422.

ON THE GEODESIC LINES ON AN OBLATE SPHEROID.

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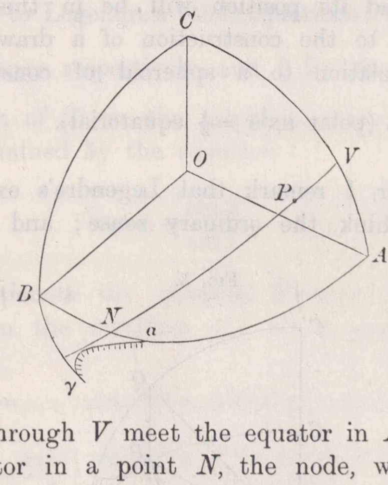
THE theory of the geodesic lines on an oblate spheroid of any excentricity whatever was investigated by Legendre [1]; and the general course of them is well known, viz. each geodesic line undulates between two parallels equidistant from the equator (being thus either a closed curve, or a curve of indefinite length, according to the distance between the two parallels): at a point of contact with the parallel the curve is, of course, at right angles to the meridian; say this is V, a vertex of the geodesic

FIG. 1.

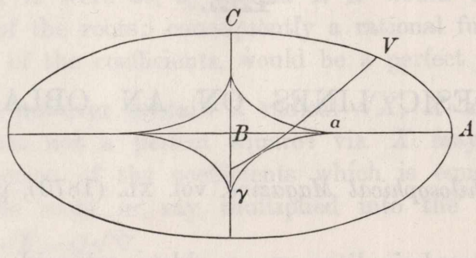


line, and let the meridian through V meet the equator in A; the geodesic line proceeds from V to meet the equator in a point N, the node, where AN is at most $=90°$; and the undulations are obtained by the repetition of this portion VN of the geodesic line alternately on each side of the equator and of the meridian.

[1] *Mém. de l'Inst.* 1806; see also the *Exer. de Calcul Intégral*, t. I. (1811), p. 178, and the *Traité des Fonctions Elliptiques*, t. I. (1825), p. 360.

I consider in the present paper the series of geodesic lines which cut at right angles a given meridian AC, or, say, a series of geodesic normals. It may be remarked that as V passes from the position A on the equator to the pole C, the angular distance AN increases from a certain determinate value (equal, as will appear, to $\frac{C}{A}\,90°$, if C, A are the polar and equatorial axes respectively) up to the value $90°$; and it thus appears that, attending only to their course after they first meet the equator, the geodesic normals have an envelope resembling in its general appearance the evolute of an ellipse (see fig. 1 and also fig. 2), the centre hereof being the point

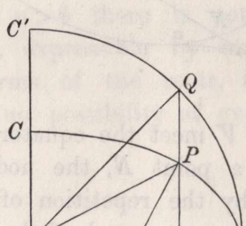
FIG. 2.



B at the distance $BA = 90°$, and the axes coinciding in direction with the equator BA and meridian BC: this is in fact a real geodesic evolute of the meridian CA. The point α is, it is clear, the intersection of the equator by the geodesic line for which V is consecutive to the point A (so that $\angle BOA = \left(1 - \frac{C}{A}\right) 90°$); and the point γ is the intersection of the meridian CB by the geodesic line for which V is consecutive to the point C; and its position will be in this way presently determined. I was anxious, with a view to the construction of a drawing and a model, to obtain some numerical results in relation to a spheroid of considerable excentricity, and I selected that for which $\frac{C}{A} = \frac{1}{2}$ (polar axis $= \frac{1}{2}$ equatorial).

Before proceeding further, I remark that Legendre's expression "reduced latitude" is used in what is not, I think, the ordinary sense; and I propose to substitute the

FIG. 3.



term "parametric latitude": viz., in fig. 3, referring the point P on the ellipse by means of the ordinate MPQ to a point Q on the circle, radius $OK (= OA$, fig. 1), and drawing the normal PT, then we have for the point P the three latitudes,

$$\lambda\ = \angle PTK, \quad \text{normal latitude,}$$
$$\lambda'' = \angle POK, \quad \text{central latitude,}$$
$$\lambda'\ = \angle QOK, \quad \text{parametric latitude;}$$

viz. λ' is the parameter most convenient for the expression of the values of the coordinates x, y ($x = A\cos\lambda'$, $y = C\sin\lambda'$) of a point P on the ellipse. The relations between the three latitudes are

