435.

ON THE SIX COORDINATES OF A LINE.

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THE notion of the six coordinates of a line was, so far as I am aware, first established in my paper "On a new analytical representation of Curves in Space," *Quart. Math. Jour.* t. III. (1860), pp. 225—236, [284]; see p. 226, where writing p, q, r, s, t, u for the six determinants of the matrix $\begin{Bmatrix} x, & y, & z, & w \\ \alpha, & \beta, & \gamma, & \delta \end{Bmatrix}$, I remark that these values give identically $ps+qt+ru=0$; and I consider a cone as represented by a homogeneous equation $V=0$ between the six coordinates (p, q, r, s, t, u); and many of the investigations of the present memoir, in which these coordinates are employed, have been in my possession for some years past. But these coordinates presented themselves independently to Prof. Plücker, and the theory of them is set forth in his most interesting and valuable memoir, "On a new Geometry of Space," *Phil. Trans.* t. CLV. (1865), pp. 725—791; the course of development there given to the theory is however altogether different from that in the present memoir. They have also more recently been made use of in a paper by Herr Lüroth, "Zur Theorie der windschiefen Flächen," *Crelle*, t. LXII. (1867), pp. 130—152.

I have in the present memoir applied these coordinates to the question of the Involution of six lines; the notion of this relation of six lines is due to Prof. Sylvester, to whom it presented itself in the year 1861, in connexion with a theorem in the *Lehrbuch der Statik*, by Möbius (Leipzig, 1837), that if four forces acting on a solid body are in equilibrium the lines along which the forces act are the generating lines of a hyperboloid. Prof. Sylvester was thereby led to consider six lines such that (regarding them as lines in a solid body) there exist along them forces which are in equilibrium; and he thence obtained, by the statical considerations reproduced in the present memoir, the construction (when five of the lines are given) of a sixth line to pass through a given point or to be situate in a given plane.

Article, Nos. 1 to 8. *The Six Coordinates of a Line; definition and general notions.*

1. Using any quadriplanar coordinates (x, y, z, w) whatever, consider a line; on the line two points the coordinates of which are $(\alpha, \beta, \gamma, \delta)$ and $(\alpha', \beta', \gamma', \delta')$ respectively; and through the line two planes, the equations whereof are $(A, B, C, D \between x, y, z, w) = 0$, and $(A', B', C', D' \between x, y, z, w) = 0$ respectively; we have

$$\begin{aligned}
&(A\ , B\ , C\ , D\ \between \alpha\ , \beta\ , \gamma\ , \delta\ ) = 0,\\
&(A\ , B\ , C\ , D\ \between \alpha', \beta', \gamma', \delta') = 0,\\
&(A', B', C', D' \between \alpha\ , \beta\ , \gamma\ , \delta\ ) = 0,\\
&(A', B', C', D' \between \alpha', \beta', \gamma', \delta') = 0.
\end{aligned}$$

2. From the first and second equations, eliminating successively A, B, C, D, we find

$$\left|\begin{array}{cccc}
0\ , & \alpha\beta' - \alpha'\beta\ , & -(\gamma\alpha' - \gamma'\alpha), & \alpha\delta' - \alpha'\delta \\
-(\alpha\beta' - \alpha'\beta), & 0\ , & \beta\gamma' - \beta'\gamma\ , & \beta\delta' - \beta'\delta \\
\gamma\alpha' - \gamma'\alpha\ , & -(\beta\gamma' - \beta'\gamma), & 0 & \gamma\delta' - \gamma'\delta \\
-(\alpha\delta' - \alpha'\delta), & -(\beta\delta' - \beta'\delta), & -(\gamma\delta' - \gamma'\delta), & 0
\end{array}\right| (A, B, C, D) = 0,$$

and from the third and fourth equations we find the like system with (A', B', C', D') in place of (A, B, C, D). Comparing the corresponding equations of the two systems, we find an equality of ratios, as will presently be mentioned.

3. From the first and third equations, eliminating successively $\alpha, \beta, \gamma, \delta$, we find

$$\left|\begin{array}{cccc}
0\ , & AB' - A'B, & -(CA' - C'A), & AD' - A'D \\
-(AB' - A'B), & 0\ , & BC' - B'C\ , & BD' - B'D \\
CA' - C'A\ , & -(BC' - B'C), & 0\ , & CD' - C'D \\
-(AD' - A'D), & -(BD' - B'D), & -(CD' - C'D), & 0
\end{array}\right| (\alpha, \beta, \gamma, \delta) = 0,$$

and from the second and fourth equations we find the like system with $(\alpha', \beta', \gamma', \delta')$ in place of $(\alpha, \beta, \gamma, \delta)$: comparing the corresponding equations of the two systems, we find the same equality of ratios as before, viz.

4. This is

$$\begin{aligned}
&\beta\gamma' - \beta'\gamma : \gamma\alpha' - \gamma'\alpha : \alpha\beta' - \alpha'\beta : \alpha\delta' - \alpha'\delta : \beta\delta' - \beta'\delta : \gamma\delta' - \gamma'\delta \\
&= AD' - A'D : BD' - B'D : CD' - C'D : BC' - B'C : CA' - C'A : AB' - A'B,
\end{aligned}$$

and putting each of these two equal sets of ratios

$$= a \quad : b \quad : c \quad : f \quad : g \quad : h,$$

then the quantities (a, b, c, f, g, h), which it is easy to see satisfy the condition

$$af + bg + ch = 0,$$

are said to be the 'six coordinates' of the line: as only the ratios of the six quantities are material, and as the last-mentioned equation establishes a single relation between these ratios, the system of the six coordinates contain four arbitrary ratios or parameters, for the determination of the particular line.

5. A line is thus determined by its six coordinates (a, b, c, f, g, h), which are such that $af+bg+ch=0$; and conversely any six quantities (a, b, c, f, g, h) satisfying this relation may be taken to be the six coordinates of a line.

6. It is proper to show that the ratios $a : b : c : f : g : h$ are independent of the particular two points on the line, or two planes through the line, used for their determination. In fact, if instead of the points

$$\begin{array}{cccc} \alpha, & \beta, & \gamma, & \delta, \\ \alpha', & \beta', & \gamma', & \delta', \end{array}$$

we have any other two points on the line, say the points

$$\begin{array}{cccc} \lambda\alpha+\mu\alpha', & \lambda\beta+\mu\beta', & \lambda\gamma+\mu\gamma', & \lambda\delta+\mu\delta', \\ \nu\alpha+\rho\alpha', & \nu\beta+\rho\beta', & \nu\gamma+\rho\gamma', & \nu\delta+\rho\delta', \end{array}$$

then the six determinants have their original values each multiplied by $\lambda\rho-\mu\nu$; and the ratios are unaltered.

