2} a_1^2, \\ & \vdots \end{aligned}$$

we see at once that in the first equation the terms contain each of them the factor $a_0^{m'+m-1}$, and in the second equation they contain each of them the factor $a_0^{m'+m-2}a_1$; omitting these factors, we find

$$\frac{\mu'}{m'}(\mu + n'\nu) \qquad - \frac{\mu}{m}(\mu' + n\nu') \qquad = \frac{\mu_1}{m' + m - 1},$$

$$\frac{m'}{\mu'}(\mu + n'\nu + \nu)(m-1) + (\mu' + \nu')(\mu + n'\nu + \nu)$$

$$- \frac{\mu}{m}(\mu' + n\nu' + \nu')(m' - 1) - (\mu + \nu)(\mu' + n\nu' + \nu') = \mu_1 + \nu_1.$$

The first of these gives

$$\mu_1 = (m' + m - 1)\left\{\frac{\mu'}{m'}(\mu + n'\nu) - \frac{\mu}{m}(\mu' + n\nu')\right\},$$

which is the value of μ_1: substituting this in the second equation, we have

$$\nu_1 = \frac{\mu'}{m'}(\mu + n'\nu + \nu)(m-1) - \frac{\mu}{m}(\mu' + n\nu' + \nu')$$

$$+ (\mu' + \nu')(\mu + n'\nu + \nu) \qquad - (\mu + \nu)(\mu' + n\nu' + \nu')$$

$$- (m' + m - 1)\frac{\mu'}{m'}(\mu + n'\nu) - (m' + m - 1)\frac{\mu}{m}(\mu' + n\nu').$$

In the left-hand column, the terms containing $(\mu + n'\nu)$ are

$$(\mu + n'\nu)\left\{\frac{\mu'}{m'}(m-1) + \mu' + \nu' - \frac{\mu'}{m'}(m' + m - 1)\right\},$$

$$= (\mu + n'\nu)\,\nu',$$

and there are besides the terms $\frac{\mu'}{m'}\nu(m-1) + \nu(\mu' + \nu')$, hence the left-hand column is

$$= \nu\nu'(n' + 1) + \mu\nu' + \mu'\nu\left(\frac{m-1}{m'} + 1\right).$$

We have therefore

$$\nu_1 = \nu\nu'(n' + 1) + \mu\nu' + \mu'\nu\left(\frac{m-1}{m'} + 1\right) - \nu\nu'(n + 1) - \mu\nu'\left(\frac{m' - 1}{m} + 1\right) - \mu'\nu,$$

that is,

$$\nu_1 = \nu\nu'(n' - n) + \frac{\mu'}{m'}(m-1)\,\nu - \frac{\mu}{m}(m' - 1)\,\nu'.$$

To complete the proof, it would be of course necessary to compare the remaining terms on the two sides respectively: but in what precedes it is shown that, the form being assumed, the expression for the alternant must of necessity be $(\mu, \nu, m' + m - 1, n' + n)$, where μ, ν have their foregoing values.

Resuming the equation

$$[(\mu', \nu'; m', n')(\mu, \nu; m, n)] = (\mu_1, \nu_1; m_1, n_1),$$

where μ_1, ν_1, m_1, n_1 have the before-mentioned values, then if we herein consider μ, ν, m, n; μ_1, ν_1, m_1, n_1 as given, we can find μ', ν', m', n'.

We have $m_1 = m' + m - 1$, $n_1 = n' + n$, that is, $m' = m_1 - m + 1$, $n' = n_1 - n$, and substituting these values in the expressions of $\frac{\mu_1}{m_1}$ and ν_1, we find

$$\frac{\mu_1}{m_1} = \frac{\mu'}{m_1 - m + 1}\{\mu + (n_1 - n)\nu\} - \frac{\mu}{m}(\mu' + n\nu'),$$

$$\nu_1 = (n_1 - 2n)\nu\nu' + \frac{m-1}{m_1 - m + 1}\nu\mu' - \frac{m_1 - m}{m}\mu\nu',$$

that is,

$$\frac{\mu_1}{m_1} = \left\{\frac{\mu}{m_1 - m + 1} + \frac{\nu(n_1 - n)}{m_1 - m + 1} - \frac{\mu}{m}\right\}\mu' - \frac{\mu n}{m}\nu',$$

$$\nu_1 = \left\{\frac{(m-1)\nu}{m_1 - m + 1}\right\}\mu' + \left\{(n_1 - 2n)\nu - \frac{m_1 - m}{m}\mu\right\}\nu',$$

which are linear equations for the determination of μ' and ν'.