$$\tan\lambda'' = \frac{C}{A}\tan\lambda' = \frac{C^2}{A^2}\tan\lambda,$$

so that λ'', λ', λ are in the order of increasing magnitude. I use in like manner l, l', l'' in regard to the vertex V. The course of a geodesic line is determined by the equation

$$\cos\lambda'\sin\alpha = \text{const.},$$

where λ' is the reduced latitude of any point P on the geodesic line, and α is at this point the azimuth of the geodesic line, or its inclination to the meridian. Hence, if l' be the parametric latitude of the vertex V, the equation is

$$\cos\lambda'\sin\alpha = \cos l'$$

(whence also, when $\lambda' = 0$, $\alpha = 90^\circ - l'$; that is, the geodesic line cuts the equator at an angle $= l'$, the parametric latitude of the vertex). The equation in question, $\cos\lambda'\sin\alpha = \cos l'$, leads at once to Legendre's other equations: viz. taking, as above, A, C for the equatorial and polar semiaxes respectively, and δ for the excentricity, $\delta = \sqrt{1 - \frac{C^2}{A^2}}$; and to determine the position of P on the meridian, using (instead of the parametric latitude λ') the angle ϕ determined by the equation

$$\cos\phi = \frac{\sin\lambda'}{\sin l'},$$

and writing, moreover, s to denote the geodesic distance VP, and Λ to denote the longitude of P measured from the meridian CA which passes through the vertex V, these are

$$ds = \quad d\phi\sqrt{C^2 + A^2\delta^2\sin^2 l'\cos^2\phi},$$

$$d\Lambda = \frac{\cos l'}{A}\,\frac{d\phi\sqrt{C^2 + A^2\delta^2\sin^2 l'\cos^2\phi}}{1 - \sin^2 l'\cos^2\phi};$$

which differential expressions are to be integrated from $\phi = 0$; and the equations then determine λ', s, and Λ, all in terms of the angle ϕ,—that is, virtually s and Λ, the length and longitude, in terms of the parametric latitude λ'.

Writing, with Legendre,

$$c^2 = \frac{A^2\delta^2 \sin^2 l'}{C^2 + A^2\delta^2 \sin^2 l'}, \quad = \quad \delta^2 \sin^2 l,$$

$$b^2 = 1 - c^2, = \frac{C^2}{C^2 + A^2\delta^2 \sin^2 l'}, \quad = 1 - \delta^2 \sin^2 l;$$

also

$$n = \tan^2 l', \quad M = \frac{C}{bA \cos l'} = \frac{C}{A \cos l'},$$

then the formulæ become

$$ds = \frac{C}{b}\, d\phi \sqrt{1 - c^2 \sin^2 \phi},$$

$$d\Lambda = M \frac{d\phi \sqrt{1 - c^2 \sin^2 \phi}}{1 + n \sin^2 \phi}.$$

Hence integrating from $\phi = 0$, and using the notations F, E, Π of elliptic functions, we have

$$s = \frac{C}{b} E(c, \phi),$$

$$\Lambda = \frac{M}{n} \{(n + c^2)\, \Pi(n, c, \phi) - c^2 F(c, \phi)\};$$

viz. these belong to any point P whatever on the geodesic line, parametric latitude of vertex $= l'$; and if we write herein $\phi = 90°$, then they will refer to the node N, or point of intersection with the equator.

The position of the point α is at once obtained by writing $l' = 0$: viz. this gives $c = 0$, $b = 1$, $M = \dfrac{C}{A}$, $n = 0$: the differential expressions are $ds = C d\phi$, $d\Lambda = \dfrac{C}{A} d\phi$. Or integrating from $\phi = 0$ to $\phi = \frac{1}{2}\pi$, we have $s = A \,.\, \dfrac{C}{A} \,.\, \frac{1}{2}\pi$, $\Lambda = \dfrac{C}{A} \,.\, \frac{1}{2}\pi$, agreeing with each other, and giving longitude of $\alpha = \dfrac{C}{A} \,.\, \frac{1}{2}\pi$; or, what is the same thing, $\angle \alpha OB = \frac{1}{2}\pi \left(1 - \dfrac{C}{A}\right)$.