And the like is the case, if instead of the planes

$$\begin{array}{cccc} A, & B, & C, & D, \\ A', & B', & C', & D', \end{array}$$

we have any other two planes through the line, say the planes

$$\begin{array}{cccc} \lambda A+\mu A', & \lambda B+\mu B', & \lambda C+\mu C', & \lambda D+\mu D', \\ \nu A+\rho A', & \nu B+\rho B', & \nu C+\rho C', & \nu D+\rho D', \end{array}$$

the determinants have their original values each multiplied by $\lambda\rho-\mu\nu$; and the ratios are unaltered.

7. It may be remarked, that the theory of the six coordinates considered as derived from the two points $(\alpha, \beta, \gamma, \delta)$, $(\alpha', \beta', \gamma', \delta')$, and as derived from the two planes (A, B, C, D), (A', B', C', D'), is precisely the same in each case; and we may confine ourselves to the first point of view, regarding therefore the six coordinates as derived from the two points $(\alpha, \beta, \gamma, \delta)$, $(\alpha', \beta', \gamma', \delta')$. I further remark, that I do not at present in anywise fix the absolute magnitudes of the coordinates $(a, b\ c, f, g, h)$: it is only the ratios that we are concerned with.

8. The values of the ratios $a : b : c : f : g : h$ of the six coordinates do however depend on the particular coordinate planes $x=0$, $y=0$, $z=0$, $w=0$, made use of for their determination; and in the sequel it will be necessary to investigate the

formulæ of transformation to a new set of coordinate planes $x_0=0$, $y_0=0$, $z_0=0$, $w_0=0$. And I shall also show in what manner the absolute magnitudes of the coordinates may be fixed. But deferring the consideration of these questions, I consider the planes $x=0$, $y=0$, $z=0$, $w=0$ as given planes, and take the six coordinates (a, b, c, f, g, h) of a line to be determined as above in reference to these given planes, the absolute values of these coordinates remaining indeterminate, and their ratios only being attended to. And I proceed to consider the various questions which present themselves in the geometry of the line, considered as thus determined by means of its six coordinates (a, b, c, f, g, h).

Article, Nos. 9 to 18. (*Various Sub-headings.*) *Elementary Theorems.*

Condition that a line may be in a given plane.

9. Taking the line to be (a, b, c, f, g, h), the equation of the given plane to be

$$(A, B, C, D\between x, y, z, w)=0;$$

then if $(\alpha, \beta, \gamma, \delta)$, $(\alpha', \beta', \gamma', \delta')$ are the coordinates of any two points on the line, we have the system of equations *ante*, No. 2, and substituting therein for $\beta\gamma'-\beta'\gamma$, &c. the values (a, b, c, f, g, h), we find

$$\left|\begin{array}{cccc} 0, & c, & -b, & f \\ -c, & 0, & a, & g \\ b, & -a, & 0, & h \\ -f, & -g, & -h, & 0 \end{array}\right| (A, B, C, D)=0;$$

which equations, equivalent to a twofold relation, are the required condition. It may be remarked that, treating (A, B, C, D) as current plane coordinates, each equation of the system is that of a point lying in the line.

Condition that a line may pass through a given point.

10. The coordinates of the given point are taken to be $(\alpha, \beta, \gamma, \delta)$. If

$$(A, B, C, D\between x, y, z, w)=0, \quad (A', B', C', D'\between x, y, z, w)=0,$$

are the equations of any two planes through the line, then we have the system of equations *ante* No. 3, and substituting therein for $AB'-A'B$, &c. their values in terms of the coordinates (a, b, c, f, g, h) of the line, we have

$$\left|\begin{array}{cccc} 0, & h, & -g, & a \\ -h, & 0, & f, & b \\ g, & -f, & 0, & c \\ -a, & -b, & -c, & 0 \end{array}\right| (\alpha, \beta, \gamma, \delta)=0;$$

which equations, equivalent to a twofold relation, are the required condition. It is obvious that, treating $(\alpha, \beta, \gamma, \delta)$ as current point coordinates, each equation of the system is the equation of a plane through the given line.

Condition for the intersection of two lines.

11. The coordinates of the lines are taken to be (a, b, c, f, g, h), and $(a_{,}, b_{,}, c_{,}, f_{,}, g_{,}, h_{,})$, respectively. If $(\alpha, \beta, \gamma, \delta)$, $(\alpha', \beta', \gamma', \delta')$, are the coordinates of any two points in the first line, and $(\alpha_{,}, \beta_{,}, \gamma_{,}, \delta_{,})$, $(\alpha_{,}', \beta_{,}', \gamma_{,}', \delta_{,}')$, are the coordinates of any two points on the second line, then the four points are in a plane, that is, we have

$$\begin{vmatrix} \alpha\,, & \beta\,, & \gamma\,, & \delta \\ \alpha'\,, & \beta'\,, & \gamma'\,, & \delta' \\ \alpha_{,}\,, & \beta_{,}\,, & \gamma_{,}\,, & \delta_{,} \\ \alpha_{,}', & \beta_{,}', & \gamma_{,}', & \delta_{,}' \end{vmatrix} = 0,$$

that is, expanding the determinant and substituting for $\beta\gamma' - \beta'\gamma$, &c. and $\beta_{,}\gamma_{,}' - \beta_{,}'\gamma_{,}$, &c. their values in terms of the coordinates of the two lines respectively, we have

$$af_{,} + bg_{,} + ch_{,} + fa_{,} + gb_{,} + hc_{,} = 0,$$

or, as this may also be written,

$$(f_{,},\ g_{,},\ h_{,},\ a_{,},\ b_{,},\ c_{,} \between a,\ b,\ c,\ f,\ g,\ h) = 0,$$

for the condition that the two lines may intersect.

12. The same result will be obtained if we take

$$(A,\ B,\ C,\ D \between x,\ y,\ z,\ w) = 0,\quad (A',\ B',\ C',\ D' \between x,\ y,\ z,\ w) = 0,$$

for the equations of any two planes through the first line, and

$$(A_{,},\ B_{,},\ C_{,},\ D_{,} \between x,\ y,\ z,\ w) = 0,\quad (A_{,}',\ B_{,}',\ C_{,}',\ D_{,}' \between x,\ y,\ z,\ w) = 0,$$

for the equations of any two planes through the second line. The four planes will meet in a point, that is, we have

$$\begin{vmatrix} A\,, & B\,, & C\,, & D \\ A'\,, & B'\,, & C'\,, & D' \\ A_{,}\,, & B_{,}\,, & C_{,}\,, & D_{,} \\ A_{,}', & B_{,}', & C_{,}', & D_{,}' \end{vmatrix} = 0.$$

or, expanding and substituting, we have the same condition as before.

13. In the case of any two lines (a, b, c, f, g, h), and $(a_{,}, b_{,}, c_{,}, f_{,}, g_{,}, h_{,})$, we may define the 'moment' of the two lines to be the function

$$af_{,} + bg_{,} + ch_{,} + fa_{,} + gb_{,} + hc_{,},$$

it being understood that we have not as yet any complete quantitative definition of the moment; this being so, we have, in what precedes, the theorem that the moment of two intersecting lines is $= 0$.

