423.

ON THE PLANE REPRESENTATION OF A SOLID FIGURE.

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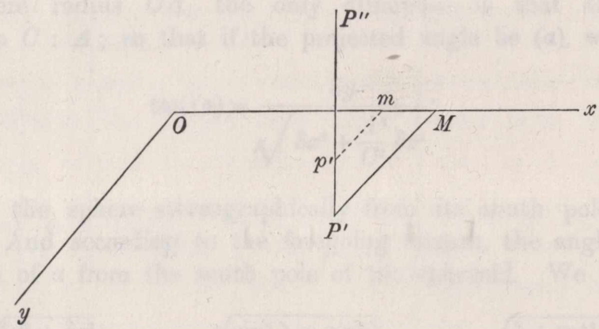
WE represent *in plano* the position of a point P whose coordinates in space are (x, y, z) by drawing these coordinates on the same scale or on different scales, and in given directions from a fixed origin in the plane; $OM = x$, $MP' = y$, $P'P'' = z$. But observe that the point P'' *alone* does not completely represent the point P; in fact P'' represents a whole series of points lying in a line; any one such point is the

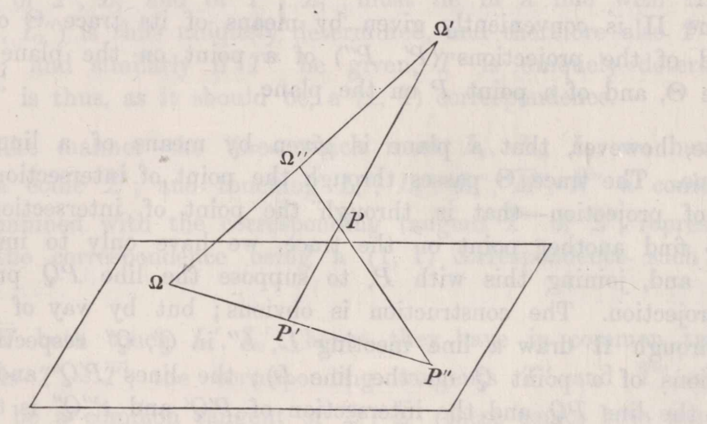


point whose coordinates are Om, mp', $p'P''$. For the complete representation of P we require the *two points* P', P'': these might be distinguished as the projection P'', and the foot-point P'. The two points P', P'' are obviously such that the line joining them is in a given direction.

The preceding is, of course, the ordinary method of orthogonal projection, or geometrical delineation of a solid figure: it may be used under various forms; for example, the coordinates x, y, z may be taken on the same scale and in directions inclined to each other at angles of 120° (isometrical projection); or the coordinates x, y may be drawn on the same scale and at their actual inclination, 90°, to each other;

and the coordinate z on the same or an altered scale in any given direction; the points P' then give a true ground-plan of the solid figure, and the lengths of the lines $P'P''$ give the altitudes of the several points P: this is also a method in ordinary use.

But it is to be observed that the points P', P'' are both of them *projections*, and that the general theory is as follows: we represent the position of the point P



by means of its projections P', P'', from two fixed points Ω', Ω'' respectively; the line joining these points passes, it is clear, through a fixed point Ω which is the intersection of the plane of projection by the line which joins the two points Ω', Ω''.

Hence we say that a point P in space is represented *in plano* by any two points P', P'' which are such that the line joining them passes through a fixed point Ω. And we have thus a *system of constructive geometry* which is the more simple on account of the generality of its basis, and which is at once applicable to any of the special projections above referred to. I establish the fundamental notions of such a geometry, and by way of illustration apply it to the solution of the well-known problem of finding the lines which meet four given lines in space.

A point P (as already mentioned) is given by its projections P', P'', which are points such that the line joining them passes through the fixed point Ω.

A line L is given by its projections L', L'', which are any two lines in the plane. We speak of the point (P', P''), meaning the point P whose projections are P', P''; and similarly of the line (L', L''), meaning the line whose projections are L', L''.

If P', P'' coincide, then the point P is in the plane of projection; and so if L', L'' coincide, then the line L is in the plane of projection.