Writing in the formulæ $l' = 90°$, we have $c = \delta$, $b = \dfrac{C}{A}$, $\dfrac{M}{n} = 0$; whence $d\Lambda = 0$, or $\Lambda = \text{const.}, = \frac{1}{2}\pi$, since the geodesic line here coincides with the meridian CB; and moreover $s = AE(\delta, \phi)$; viz. this is merely the expression of the distance from C of a point P on the meridian CB. But we do not thus obtain the position of the point γ.

To find it we must consider a position of V consecutive to C, say, $l' = \frac{1}{2}\pi - \epsilon$, where ϵ is indefinitely small; n is thus indefinitely large, and the integral $\Pi(n, c, \phi)$ is not conveniently dealt with. But it may be replaced by an expression depending on

$\Pi\left(\frac{c^2}{n}, c, \phi\right)$, where $\frac{c^2}{n}$ is indefinitely small; viz. (Legendre, *Fonct. Ellipt.* vol. I. p. 69) we have

$$\Pi(n, c, \phi) = F(c, \phi) + \frac{1}{\sqrt{\alpha}}\tan^{-1}\frac{\sqrt{\alpha}\tan\phi}{\sqrt{1-c^2\sin^2\phi}} - \Pi\left(\frac{c^2}{n}, c, \phi\right),$$

where

$$\alpha = (1+n)\left(1+\frac{c^2}{n}\right).$$

We thus have

$$\Lambda = \frac{M}{n}\left\{nF(c, \phi) + \frac{\sqrt{n}\sqrt{c^2+n}}{\sqrt{1+n}}\tan^{-1}\frac{\sqrt{\alpha}\tan\phi}{\sqrt{1-c^2\sin^2\phi}} - (c^2+n)\,\Pi\left(\frac{c^2}{n}, c, \phi\right)\right\},$$

where, $\frac{c^2}{n}$ being small,

$$\Pi\left(\frac{c^2}{n}, c, \phi\right) = \int\frac{d\phi}{\left(1+\frac{c^2}{n}\sin^2\phi\right)\sqrt{1-c^2\sin^2\phi}},$$

$$= \int\frac{\left(1-\frac{c^2}{n}\sin^2\phi\right)d\phi}{\sqrt{1-c^2\sin^2\phi}}, = \left(1-\frac{1}{n}\right)F(c, \phi) + \frac{1}{n}E(c, \phi).$$

And expanding also the $\tan^{-1}$ term, we thus have

$$\Lambda = \frac{M}{n}\left\{nF(c, \phi) + \frac{\sqrt{n}\sqrt{c^2+n}}{\sqrt{1+n}}\left[\tfrac{1}{2}\pi - \frac{\sqrt{1-c^2\sin^2\phi}}{\tan\phi}\,\frac{\sqrt{n}}{\sqrt{(1+n)(c^2+n)}}\right]\right.$$

$$\left. - (c^2+n)\left[\left(1-\frac{1}{n}\right)F(c, \phi) + \frac{1}{n}E(c, \phi)\right]\right\},$$

$$= \frac{M}{n}\left\{\left(b^2+\frac{c^2}{n}\right)F(c, \phi) - \left(1+\frac{c^2}{n}\right)E(c, \phi) + \frac{\sqrt{n}\sqrt{c^2+n}}{\sqrt{1+n}}\cdot\tfrac{1}{2}\pi - \frac{n}{n+1}\cot\phi\sqrt{1-c^2\sin^2\phi}\right\},$$

which, in the term in { } neglecting negative powers of n, becomes

$$\Lambda = \frac{M}{n}\left\{\sqrt{n}\cdot\tfrac{1}{2}\pi + b^2F(c, \phi) - E(c, \phi) - \cot\phi\sqrt{1-c^2\sin^2\phi}\right\}.$$

We may moreover write $c = \delta$, $b = \frac{C}{A}$, $\phi = 90° - \lambda'$, $n = \frac{1}{\epsilon^2}$, $M = \epsilon$, and therefore $\frac{M}{n} = \epsilon$, so that the formula is

$$\Lambda = \epsilon\left\{\frac{1}{\epsilon}\cdot\tfrac{1}{2}\pi + b^2F(c, 90°-\lambda') - E(c, 90°-\lambda') - \tan\lambda'\sqrt{1-c^2\cos^2\lambda'}\right\},$$

$$= \tfrac{1}{2}\pi - \epsilon\{\tan\lambda'\sqrt{1-c^2\cos^2\lambda'} + E(c, 90°-\lambda') - b^2F(c, 90°-\lambda')\},$$

where I retain c, b as standing for $\sqrt{1-\frac{C^2}{A^2}}$, $\frac{C}{A}$ respectively.