Plane through two intersecting lines.

14. Let $(A, B, C, D \between x, y, z, w) = 0$ be the equation of the plane through the two intersecting lines (a, b, c, f, g, h) and $(a_,, b_,, c_,, f_,, g_,, h_,)$. We have two systems of equations, as in No. 9, and comparing the corresponding equations of the two systems, we find in the first instance

$$\begin{aligned} A : B : C : D &= \lambda \quad : bf_, - b_, f : cf_, - c_, f : -(bc_, - b_, c) \\ &= ag_, - a_, g : \quad \mu \quad : cg_, - c_, g : -(ca_, - c_, a) \\ &= ah_, - a_, h : bh_, - b_, h : \quad \nu \quad : -(ab_, - a_, b) \\ &= gh_, - g_, h : hf_, - h_, f : fg_, - f_, g : \quad \rho \quad , \end{aligned}$$

where λ, μ, ν, ρ, are in the first instance unknown; the different sets of ratios are of course identical in virtue of the relation

$$(f_,, g_,, h_,, a_,, b_,, c_, \between a, b, c, f, g, h) = 0,$$

and comparing them we have equations which lead to the values of λ, μ, ν, ρ; and we thus obtain more completely,

$$\begin{aligned} A : B : C : D &= f_, a + b_, g + c_, h : bf_, + b_, f \quad : cf_, - c_, f \quad : -(bc_, - b_, c) \\ &= ag_, - a_, g \quad : a_, f + g_, b + c_, h : cg_, - c_, g \quad : -(ca_, - c_, a) \\ &= ah_, - a_, h \quad : bh_, - b_, h \quad : a_, f + b_, g + h_, c : -(ab_, - a_, b) \\ &= gh_, - g_, h \quad : hf_, - h_, f \quad : fg_, - f_, g \quad : af_, + bg_, + ch_, . \end{aligned}$$

15. It is in these equations easy to verify the identity of the different sets of values: we ought, for instance, to have

$$\frac{a_, f + b_, g + h_, c}{fg_, - f_, g} = -\frac{ab_, - a_, b}{af_, + bg_, + ch_,};$$

that is,

$$(h_, c + a_, f + b_, g)(h_, c + af_, + bg_,) + (ab_, - a_, b)(fg_, - f_, g) = 0,$$

and, observing that

$$\begin{aligned} &(a_, f + b_, g)(af_, + bg_,) + (ab_, - a_, b)(fg_, - f_, g) \\ &= (af + bg)(a_, f_, + b_, g_,), = ch \cdot c_, h_,, \end{aligned}$$

the left-hand side is

$$\begin{aligned} &= ch_, (ch_, + af_, + bg_, + a_, f + b_, g + c_, h), \\ &= ch_, (af_, + bg_, + ch_, + fa_, + gb_, + hc_,), = 0. \end{aligned}$$

Point on two intersecting lines.

16. Let $(\alpha, \beta, \gamma, \delta)$ be the coordinates of the point of intersection of the two intersecting lines (a, b, c, f, g, h) and $(a_,, b_,, c_,, f_,, g_,, h_,)$. We have two systems of

equations such as in No. 10, and comparing the corresponding equations of the two systems, we find

$$\begin{array}{rllll} \alpha : \beta : \gamma : \delta = & L & : ag_{,} - a_{,}g & : ah_{,} - a_{,}h & : gh_{,} - g_{,}h \\ = & bf_{,} - b_{,}f & : M & : bh_{,} - b_{,}h & : hf_{,} - h_{,}f \\ = & cf_{,} - c_{,}f & : cg_{,} - c_{,}g & : N & : fg_{,} - f_{,}g \\ = & -(bc_{,} - b_{,}c) & : -(ca_{,} - c_{,}a) & : -(ab_{,} - a_{,}b) & : P \end{array},$$

where L, M, N, P, are in the first instance unknown; but, comparing the different sets of values, we have equations for finding the values of these quantities, and we thus obtain the more complete system

$$\begin{array}{rllll} \alpha : \beta : \gamma : \delta = & f_{,}a + b_{,}g + c_{,}h & : ag_{,} - a_{,}g & : ah_{,} - a_{,}h & : gh_{,} - g_{,}h \\ = & bf_{,} - b_{,}f & : a_{,}f + g_{,}b + c_{,}h & : bh_{,} - b_{,}h & : hf_{,} - h_{,}f \\ = & cf_{,} - c_{,}f & : cg_{,} - c_{,}g & : a_{,}f + b_{,}g + h_{,}c & : fg_{,} - f_{,}g \\ = & -(bc_{,} - b_{,}c) & : -(ca_{,} - c_{,}a) & : -(ab_{,} - a_{,}b) & : f_{,}a + g_{,}b + h_{,}c, \end{array}$$

where it is to be observed that the right-hand side considered as a matrix is the transposed matrix of that which occurs in No. 14 in the formula for $A : B : C : D$. The verification of the identity of the different sets of values can of course be effected as in No. 15.

Expression for an arbitrary plane through a line.

17. The condition in order that the plane $(A, B, C, D \between x, y, z, w) = 0$, may pass through the line (a, b, c, f, g, h), is the twofold relation given, No. 9; it is satisfied by any one of the four systems

$$\begin{array}{rl} A : B : C : D = & 0 : h : -g : a, \\ \text{or} = & -h : 0 : f : b, \\ \text{or} = & g : -f : 0 : c, \\ \text{or} = & -a : -b : -c : 0; \end{array}$$

and consequently also by

$$\begin{array}{rl} A : B : C : D = & (\ 0,\ -h,\ g,\ -a \between \xi, \eta, \zeta, \omega) \\ & : (\ h,\ 0,\ -f,\ -b \between \xi, \eta, \zeta, \omega) \\ & : (-g,\ f,\ 0,\ -c \between \xi, \eta, \zeta, \omega) \\ & : (\ a,\ b,\ c,\ 0 \between \xi, \eta, \zeta, \omega); \end{array}$$

or, what is the same thing, by

$$\begin{array}{rl} A : B : C : D = & (\ 0,\ h,\ -g,\ a \between \xi, \eta, \zeta, \omega) \\ & : (-h,\ 0,\ f,\ b \between \xi, \eta, \zeta, \omega) \\ & : (\ g,\ -f,\ 0,\ c \between \xi, \eta, \zeta, \omega) \\ & : (-a,\ -b,\ -c,\ 0 \between \xi, \eta, \zeta, \omega) \end{array}$$

where $(\xi, \eta, \zeta, \omega)$ are arbitrary: there is, however, no loss of generality in putting any two of these quantities $= 0$.

Expression for an arbitrary point in a line.