If through Ω we draw a line meeting L', L'' in the points P', P'' respectively, these are the projections of a point P on the line L. In particular the intersection of L', L'' (considered as two coincident points) represents the intersection of the line L with the plane of projection.

The line through the points (P', P'') and (Q', Q'') has for its projections the lines $P'Q'$ and $P''Q''$.

Two lines (L', L'') and (M', M'') intersect each other if only the intersections $L'M'$ and $L''M''$ are the projections of a point P—that is, if the line through the points $L'M'$ and $L''M''$ passes through Ω. And then clearly P is the intersection of the two lines.

A plane Π is conveniently given by means of its trace Θ on the plane of projection, and of the projections (P', P'') of a point on the plane; or, say, by means of the trace Θ, and of a point P on the plane.

Suppose, however, that a plane is given by means of a line L and a point P on the plane. The trace Θ passes through the point of intersection of the line L with the plane of projection—that is, through the point of intersection of the projections L', L''. To find another point on the trace, we have only to imagine on the line L a point Q, and, joining this with P, to suppose the line PQ produced to meet the plane of projection. The construction is obvious; but by way of illustration I give it in full. Through Ω draw a line meeting L', L'' in Q', Q'' respectively (then these are the projections of a point Q on the line L); the lines $P'Q'$ and $P''Q''$ are the projections of the line PQ, and the intersection of $P'Q'$ and $P''Q''$ is therefore the required point on the trace Θ.

The line of intersection of two planes passes through the point of intersection of their traces Θ_1, Θ_2; whence, if the planes have in common a point P, the line of intersection is the line joining P with the intersection of the traces Θ_1, Θ_2.

In what precedes we have the solution of the following problem:—"Given a point P, and two lines L_1, L_2, to find a line through P meeting the two lines L_1, L_2." The required line is in fact the line of intersection of the planes (P, L_1) and (P, L_2); we have seen how to construct the traces Θ_1 and Θ_2 of these planes respectively; and the required line is the line joining P with the intersection of Θ_1 and Θ_2.

I proceed now to the problem to find the two lines, each of them meeting four given lines, L_1, L_2, L_3, L_4 (these being, of course, given by means of their projections (L_1', L_1'') &c.). The question is in effect to find on the line L_1 a point P such that, drawing from it a line to meet L_2, L_3, and also a line to meet L_2, L_4, these shall be one and the same line.

Now, considering in the first instance P as an arbitrary point on the line L_1, the line from P to meet L_2, L_3 is any line whatever meeting the lines L_1, L_2, L_3: say it is a generating line of the hyperboloid whose directrices are L_1, L_2, L_3, or of the hyperboloid $L_1L_2L_3$. Hence projecting from any point Ω' whatever, the generating lines and directrices are projected into tangents of one and the same conic. We know the projections L_1', L_2', L_3' of the directrices; to find two other tangents of the conic, we take two arbitrary positions of P on the line L_1, and construct as above the projections M', N' of the lines from these to meet the lines L_2, L_3. The conic is then given as the conic touching the five lines L_1', L_2', L_3', M', N': say this is the conic Σ'. Similarly, instead of Ω', considering the point Ω'', we have the lines L_1'', L_2'', L_3'' and the lines M'', N'', which are the other projections of the lines through the two positions of P; and touching these five lines we have a conic Σ''. Each tangent T' of Σ', combined with the *corresponding* tangent T'' of Σ'', represents a line T meeting L_1, L_2, L_3; to establish the correspondence, observe that, inasmuch as the line T meets L_1, the intersections of T', L_1' and of T'', L_1'' must lie in a line with Ω; if T' be given, the point (T'', L_1'') is thus uniquely determined, and therefore also T'' (since L_1'' is a tangent of Σ''); and similarly if T'' be given, T' is uniquely determined; the correspondence T', T'' is thus, as it should be, a (1, 1) correspondence.

Considering in like manner the lines which meet L_1, L_2, L_4, we have touching L_1', L_2', L_4', $\overline{M}'$, $\overline{N}'$ a conic $\overline{\Sigma}'$; and touching L_1'', L_2'', L_4'', $\overline{M}''$, $\overline{N}''$ a conic $\overline{\Sigma}''$; each tangent T' of $\overline{\Sigma}'$, combined with the corresponding tangent $\overline{T}''$ of $\overline{\Sigma}''$, represents a line meeting L_1, L_2, L_4, the correspondence being a (1, 1) correspondence such as in the former case.