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Writing herein $\lambda' = 0$, we have

$$\Lambda = \tfrac{1}{2}\pi - \epsilon\,(E, c - b^2 F, c),$$

where the coefficient $E, c - b^2 F, c$ is

$$= \int_0^{\frac{1}{2}\pi} d\theta \left(\sqrt{1 - c^2 \sin^2\theta} - \frac{1 - c^2}{\sqrt{1 - c^2 \sin^2\theta}}\right) = c^2 \int_0^{\frac{1}{2}\pi} \frac{\cos^2\theta\, d\theta}{\sqrt{1 - c^2\sin^2\theta}}$$

consequently positive; that is, Λ, the longitude of the node, is less than $90°$, as it should be. Hence in order that Λ may be $= 90°$, we must have λ' negative, say, $\lambda' = -\mu'$, where μ' is positive; and, observing that we may under the signs E, F write $90° - \mu'$ instead of $90° + \mu'$, we thus have

$$\tfrac{1}{2}\pi = \tfrac{1}{2}\pi + \epsilon\,\{\sqrt{1 - c^2\cos^2\mu'}\tan\mu' - E\,(c,\ 90° - \mu') + b^2 F\,(c,\ 90° - \mu')\};$$

that is, we must have

$$\tan\mu' \sqrt{1 - c^2\cos^2\mu'} = E\,(c,\ 90° - \mu') - b^2 F\,(c,\ 90° - \mu');$$

viz. μ' is here the parametric latitude (south) of the intersection of the meridian CB with the consecutive geodesic line—that is, of the point γ. As μ' increases from 0 to $90°$, the left-hand side increases from 0 to ∞; and the right-hand side, beginning from a positive value and either attaining a maximum or not, ultimately decreases to 0; there is consequently a real root, which is easily found by trial.

Thus $\dfrac{C}{A} = \tfrac{1}{2}$, $C = \tfrac{1}{2}\sqrt{3}$ (angle of modulus $= 60°$), $b = \tfrac{1}{2}$; or the equation is

$$\tan\mu' \sqrt{1 - \tfrac{3}{4}\cos^2\mu'} = E\,(90° - \mu') - \tfrac{1}{4}F\,(90° - \mu').$$

Using Legendre's Table IX., we have

μ'.	$90° - \mu'$.	E.	F.	$E - \frac{1}{4}F$.	$\tan\mu' \sqrt{1 - \frac{3}{4}\cos^2\mu'}$.
0°	90°	1·21105	2·15651	·6719	·0
10	80	1·12248	1·81252	·6693	
20	70	1·02663	1·49441	·6530	
30	60	·91839	1·21253	·6153	·3819
40	50	·79538	·96465	·5542	·6278

so that we see the required value is between $30°$ and $40°$; and a rough interpolation gives the value $\mu' = 37°\,40'$. But repeating the calculation with the values $37°$ and $38°$, we have

μ'.	$90° - \mu'$.	E.	F.	$E - \frac{1}{4}F$.	$\tan\mu' \sqrt{1 - \frac{3}{4}\cos^2\mu'}$.
37°	53°	·833879	1·035870	·57419	·54425
38	52	·821197	1·011849	·56823	·57108

whence, interpolating, $\mu' = 37°\,55'$.

The semiaxes of the geodesic evolute, measured according to their longitude and parametric latitude respectively, are thus $B\alpha$, long. of $\alpha = 45°$; $B\gamma$, param. lat. $= 37° \, 55'$. But measuring them according to their geodesic distance, the equatorial radius A being taken $= 1$, we have

$$B\alpha = \tfrac{1}{4}\pi \qquad\qquad = \cdot 78540,$$

$$B\gamma = \left(\frac{C}{b} - 1\right)\{E, - E(52° \, 5')\} = 1\cdot 21106 - \cdot 82225 = \cdot 38881.$$

Reverting to the general formulæ for s, Λ, but writing therein $A = 1$, and therefore $C = \sqrt{1-\delta^2}$; writing also $\phi = 90°$ (that is, making the formulæ to refer to the node N of the geodesic line), we have

$$s = \frac{\sqrt{1-\delta^2}}{\sqrt{1-\delta^2 \sin^2 l}} E, c,$$

$$\Lambda = \frac{\sqrt{1-\delta^2}}{n \cos l} \{(n + c^2) \Pi, (n, \; c) - c^2 F, c\};$$

but for the calculation of the second of these formulæ by means of Legendre's Tables it is necessary to express $\Pi, (n, \; c)$ in terms of the functions E, F.