18. The condition in order that the point $(\alpha, \beta, \gamma, \delta)$, may lie in the line (a, b, c, f, g, h), is the twofold relation given, No. 10; it is satisfied by any one of the four systems

$$\begin{array}{rl} \alpha : \beta : \gamma : \delta = & 0 : \ \ c : -b : f, \\ \text{or } = & -c : \ \ 0 : \ \ a : g, \\ \text{or } = & b : -a : \ \ 0 : h, \\ \text{or } = & -f : -g : -h : 0 ; \end{array}$$

and consequently, also by

$$\begin{array}{rl} \alpha : \beta : \gamma : \delta = & (\ \ 0, \ -c, \ \ b, \ -f \between x, y, z, w) \\ : & (\ \ c, \ \ 0, \ -a, \ -g \between x, y, z, w) \\ : & (-b, \ \ a, \ \ 0, \ -h \between x, y, z, w) \\ : & (\ \ f, \ \ g, \ \ h, \ \ 0 \between x, y, z, w) ; \end{array}$$

or, what is the same thing, by

$$\begin{array}{rl} \alpha : \beta : \gamma : \delta = & (\ \ 0, \ \ c, \ -b, \ \ f \between x, y, z, w) \\ : & (-c, \ \ 0, \ \ a, \ \ g \between x, y, z, w) \\ : & (\ \ b, \ -a, \ \ 0, \ \ h \between x, y, z, w) \\ : & (-f, \ -g, \ -h, \ \ 0 \between x, y, z, w) \end{array}$$

where (x, y, z, w) are arbitrary: there is, however, no loss of generality in putting two of these quantities $=0$.

Article Nos. 19 to 25. *Geometrical considerations in regard to three, four, five, and six lines.*

Before proceeding further, I will establish certain geometrical notions in regard to three, four, five, and six lines. I use the term 'tractor' to denote a line which meets any given lines.

19. Three given lines have an infinity of tractors; viz. these are the generating lines of a hyperboloid having the three given lines for directrices.

20. Four given lines may be directrices (generating lines) of the same hyperboloid, viz. every tractor of any three of the four lines is then a tractor of all the four lines. But in general, four given lines have a pair of tractors; viz. considering the tractors of any three of the four lines, these form a hyperboloid having the three lines for directrices; the fourth line meets this hyperboloid in two points, and the generating line through either of these points is a line meeting each of the four given lines, that is, it is a tractor of the four given lines.

21. The fourth line may however touch the hyperboloid; and in this case, instead of a pair of tractors, the four lines have a twofold tractor. The relation of the four lines to each other is a symmetrical one; and we have thence the theorem, that if any one of four given lines touch the hyperboloid through the other three lines, then will each of the four given lines touch the hyperboloid through the other three lines. But the relation to each other of four lines having a twofold tractor may be otherwise expressed as follows; viz. considering a tractor of the four given lines, each line determines with the tractor a point, the intersection of the line and tractor; and it also determines a plane, viz. the plane containing the line and tractor; we have therefore a range of four points on the tractor, and a pencil of four planes through the tractor; and if the tractor be a two-fold tractor, the range and pencil will be homographic; and conversely, if the range and pencil are homographic, the tractor will be a twofold tractor. This is easily obtained as a limiting case from the general one where the four lines have a pair of tractors; each line determines with the one tractor a point and a plane as above, and this plane intersects the second tractor in a point; we have thus through the first tractor a pencil of planes, and on the second tractor a range of points, and these two are homographic. But, in the case of a twofold tractor, the range on the second tractor coincides with that on the first tractor; that is, the range of points on the tractor is homographic with the pencil of planes through the tractor.

22. Given any four lines, and a point O, then either in the general case where the four lines have a pair of tractors, or in the special case where they have a twofold tractor, there exists and can be found through the point O a single fifth line such that the five lines have (as the case may be) a pair of tractors, or a twofold tractor. And similarly, given the four lines and a plane Ω, there exists and can be found in the plane Ω a single fifth line such that the five lines have (as the case may be) a pair of tractors, or a twofold tractor.

23. Five given lines have not in general any tractor; the five lines may be directrices (generating lines) of the same hyperboloid, and they have then an infinity of tractors; or they may have a pair of tractors, viz. the fifth line may be a line meeting the tractors of the other four lines; or (as a particular case of the last relation) the five lines may have a twofold tractor; or the five lines may have a single tractor.

24. Given any five lines and a point O; then, selecting any four of the given lines, we may through O draw a line having with the four lines a pair of tractors. Treating in this manner each of the five sets of four lines, we obtain through the point O five lines constructed as above; we have the theorem which will be proved in the sequel, that these five lines lie in a plane Ω. And similarly, given the five lines, and a plane Ω, then selecting any four of the five lines, we may in the plane Ω draw a line having with the four lines a pair of tractors; treating in this manner each of the five sets of four lines, we obtain in the plane Ω five lines; and we have then the theorem that these five lines meet in a point O.

25. In the case of six given lines, we may have between the lines the like relations to those for the case of five given lines; or we may have the more general relation of the involution of six lines, depending on the last-mentioned theorems, viz. given any five lines, and the point O or the plane Ω, then determining in the one case the plane Ω and in the other case the point O, and taking as a sixth line any line whatever through the point O and in the plane Ω, the six lines are said to be in involution, or to form an involution of six lines. I now revert to the analytical theory of the line.

Article Nos. 26 to 51. (*Various sub-headings.*) *Cases of a linear relation or linear relations between the six Coordinates.*

26. If the coordinates (a, b, c, f, g, h) of a line are regarded as variable quantities connected by a single equation or by two or three equations, we have a system of lines with three or two arbitrary parameters or with a single arbitrary parameter; and so if there are four equations the system consists of a determinate number of lines. For a linear relation, the coefficients may be either (F, G, H, A, B, C), not the coordinates of a line, that is, not satisfying the relation $AF + BG + CH = 0$, or they may be the coordinates of a line, satisfying the relation in question. I consider the several cases in order as follows:

Linear relation $(F, G, H, A, B, C \between a, b, c, f, g, h) = 0$, *where* (A, B, C, F, G, H) *are not the coordinates of a line.*

27. Considering any six lines which satisfy the relation in question, we may eliminate the coefficients F, G, H, A, B, C, and thus obtain an equation $\nabla = 0$, where ∇ is the determinant formed with the coordinates of the six lines; this equation, regarding therein the coordinates of five of the six lines as given, is in regard to the coordinates of the remaining line, say the original line (a, b, c, f, g, h), a linear relation equivalent to the original linear relation $(F, G, H, A, B, C \between a, b, c, f, g, h) = 0$. The equation in its new form, viz. the equation $\nabla = 0$, establishes between the six lines a relation which is in fact the relation of involution already referred to; viz. it will be shown in the sequel that, starting from the equation $\nabla = 0$ as the definition of the relation of involution, we are led to a construction for a line in involution with five given lines the same as the construction explained *ante* No. 25.

