427.

ON GAUSS'S PENTAGRAMMA MIRIFICUM.

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TAKE on a sphere (in the northern hemisphere) two points, A, B, whose longitudes differ by 90°, and refer them to the equator by the meridians AE and BC respectively; join A, B by an arc of great circle, and take in the southern hemisphere the pole D of this circle; and join D with E and C respectively by arcs of great circle. We have a spherical pentagon $ABCDE$, which is in fact the "Pentagramma mirificum," considered by Gauss, as appearing vol. III. pp. 481—490 of the *Collected Works*. Among its properties we have

$$\left.\begin{array}{l}\text{the distance of any two non-adjacent summits}\\ \text{the inclination of any two non-adjacent sides}\end{array}\right\} = 90^\circ;$$

so that each summit is the pole of the opposite side, or the pentagon is its own reciprocal.

Each angle is the supplement of the opposite side.

If the squared tangents of the sides (or angles) taken in order are α, β, γ, δ, ϵ, then

$$1+\alpha=\gamma\delta,\quad 1+\beta=\delta\epsilon,\quad 1+\gamma=\epsilon\alpha,\quad 1+\delta=\alpha\beta,\quad 1+\epsilon=\beta\gamma,$$

equivalent to three independent equations, so that any three of the quantities may be expressed in terms of the remaining two. (This agrees with the foregoing construction, where the arbitrary quantities are the latitudes of A, B respectively.)

Projecting from the centre of the sphere upon any plane, we have a plane pentagon which is such that the perpendiculars let fall from the summits upon the opposite sides respectively meet in a point. This (as easily seen) implies that the two portions into which each perpendicular is divided by the point in question have the same product.

Conversely, starting from the plane pentagon, and erecting from the point of intersection a perpendicular to the plane, the length of this perpendicular being equal to the square root of the product in question, we have the centre of a sphere such that the projection upon it of the plane polygon is the pentagramma mirificum.

I remark as to the analytical theory, that, taking the origin at the intersection of the perpendiculars, and for the coordinates of the summits $(\alpha_1, \beta_1), \ldots (\alpha_5, \beta_5)$ respectively, then we have

$$\alpha_1\alpha_4 + \beta_1\beta_4 = \alpha_2\alpha_5 + \beta_2\beta_5 = \alpha_3\alpha_1 + \beta_3\beta_1 = \alpha_4\alpha_2 + \beta_4\beta_2 = \alpha_5\alpha_3 + \beta_5\beta_3, \quad = -\gamma^2,$$

where γ^2 is the above-mentioned product, or γ is the radius of the sphere.

Cambridge, September 14, 1871.