The proper formula is given in *Fonct. Ellipt.* vol. I. p. 137; viz. this is

$$\frac{\Delta(b, \; \theta)}{\sin\theta \cos\theta} \Pi, (n, \; c) = \tfrac{1}{2}\pi + \frac{\sin\theta}{\cos\theta} \Delta(b, \; \theta) F, c + F, c F(b, \; \theta) - F, c E(b, \; \theta) - E, c F(b, \; \theta),$$

where $\Delta(b, \; \theta) = \sqrt{1 - b^2 \sin^2\theta}$. θ is an angle given by the equation $\cot\theta = \sqrt{n}$; we have $n = \tan^2 l'$; consequently $\theta = 90° - l'$. Substituting this value, except that for shortness I retain $E(b, \; \theta)$, $F(b, \; \theta)$ in place of $E(b, \; 90° - l')$, $F(b, \; 90° - l')$, we have

$$\Delta(b, \; \theta) = \sqrt{1 - b^2 \cos^2 l'},$$

$$= \sqrt{1 - (1 - \delta^2 \sin^2 l) \cos^2 l'}, \; = \sin l;$$

and thence

$$\tan\theta \Delta(b, \; \theta) = \cot l \sin l = \frac{\cos l}{\sqrt{1-\delta^2}};$$

whence

$$\Pi, (n, \; c) = \frac{\sin l' \cos l'}{\sin l} \left\{\tfrac{1}{2}\pi + F, c\left[\frac{\cos l}{\sqrt{1-\delta^2}} + F(b, \; \theta) - E(b, \; \theta)\right] - E, c F(b, \; \theta)\right\}.$$

But

$$n + c^2 = \tan^2 l' + \delta^2 \sin^2 l = \sin^2 l \sec^2 l.$$

Hence

$$(n + c^2) \Pi, (n, \; c) - c^2 F, c = \sin^2 l \{\sec^2 l' \Pi, (n, \; c) - \delta^2 F, c\};$$

and multiplying this by

$$\frac{\sqrt{1-\delta^2}}{n\cos l},=\frac{\sqrt{1-\delta^2}\cos l}{\tan^2 l'\cos^2 l}$$

the exterior factor is

$$\frac{\sqrt{1-\delta^2}\cos l\tan^2 l}{\tan^2 l'},=\frac{\cos l}{\sqrt{1-\delta^2}},$$

and we have

$$\Lambda=\frac{\cos l}{\sqrt{1-\delta^2}}\{\sec^2 l'\Pi_,(n,\ c)-\delta^2 F_, c\},$$

which is the formula I used in the calculations. It would, however, have been better to reduce a step further; viz. we have

$$\sec^2 l'\Pi_,(n,\ c)=\frac{\tan l'}{\tan l\cos l}\{\ \},$$

$$=\frac{\sqrt{1-\delta^2}}{\cos l}\left\{\tfrac{1}{2}\pi+F_,c\left[\frac{\cos l}{\sqrt{1-\delta^2}}+F(b,\ \theta)-E(b,\ \theta)\right]-E_,c\,F(b,\ \theta)\right\},$$

$$=\frac{\sqrt{1-\delta^2}}{\cos l}\{\tfrac{1}{2}\pi+F_,c\,[F(b,\ \theta)-E(b,\ \theta)]-E_,c\,F(b,\ \theta)\}+F_,c,$$

and thence

$$\sec^2 l'\Pi_,(n,\ c)-\delta^2 F_,c=\frac{\sqrt{1-\delta^2}}{\cos l}\{\tfrac{1}{2}\pi+F_,c\,[\sqrt{1-\delta^2}\cos l+F(b,\ \theta)-E(b,\ \theta)]-E_,cF(b,\ \theta)\};$$

or, finally,

$$\Lambda=\tfrac{1}{2}\pi+F_,c\,F(b,\ \theta)-F_,c\,E(b,\ \theta)-E_,cF(b,\ \theta)+\sqrt{1-\delta^2}\cos l F_,c.$$

It is easy with this expression of Λ to obtain the results already found for the extreme values $l'=0°$, $l'=90°$.