Linear relation $(F, G, H, A, B, C \between a, b, c, f, g, h) = 0$, *where* (A, B, C, F, G, H) *are the coordinates of a line.*

28. The linear relation expresses that the two lines $(a, b, c, f, g, h \between A, B, C, F, G, H)$ intersect, or what is the same thing, that the line (a, b, c, f, g, h) is any line whatever meeting the line (A, B, C, F, G, H).

Two linear relations $(F, G, H, A, B, C \between a, b, c, f, g, h) = 0$,

$$(F_1, G_1, H_1, A_1, B_1, C_1 \between a, b, c, f, g, h) = 0,$$

where the two sets of coefficients respectively are or are not the coordinates of a line.

29. If the two sets of coefficients are each of them the coordinates of a line, then the two equations express that the line (a, b, c, f, g, h) is any line whatever cutting each of the two given lines. And the general case is in fact reducible to this particular one; for suppose that neither set of coefficients belongs to a line, then we may from the two given linear relations form the relation

$$(\lambda F+\lambda_1 F_1,\ \lambda G+\lambda_1 G_1,\ \lambda H+\lambda_1 H_1,\ \lambda A+\lambda_1 A_1,\ \lambda B+\lambda_1 B_1,\ \lambda C+\lambda_1 C_1 \between a, b, c, f, g, h)=0,$$

and if the ratio $\lambda : \lambda_1$ be properly determined, then $(\lambda A+\lambda_1 A_1, \ldots)$ will be the coordinates of a line. This will in fact be the case if

$$(\lambda A+\lambda_1 A_1)(\lambda F+\lambda_1 F_1)+(\lambda B+\lambda_1 B_1)(\lambda G+\lambda_1 G_1)+(\lambda C+\lambda_1 C_1)(\lambda H+\lambda_1 H_1)=0,$$

that is, if

$$(AF+BG+CH,\ AF_1+BG_1+CH_1+FA_1+GB_1+CH_1,\ A_1F_1+B_1G_1+C_1H_1 \between \lambda, \lambda_1)^2=0,$$

a quadric equation giving two values of the ratio $\lambda : \lambda_1$, that is, two linear relations in each of which the coefficients are the coordinates of a line: we have thus two derived lines, and the line (a, b, c, f, g, h) meets each of these derived lines.

There is no real difference if one or the other of the given systems of coefficients, say the system (A, B, C, F, G, H), are the coordinates of a line. We have then $AF+BG+CH=0$; the quadric equation in $\lambda : \lambda_1$ has a root $\lambda_1 : \lambda=0$, and rejecting it, the other root is determined by a simple equation: this only means that the line (A, B, C, F, G, H) is itself one of the two derived lines.

But there is a real difference in the case where the equation in $\lambda : \lambda_1$ has equal roots; to explain this special case, observe that if in the general case we consider the two derived lines as a pair of tractors of any four lines, then the linear relations express that the line (a, b, c, f, g, h) has with these four lines a pair of tractors; and in the special case under consideration the linear relations express that the line (a, b, c, f, g, h) has with the four lines, or (what is the same thing) with any three of them, that is with some three lines, a twofold tractor. According to what precedes (No. 21), the construction of the line (a, b, c, f, g, h) is in fact as follows, viz. if on the twofold tractor considered as given, we take a series of points p, and through the tractor, homographic with the range, a pencil of planes P, then the sought-for line will be any line through a point p, in the corresponding plane P. But it is proper to give an analytical proof of the construction.

30. I observe that we may without loss of generality assume $A_1F_1+B_1G_1+C_1H_1=0$, and this being so, the condition for the equality of the roots of the quadric equation is

$$AF_1+BG_1+CH_1+FA_1+BG_1+CH_1=0,$$

that is, writing $(a_1, b_1, c_1, f_1, g_1, h_1)$ in place of $(A_1, B_1, C_1, F_1, G_1, H_1)$, the case in question may be taken to be that of

Two linear relations

$$(f_1, g_1, h_1, a_1, b_1, c_1 \between a, b, c, f, g, h)=0,$$
$$(F, G, H, A, B, C \between a, b, c, f, g, h)=0,$$

where $(a_1, b_1, c_1, f_1, g_1, h_1)$ *are,* (A, B, C, F, G, H) *are not, the coordinates of a point, and where*

$$(f_1, g_1, h_1, a_1, b_1, c_1 \between A, B, C, F, G, H) = 0\,;$$

that is, where the twofold derived line is in fact the original line

$$(a_1, b_1, c_1, f_1, g_1, h_1).$$

31. To simplify, we may take $x = 0$, $y = 0$ for the equations of the line; the coordinates of the line then are $(a_1, b_1, c_1, f_1, g_1, h_1) = (0, 0, 0, 0, 0, 1)$. Taking moreover $x = 0$, $y = 0$, $\frac{z}{\gamma} = \frac{w}{\delta}$ for the coordinates of the point p, and $\frac{x}{\alpha} = \frac{y}{\beta}$ for the equation of the plane P, the homographic relation of the point and plane is given by an equation of the form

$$-F\beta\gamma + G\alpha\gamma - A\alpha\delta - B\beta\delta = 0,$$

or, as this may be written,

$$(F, G, H, A, B, 0 \between -\beta\gamma, \alpha\gamma, 0, -\alpha\delta, -\beta\delta, \omega) = 0,$$

where H and ω, being each multiplied by 0, do not really enter into the equation.

The equations of any line whatever through the point p and in the plane P may be written $\beta x - \alpha y = 0$, $A'x + B'y + \delta z - \gamma\omega = 0$, where A', B' are arbitrary: hence arranging the coefficients in the order

$$\begin{matrix} \beta, & -\alpha, & 0, & 0, \\ A', & B', & \delta, & -\gamma, \end{matrix}$$

the coordinates (a, b, c, f, g, h) of the line in question are

$$(-\beta\gamma, \alpha\gamma, 0, -\alpha\delta, -\beta\delta, A'\alpha + B'\beta)\,;$$

so that we have

$$\begin{aligned}&(f_1, g_1, h_1, a_1, b_1, c_1 \between a, b, c, f, g, h)\\ &= (0, 0, 1, 0, 0, 0 \between a, b, c, f, g, h), = c, = 0\,;\end{aligned}$$

and morever the homographic relation, replacing therein the arbitrary quantity ω by $A'\alpha + B'\beta$, becomes

$$(F, G, H, A, B, 0 \between a, b, c, f, g, h) = 0.$$

Hence the linear relations satisfied by the coordinates (a, b, c, f, g, h) of the line in question are

$$(f_1, g_1, h_1, a_1, b_1, c_1 \between a, b, c, f, g, h) = 0,$$
$$(F, G, H, A, B, C \between a, b, c, f, g, h) = 0,$$

with the coefficients

$$(f_1, g_1, h_1, a_1, b_1, c_1) = (0, 0, 1, 0, 0, 0),$$
$$(A, B, C, F, G, H) = (A, B, 0, F, G, H),$$

values which satisfy the condition

$$(f_1, g_1, h_1, a_1, b_1, c_1 \between A, B, C, F, G, H) = 0.$$

Hence the line (a, b, c, f, g, h) through the point p and in the plane P is a line the coordinates of which satisfy two linear relations as mentioned in the heading; and the theorem is thus proved. The demonstration would be simplified by taking, as is allowable, the homographic relation to be $\frac{\alpha}{\beta} = k\frac{\gamma}{\delta}$.

