707.

ON THE COLOURING OF MAPS.

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THE theorem that four colours are sufficient for any map, is mentioned somewhere by the late Professor De Morgan, who refers to it as a theorem known to map-makers. To state the theorem in a precise form, let the term "area" be understood to mean a simply or multiply connected* area: and let two areas, if they touch along a line, be said to be "attached" to each other; but if they touch only at a point or points, let them be said to be "appointed" to each other. For instance, if a circular area be divided by radii into sectors, then each sector is attached to the two contiguous sectors, but it is appointed to the several other sectors. The theorem then is, that if an area be partitioned in any manner into areas, these can be, with four colours only, coloured in such wise that in every case two attached areas have distinct colours; appointed areas may have the same colour. Detached areas may in a map represent parts of the same country, but this relation is not in anywise attended to: the colours of such detached areas will be the same, or different, as the theorem may require.

It is easy to see that four colours are wanted; for instance, we have a circle divided into three sectors, the whole circle forming an *enclave* in another area; then we require three colours for the three sectors, and a fourth colour for the surrounding area: if the circle were divided into four sectors, then for these two colours would

* An area is "connected" when every two points of the area can be joined by a continuous line lying wholly within the area; the area within a non-intersecting closed curve, or say an area having a single boundary, is "simply connected"; but if besides the exterior boundary there is one or more than one interior boundary (that is, if there is within the exterior boundary one or more than one *enclave* not belonging to the area), then the area is "multiply connected." The theorem extends to multiply connected areas, but there is no real loss of generality in taking, and we may for convenience take the areas of the theorem to be each of them a simply connected area.

be sufficient, and taking a third colour for the surrounding area, three colours only would be wanted; and so in general according as the number of sectors is even or odd, three colours or four colours are wanted. And in any tolerably simple case it can be seen that four colours are sufficient. But I have not succeeded in obtaining a general proof: and it is worth while to explain wherein the difficulty consists. Supposing a system of n areas coloured according to the theorem with four colours only, if we add an $(n+1)$th area, it by no means follows that we can *without altering the original colouring* colour this with one of the four colours. For instance, if the original colouring be such that the four colours all present themselves in the exterior boundary of the n areas, and if the new area be an area enclosing the n areas, then there is not any one of the four colours available for the new area.

The theorem, if it is true at all, is true under more stringent conditions. For instance, if in any case the figure includes four or more areas meeting in a point (such as the sectors of a circle), then if (introducing a new area) we place at the point a small circular area, cut out from and attaching itself to each of the original sectorial areas, it must according to the theorem be possible with four colours only to colour the new figure; and this implies that it must be possible to colour the original figure so that only three colours (or it may be two) are used for the sectorial areas. And in precisely the same way (the theorem is in fact really the same) it must be possible to colour the original figure in such wise that only three colours (or it may be two) present themselves in the exterior boundary of the figure.

But now suppose that the theorem *under these more stringent conditions* is true for n areas: say that it is possible with four colours only, to colour the n areas in such wise that not more than three colours present themselves in the external boundary: then it might be easy to prove that the $n+1$ areas could be coloured with four colours only: but this would be insufficient for the purpose of a general proof; it would be necessary to show further that the $n+1$ areas could be with the four colours only coloured *in accordance with the foregoing boundary condition;* for without this we cannot from the case of the $n+1$ areas pass to the next case of $n+2$ areas. And so in general, whatever more stringent conditions we import into the theorem as regards the n areas, it is necessary to show not only that the $n+1$ areas can be coloured with four colours only, but that they can be coloured in accordance with the more stringent conditions. As already mentioned, I have failed to obtain a proof.