As Legendre's Tables have for argument, not the modulus c, but the angle of the modulus, say χ (that is, $\sin\chi=c=\delta\sin l$), it is convenient to replace $\sqrt{1-\delta^2\sin^2 l}$ by its value $\cos\chi$; and the formulæ thus are

$$s=\frac{\sqrt{1-\delta^2}}{\cos\chi}E_,c,$$

$$\Lambda=\tfrac{1}{2}\pi+F_,c\,[\sqrt{1-\delta^2}\cos l+F(b,\ \theta)-E(b,\ \theta)]-E_,c\,F(b,\ \theta),$$

where

$$C=\sin\chi=\delta\sin l,\quad \tan l'=\sqrt{1-\delta^2}\tan l,\quad \theta=90°-l';$$

and in the case intended to be numerically discussed, $\delta=\tfrac{1}{2}\sqrt{3}$, $\sqrt{1-\delta^2}=\tfrac{1}{2}$. I take l' as the argument, giving it the values $0°$, $10°$, ... $90°$, and perform the calculation as shown in the Table.

l'.	θ.	l.	$\chi = \angle c$.	$90° - \chi = \angle b$.	$\frac{1}{2}\cos l$.	F,c, log.	E,c, log.	F,c, nat.	E,c, nat.	$F(b, \theta)$, nat.	$E(b, \theta)$, nat.
0°											
10	80°	19° 26′	16°·75	73°·25	·47151	20548	18686	1·6050	1·5376	2·08962	1·03762
20	70	36 3	30·63	59·37	·40425	22814	16532	1·6910	1·4633	1·48840	1·02962
30	60	49 6	40·90	49·10	·32737	25478	14167	1·7980	1·3857	1·16024	·95214
40	50	59 13	48·07	41·93	·25589	27626	12163	1·8891	1·3232	·92141	·82827
50	40	67 14	52·98	37·02	·19349	29935	10645	1·9923	1·2777	·71820	·67903
60	30	73 54	56·32	33·68	·13866	31479	09557	2·0644	1·2461	·53083	·51655
70	20	79 41	58·43	31·57	·08954	32540	08850	2·1154	1·2260	·35099	·34716
80	10	84 58	59·62	30·38	·04387	33169	08446	2·1463	1·2147	·17475	·17431
90			60·0				08316				

l'.	$\frac{1}{2}\cos l + F(b, \theta) - E(b, \theta)$, nat.	Do. log.	Add log F,c.	(1) Nat.	Log $F(b, \theta)$.	Add log E,c.	(2) Nat.	$\frac{1}{2}\pi + (1) - (2)$.	Do. in (°). *	Log E,c $-\log\cos\chi$.	Nat.	$\frac{1}{2}$ do., = length. *
0°								·7854	45°			·7854
10	1·52308	·18272	·38820	2·4446	·32007	·50693	3·2132	·8022	45 58′	20569	1·6058	·8029
20	·86318	$\bar{1}$·93610	·16424	1·4596	·17272	·33804	2·1779	·8525	48 51	23075	1·7012	·8506
30	·53547	$\bar{1}$·72874	$\bar{1}$·98352	·9627	·06455	·20622	1·6078	·9257	53 2	26323	1·8333	·9166
40	·34903	$\bar{1}$·54286	$\bar{1}$·81912	·6594	$\bar{1}$·96445	·08608	1·2192	1·0110	57 56	29668	1·9801	·9900
50	·23266	$\bar{1}$·36672	$\bar{1}$·66607	·4635	$\bar{1}$·85625	$\bar{1}$·96270	·9177	1·1166	63 59	32682	2·1224	1·0612
60	·15292	$\bar{1}$·18446	$\bar{1}$·49925	·3157	$\bar{1}$·72496	$\bar{1}$·82053	·6615	1·2250	70 11	35178	2·2479	1·1239
70	·09327	$\bar{2}$·96974	$\bar{1}$·29514	·1973	$\bar{1}$·54529	$\bar{1}$·63379	·4303	1·3378	76 39	36939	2·3410	1·1705
80	·04431	$\bar{2}$·64650	$\bar{2}$·97819	·0951	$\bar{1}$·24242	$\bar{1}$·32688	·2123	1·4536	83 17	38454	2·4240	1·2120
90								1·5708	90			1·2111

χ, $90° - \chi$ in degrees and decimals of a degree, to correspond with Legendre's Tables.

where the columns marked with an * show respectively the longitude of the node, and the length (or distance of node from vertex), for the geodesic lines belonging to the different values of the argument l'.