32. It appears from the foregoing examination of the case of two linear relations that in the following cases of three or more linear relations there is no real loss of generality in assuming that the coefficients of each set are the coordinates of a line; for if originally this be not so, we have only to replace the given relations by linear functions of these relations, and to assign such values to the multipliers $\lambda, \lambda_1, \lambda_2 \ldots$ as in each case to make the new coefficients to be the coordinates of a line; and as there are two or more arbitrary ratios $\lambda : \lambda_1 : \lambda_2 \ldots$ to be assigned at pleasure and only a single condition to be satisfied, no cases of failure can arise. The remaining cases may consequently be stated in a more simple form.

Three linear relations, the coefficients of each set being the coordinates of a line.

33. The three relations express that the line (a, b, c, f, g, h) meets each of the three given lines; that is, that the line is any generating line of a hyperboloid having the three given lines for directrices.

Four linear relations, the coefficients of each set being the coordinates of a line.

34. The four relations express that the line (a, b, c, f, g, h) meets each of four given lines; or what is the same thing, that the line is a tractor of four given lines. It is to be noticed that the four linear relations serve to express the ratios $a : b : c : f : g : h$ linearly in terms of any one of these ratios, or what is the same thing, to express the several ratios in terms of an arbitrary ratio $u : v$. Substituting the resulting values in the equation

$$af + bg + ch = 0,$$

we have a quadric equation for the determination of the remaining ratio, or of the ratio $u : v$; and then each of the ratios of the coordinates can be expressed rationally in terms of either root of the quadric equation; we thus obtain the coordinates of each of the two tractors of the four given lines; or we have a complete analytical solution of the problem, to find the tractors of four given lines. The quadric equation may have equal roots; that is, the four given lines may have a twofold tractor, which is then determined linearly.

35. The theory of the linear relations of the coordinates (a, b, c, f, g, h) of a line may be considered in a different manner. It will be convenient to take the different cases in a reverse order, beginning with the extreme case (not before mentioned) of a fivefold relation and ascending to the case of a onefold or single relation.

Case of the fivefold relation.

36. The fivefold relation

$$\left\|\begin{matrix} a, & b, & c, & f, & g, & h \\ a_1, & b_1, & c_1, & f_1, & g_1, & h_1 \end{matrix}\right\| = 0,$$

expresses that the quantities (a, b, c, f, g, h) are proportional to $(a_1, b_1, c_1, f_1, g_1, h_1)$. As the former set are by hypothesis the coordinates of a line, the given set $(a_1, b_1, c_1, f_1, g_1, h_1)$ must, it is clear, also be the coordinates of a line, and the relation then expresses that the line (a, b, c, f, g, h) coincides with the given line.

Case of the fourfold relation.

37. The fourfold relation is

$$\left\|\begin{matrix} a, & b, & c, & f, & g, & h \\ a_1, & b_1, & c_1, & f_1, & g_1, & h_1 \\ a_2, & b_2, & c_2, & f_2, & g_2, & h_2 \end{matrix}\right\| = 0,$$

or what is the same thing, we have the six equations $\lambda a + \lambda_1 a_1 + \lambda_2 a_2 = 0$, &c., involving the indeterminate quantities λ, λ_1, λ_2. If the coefficients

$$(a_1, b_1, c_1, f_1, g_1, h_1), \quad (a_2, b_2, c_2, f_2, g_2, h_2)$$

are not either set the coordinates of a line; then substituting the foregoing values $-\lambda a = \lambda_1 a_1 + \lambda_2 a_2$, &c. in the equation $af + bg + ch = 0$, we have a quadric equation in $(\lambda_1 : \lambda_2)$: and for each root of this equation, the coefficients $\lambda_1 a_1 + \lambda_2 a_2$, &c. will be the coordinates of a line. There are thus in general two derived lines; and the fourfold relation expresses that the line (a, b, c, f, g, h) coincides with one or other of these derived lines. There is no real difference if one or the other of the two sets $(a_1, b_1, c_1, f_1, g_1, h_1)$, $(a_2, b_2, c_2, f_2, g_2, h_2)$, or if each set, are the coordinates of a line; one of the derived lines or both of them will in these cases coincide with one or both of the given lines. And if the quadric equation has equal roots, then instead of two derived lines there is a twofold derived line, and the line (a, b, c, f, g, h) must coincide with this twofold line.

38. A case presenting peculiarity is however that in which the coefficients of the quadric equation vanish identically; this is only so when the coefficients $(a_1, b_1, c_1, f_1, g_1, h_1)$ and $(a_2, b_2, c_2, f_2, g_2, h_2)$ are the coordinates of two intersecting lines. The equations $-\lambda a = \lambda_1 a_1 + \lambda_2 a_2$, &c. here show that every line whatever which meets each of the two lines $(a_1, b_1, c_1, f_1, g_1, h_1)$ and $(a_2, b_2, c_2, f_2, g_2, h_2)$ meets also the line (a, b, c, f, g, h); that is, the line (a, b, c, f, g, h) is any line whatever in the plane and through the point of intersection of the two intersecting lines. We see moreover that not only

$$a_1 f_2 + b_1 g_2 + c_1 h_2 + a_2 f_1 + b_2 g_1 + c_2 h_1 = 0,$$

but also that $af_1 + bg_1 + ch_1 + fa_1 + gb_1 + ch_1 = 0$ and $af_2 + bg_2 + ch_2 + fa_2 + gb_2 + ch_2 = 0$; that is, the moment of each pair of lines is $= 0$. It may be remarked that the ratios $\lambda : \lambda_1 : \lambda_2$ may be determined from any two of the six equations

$$\lambda a + \lambda_1 a_1 + \lambda_2 a_2 = 0, \ldots \lambda h + \lambda_1 h_1 + \lambda_2 h_2 = 0;$$

but that in consequence of the moments being each $= 0$, there is not for the determination of these ratios any such set of equations as occur in the cases subsequently considered of a threefold relation, &c.

39. In what follows we have three or more sets $(a_1, b_1, c_1, f_1, g_1, h_1)$, &c.; and we may without loss of generality assume that each of these are the coordinates of a line: for replacing the several coefficients $a_1, \ldots$ by linear functions $\mu_1 a_1 + \mu_2 a_2 + \mu_3 a_3 +$ &c., &c., the multipliers may be determined so that these are the coordinates of a point: and since for each set there is only a single condition to be satisfied by the two or more ratios $\mu_1 : \mu_2 : \mu_3 \ldots$, it is easy to see that no cases of failure will arise.

Case of the threefold relation.