The conics Σ', $\overline{\Sigma}'$ both touch L_1', L_2'; hence they have in common two tangents. Say one of these is $T' = \overline{T}'$, the corresponding tangents T'' and $\overline{T}''$ will coincide with each other and be a common tangent of Σ'', $\overline{\Sigma}''$ (these conics both touch L_1'', L_2'', and have thus in common two tangents). We have thus $T' = \overline{T}'$, and $T'' = \overline{T}''$, as the projections of a line meeting L_1, L_2, L_3, L_4; and taking the other common tangents of Σ', $\overline{\Sigma}'$ and of Σ'', $\overline{\Sigma}''$, we have the projections of the other line meeting L_1, L_2, L_3, L_4.

The whole process is:—Construct M', M'' and N', N'' each of them the projections of a line through a point P of L_1, which meets L_2, L_3; and $\overline{M}'$, $\overline{M}''$ and N', $\overline{N}''$ each of them the projections of a line through a point P of L_1, which meets L_2, L_4; we have then the conics

Σ', Σ'' touching L_1', L_2', L_3', M', N', and L_1'', L_2'', L_3'', M'', N'' respectively,

$\overline{\Sigma}'$, $\overline{\Sigma}''$ „ L_1', L_2', L_4', $\overline{M}'$, $\overline{N}'$, „ L_1'', L_2'', L_4'', $\overline{M}''$, $\overline{N}''$ „ ;

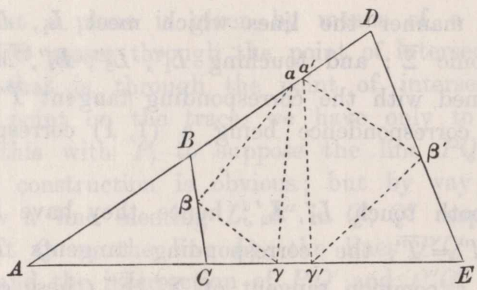
and then the projections of each of the required lines are $T' = \overline{T}'$, a common tangent of Σ', $\overline{\Sigma}'$, and $T'' = \overline{T}''$, the corresponding common tangent of Σ'', $\overline{\Sigma}''$.

It is material to remark how the construction is simplified when there is given one of the lines, say, M, which meets L_1, L_2, L_3, L_4. Here M is a common directrix of the two hyperboloids; we may for the hyperbolas Σ' and Σ'' consider, instead of L_1, L_2, L_3 and two new generating lines, the lines L_1, L_2, L_3, M, and a single new generating line N; and similarly for the hyperbolas $\overline{\Sigma}'$, $\overline{\Sigma}''$ the lines L_1, L_2, L_4, M and a single new generating line $\overline{N}$. Σ', $\overline{\Sigma}'$ have thus in common the three tangents L_1', L_2', M', and therefore only a single other common tangent, $T' = \overline{T}'$; and similarly Σ'', $\overline{\Sigma}''$ have in common the three tangents L_1'', L_2'', M'', and therefore only a single other common tangent, $T'' = \overline{T}''$; and we have thus the other line cutting the four given lines.

I take the opportunity of mentioning the following theorem:

"If in a given triangle we inscribe a variable triangle of given form, the envelope of each side of the variable triangle is a conic touching the two sides (of the given triangle) which contain the extremities of the variable side in question."

We have thence a solution of the problem (*Principia*, Book I. Sect. V. Lemma XXVII.), in a given quadrilateral to inscribe a quadrangle of given form. The question in effect is: in the triangle ABC to inscribe a triangle $\alpha\beta\gamma$ of given form; and in the triangle ADE a triangle $\alpha'\beta'\gamma'$ of given form, in such wise that the sides $\alpha\gamma$, $\alpha'\gamma'$



may be coincident. The envelope of $\alpha\gamma$ is a conic touching AD, AE, and the envelope of $\alpha'\gamma'$ a conic also touching AD, AE: there are thus two other common tangents, either of which may be taken for the position of the side $\alpha\gamma = \alpha'\gamma'$; and the problem admits accordingly of two solutions.