The remarks which follow have reference to the stereographic projection of the figure on the plane of the equator, the centre of projection being the pole (say the South Pole) of the spheroid. It is to be remarked that if a point P of the spheroid is projected as above, by means of an ordinate into the point Q of the sphere radius $OK\,(=OA)$, then projecting stereographically as to the spheroid and the sphere from the south poles thereof respectively, the points P and Q have the same projection. And it is hence easy to show that an azimuth α at a point of the meridian $\left(\text{parametric latitude } \lambda', \text{ normal latitude } \lambda, \text{ and therefore } \tan\lambda' = \frac{C}{A}\tan\lambda\right)$ is projected into an angle (α) such that

$$\tan(\alpha) = \frac{\sin\lambda'}{\sin\lambda}\tan\alpha.$$

In fact in fig. 3, if we take therein OK, OC for the axes of x, z respectively, and the axis of y at right angles to the plane of the paper, and if we have at P on the surface of the spheroid an element of length PR at the inclination α to the meridian PK, then if x, y, z are the coordinates of P, and $x+\delta x$, $y+\delta y$, $z+\delta z$ those of R, we have

$$\delta x = \rho\cos\alpha\sin\lambda,$$
$$\delta z = -\rho\cos\alpha\cos\lambda,$$
$$\delta y = \rho\sin\alpha,$$

and thence

$$\tan\alpha = \frac{\delta y}{\sqrt{\delta x^2+\delta z^2}}.$$

Now, if the meridian and the points P, R are referred by lines parallel to Oz to the surface of the sphere radius OA, the only difference is that the ordinates z are increased in the ratio $C : A$; so that if the projected angle be (α), we have

$$\tan(\alpha) = \frac{\delta y}{\sqrt{\delta x^2 + \frac{A^2}{C^2}\delta z^2}};$$

and then projecting the sphere stereographically from its south pole, the angle in the projection is $=(\alpha)$. And according to the foregoing remark, the angle (α) thus obtained is also the projection of α from the south pole of the spheroid. We have thus

$$\frac{\tan(\alpha)}{\tan\alpha} = \frac{\sqrt{\delta x^2+\delta z^2}}{\sqrt{\delta x^2+\frac{A^2}{C^2}\delta z^2}}, \quad = \frac{\sqrt{\sin^2\lambda+\cos^2\lambda}}{\sqrt{\sin^2\lambda+\frac{A^2}{C^2}\cos^2\lambda}}, \quad = \sqrt{\frac{1+\cot^2\lambda}{1+\cot^2\lambda'}}, \quad = \frac{\sin\lambda'}{\sin\lambda},$$

which is the required relation.

The foregoing equations,

$$\cos\lambda'\sin\alpha = \cos l', \quad \tan\lambda' = \frac{C}{A}\tan\lambda,$$

$$\tan(\alpha) = \frac{\sin\lambda'}{\sin\lambda}\tan\alpha,$$

determine in the stereographic projection the inclination (α) to the radius, or projection of the meridian, of the geodesic line (parametric latitude of vertex $= l'$) at the point the parametric latitude of which is $= \lambda'$; viz. they enable the construction (in the projection) of the direction of the successive elements of the geodesic line. There would be no difficulty in performing the construction geometrically; but it would, I think, be more convenient to calculate (α) numerically for a given value of l' and for the successive values of λ'. Observe that for $\lambda' = 0$ we have (as above) $90° - \alpha = l'$, and then $\dfrac{\sin\lambda'}{\sin\lambda} = \dfrac{\tan\lambda'}{\tan\lambda} = \dfrac{C}{A}$, consequently $\tan(\alpha) = \dfrac{C}{A}\cot l'$: but we have also $\cot l' = \dfrac{A}{C}\cot l$, so that this equation becomes $\tan(\alpha) = \cot l$, or we have $90° - (\alpha) = l$; viz. in the projection, the geodesic line cuts the equator at an angle $l =$ the normal latitude of the vertex of the geodesic line.

The preceding formulæ and results have enabled me to construct a drawing, on a large scale, of the stereographic projection of the geodesic lines for the spheroid, polar axis $= \frac{1}{2}$ equatorial axis.