40. The threefold relation is

$$\left\| \begin{matrix} a, & b, & c, & f, & g, & h \\ a_1, & b_1, & c_1, & f_1, & g_1, & h_1 \\ a_2, & b_2, & c_2, & f_2, & g_2, & h_2 \\ a_3, & b_3, & c_3, & f_3, & g_3, & h_3 \end{matrix} \right\| = 0,$$

where $(a_1, \ldots)$, $(a_2, \ldots)$ $(a_3, \ldots)$ are each the coordinates of a line. Here writing

$$\lambda a + \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 = 0 \ldots,$$

it is clear that every line which meets each of the lines $(a_1, \ldots)$, $(a_2, \ldots)$, $(a_3, \ldots)$ will also meet the line (a, b, c, f, g, h); the lines which meet the first-mentioned three lines are the generating lines of a hyperboloid having these three lines for directrices, and it hence appears that the line (a, b, c, f, g, h) is any directrix line whatever of the hyperboloid in question.

41. Using the notations 01, 02, 12, &c. to denote the moments of the several pairs of lines, viz.

$$01 = af_1 + bg_1 + ch_1 + fa_1 + gb_1 + hc_1,$$

$$12 = a_1 f_2 + b_1 g_2 + c_1 h_2 + f_1 a_2 + g_1 b_2 + h_1 c_2,$$

&c.,

then from the equations $\lambda a + \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 = 0$, &c., we deduce

$$\begin{array}{l} \quad . \quad\;\; \lambda_1 01 + \lambda_2 02 + \lambda_3 03 = 0, \\ \lambda 10 + \quad . \quad + \lambda_2 12 + \lambda_3 13 = 0, \\ \lambda 20 + \lambda_1 21 \quad . \quad + \lambda_3 23 = 0, \\ \lambda 30 + \lambda_1 31 + \lambda_2 32 \quad . \quad = 0, \end{array}$$

and hence eliminating $\lambda, \lambda_1, \lambda_2, \lambda_3$, we find

$$\begin{vmatrix} . & 01, & 02, & 03 \\ 10, & . & 12, & 13 \\ 20, & 21, & . & 23 \\ 30, & 31, & 32, & . \end{vmatrix} = 0,$$

a relation between the moments satisfied in virtue of the given threefold relation; but which as a mere onefold relation is of course not equivalent to the threefold relation. It will subsequently appear that the equation expresses that any one of the four lines, say the line (a, b, c, f, g, h) touches the hyperboloid having the other three lines for generatrices; this condition is satisfied in virtue of the threefold relation which, as we have seen, expresses that the line (a, b, c, f, g, h) lies wholly in the hyperboloid in question.

42. The last mentioned determinant is the Norm of

$$\sqrt{01 \,.\, 23} + \sqrt{02 \,.\, 31} + \sqrt{03 \,.\, 12};$$

so that the equation may be written

$$\sqrt{01 \,.\, 23} + \sqrt{02 \,.\, 31} + \sqrt{03 \,.\, 12} = 0,$$

or, what is the same thing,

$$\sqrt{01}\sqrt{23} + \sqrt{02}\sqrt{31} + \sqrt{03}\sqrt{12} = 0,$$

it being of course understood that the signs of the radicals must be determined in accordance with this equation; we then find

$$\lambda : \lambda_1 : \lambda_2 : \lambda_3 = \sqrt{23 \,.\, 31 \,.\, 12} : \sqrt{02 \,.\, 03 \,.\, 23} : \sqrt{03 \,.\, 01 \,.\, 31} : \sqrt{01 \,.\, 02 \,.\, 12},$$

or say

$$= \sqrt{23}\sqrt{31}\sqrt{12} : \sqrt{02}\sqrt{03}\sqrt{23} : \sqrt{03}\sqrt{01}\sqrt{31} : \sqrt{01}\sqrt{02}\sqrt{12};$$

in fact, substituting these last values in the linear equations for $\lambda, \lambda_1, \lambda_2, \lambda_3$, we find that the equations are all satisfied in virtue of the single equation

$$\sqrt{01}\sqrt{23} + \sqrt{02}\sqrt{31} + \sqrt{03}\sqrt{12} = 0.$$

Case of the twofold relation.

43. We have here

$$\begin{Vmatrix} a, & b, & c, & f, & g, & h \\ a_1, & b_1, & c_1, & f_1, & g_1, & h_1 \\ a_2, & b_2, & c_2, & f_2, & g_2, & h_2 \\ a_3, & b_3, & c_3, & f_3, & g_3, & h_3 \\ a_4, & b_4, & c_4, & f_4, & g_4, & h_4 \end{Vmatrix} = 0,$$

where $(a_1, \ldots)(a_2, \ldots)(a_3, \ldots)(a_4, \ldots)$, are each the coordinates of a line. Here, writing

$$\lambda a + \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 + \lambda_4 a_4 = 0;$$

it is clear that every line which meets each of the four given lines, will also meet the line (a, b, c, f, g, h); but the only lines meeting the four given lines are two determinate lines, the tractors of the four given lines; and the conclusion is, that the line (a, b, c, f, g, h) is any line whatever which meets the two tractors.

44. If, however, the four given lines have a twofold tractor, then the line (a, b, c, f, g, h) is still a line having two conditions imposed upon it; it is in fact a line determined as in No. 21, viz. if on the tractor we take a series of points p, and through the tractor a series of planes P, corresponding homographically to the points, then the line (a, b, c, f, g, h) is any line through a point p, in the corresponding plane P.

45. Using as before 01, 02, ... 12, &c. to denote the moments of the several pairs of lines, we have

$$\begin{array}{r} \cdot \quad \lambda_1 01 + \lambda_2 02 + \lambda_3 03 + \lambda_4 04 = 0, \\ \lambda 10 \quad \cdot \quad + \lambda_2 12 + \lambda_3 13 + \lambda_4 14 = 0, \\ \lambda 20 + \lambda_1 21 \quad \cdot \quad + \lambda_3 23 + \lambda_4 24 = 0, \\ \lambda 30 + \lambda_1 31 + \lambda_2 32 \quad \cdot \quad + \lambda_4 34 = 0, \\ \lambda 40 + \lambda_1 41 + \lambda_2 42 + \lambda_3 43 \quad \cdot \quad = 0, \end{array}$$

and thence also

$$\begin{vmatrix} \cdot & 01, & 02, & 03, & 04 \\ 10, & \cdot & 12, & 13, & 14 \\ 20, & 21, & \cdot & 23, & 24 \\ 30, & 31, & 32, & \cdot & 34 \\ 40, & 41, & 42, & 43, & \cdot \end{vmatrix} = 0,$$

a relation between the moments satisfied in virtue of the original twofold relation; but which, as a single equation, is of course not equivalent to the twofold relation. It is in fact easy to see that this equation expresses that the five lines have a common tractor; this is true, since in virtue of the twofold relation there are really two common tractors.

I have not obtained from the linear equations any symmetrical expressions for the ratios $\lambda : \lambda_1 : \lambda_2 : \lambda_3$.

Case of a onefold relation.

46. The onefold relation is

$$\begin{vmatrix} a, & b, & c, & f, & g, & h \\ a_1, & b_1, & c_1, & f_1, & g_1, & h_1 \\ a_2, & b_2, & c_2, & f_2, & g_2, & h_2 \\ a_3, & b_3, & c_3, & f_3, & g_3, & h_3 \\ a_4, & b_4, & c_4, & f_4, & g_4, & h_4 \\ a_5, & b_5, & c_5, & f_5, & g_5, & h_5 \end{vmatrix} = 0,$$

where $(a_1, \ldots)$, $(a_2, \ldots)$, $(a_3, \ldots)$, $(a_4, \ldots)$, $(a_5, \ldots)$, are each the coordinates of points in a line. The preceding mode of dealing with the question is inapplicable, since there is not in general any line which meets the five given lines; in the particular case, however, where the five given lines are met by a single line, say when they have a common tractor, then the line (a, b, c, f, g, h) is any line meeting this common tractor. The general case is that of the involution of six lines, mentioned No. 25, and the consideration of which was deferred.

47. The onefold relation implies that we can find multipliers λ, μ, ν, ρ, σ, τ, such that

$$\lambda a + \mu b + \nu c + \rho f + \sigma g + \tau h = 0,$$
$$\lambda a_1 + \mu b_1 + \nu c_1 + \rho f_1 + \sigma g_1 + \tau h_1 = 0,$$
$$\vdots$$
$$\lambda a_5 + \mu b_5 + \nu c_5 + \rho f_5 + \sigma g_5 + \tau h_5 = 0,$$

we may by means of the last five equations determine the ratios of λ, μ, ν, ρ, σ, τ, viz. these quantities will be proportional to the determinants formed out of the matrix

$$\left|\begin{array}{cccccc} a_1, & b_1, & c_1, & f_1, & g_1, & h_1 \\ a_2, & b_2, & c_2, & f_2, & g_2, & h_2 \\ a_3, & b_3, & c_3, & f_3, & g_3, & h_3 \\ a_4, & b_4, & c_4, & f_4, & g_4, & h_4 \\ a_5, & b_5, & c_5, & f_5, & g_5, & h_5 \end{array}\right|$$

and the first equation is then a linear relation in (a, b, c, f, g, h), expressing the relation that exists between these coordinates.

48. Consider an arbitrary point O on the line (a, b, c, f, g, h); taking this point as origin, the coordinates of O are 0, 0, 0, 1; and if x, y, z, w, are the coordinates of any other point on the line, then writing

$$\begin{array}{cccc} x, & y, & z, & w, \\ 0, & 0, & 0, & 1, \end{array}$$

we find

$$a : b : c : f : g : h = 0 : 0 : 0 : x : y : z;$$

and the equation

$$\lambda a + \mu b + \nu c + \rho f + \sigma g + \tau h = 0$$

becomes simply $\rho x + \sigma y + \tau z = 0$; viz. this equation expresses that the line (a, b, c, f, g, h), assumed to pass through a given point O, lies in a determinate plane Ω through this point.

49. To construct this plane Ω, I consider any four of the five given lines, say the lines 2, 3, 4, 5, and I endeavour to find the line OQ_1 through O, which has with these lines a pair of tractors; *quà* line through O, the coordinates of the line in question may be taken to be 0, 0, 0, F_1, G_1, H_1 (where F_1, G_1, H_1, are in fact the

coordinates x, y, z, of any point on the line OQ_1); and then the condition for the pair of tractors may be written

$$\begin{aligned}
p_2a_2+p_3a_3+p_4a_4+p_5a_5&=0,\\
p_2b_2+p_3b_3+p_4b_4+p_5b_5&=0,\\
p_2c_2+p_3c_3+p_4c_4+p_5c_5&=0,\\
p_2f_2+p_3f_3+p_4f_4+p_5f_5&=F_1,\\
p_2g_2+p_3g_3+p_4g_4+p_5g_5&=G_1,\\
p_2h_2+p_3h_3+p_4h_4+p_5h_5&=H_1,
\end{aligned}$$

where p_2, p_3... are arbitrary coefficients; and we hence deduce

$$\rho F_1+\sigma G_1+\tau H_1=0;$$

but in precisely the same way, if the line OQ_2 have with the lines 1, 3, 4, 5, a pair of tractors, and if F_2, G_2, H_2, be the coordinates of a point on the line OQ_2, and similarly for the lines OQ_3, OQ_4, OQ_5, and the coordinates (F_3, G_3, H_3), (F_4, G_4, H_4) (F_5, G_5, H_5), we have

$$\begin{aligned}
\rho F_2+\sigma G_2+\tau H_2&=0,\\
\rho F_3+\sigma G_3+\tau H_3&=0,\\
\rho F_4+\sigma G_4+\tau H_4&=0,\\
\rho F_5+\sigma G_5+\tau H_5&=0,
\end{aligned}$$

and these equations show that the five lines OQ_1, OQ_2, OQ_3, OQ_4, OQ_5, lie in the plane

$$\rho x+\sigma y+\tau z=0;$$

so that this plane is given as the plane through the lines OQ_1, OQ_2, OQ_3, OQ_4, OQ_5; and we have thus (given the lines 1, 2, 3, 4, 5, and the arbitrary point O) the construction of the line (a, b, c, f, g, h) through O in involution with the given lines.

50. The original onefold relation may be replaced by the six equations

$$\begin{aligned}
\lambda a+\lambda_1a_1+\lambda_2a_2+\lambda_3a_3+\lambda_4a_4+\lambda_5a_5&=0,\\
\lambda b+\lambda_1b_1+\lambda_2b_2+\lambda_3b_3+\lambda_4b_4+\lambda_5b_5&=0,\\
&\vdots\\
\lambda h+\lambda_1h_1+\lambda_2h_2+\lambda_3h_3+\lambda_4h_4+\lambda_5h_5&=0,
\end{aligned}$$

and hence denoting as before the moments by 01, 02, 12, &c. we have

$$\begin{array}{llllllll}
. & +\lambda_1 01 & +\lambda_2 02 & +\lambda_3 03 & +\lambda_4 04 & +\lambda_5 05 & =0,\\
\lambda 10 & . & +\lambda_2 12 & +\lambda_3 13 & +\lambda_4 14 & +\lambda_5 15 & =0,\\
\lambda 20 & +\lambda_1 21 & . & +\lambda_3 23 & +\lambda_4 24 & +\lambda_5 25 & =0,\\
\lambda 30 & +\lambda_1 31 & +\lambda_2 32 & . & +\lambda_4 34 & +\lambda_5 35 & =0,\\
\lambda 40 & +\lambda_1 41 & +\lambda_2 42 & +\lambda_3 43 & . & +\lambda_5 45 & =0,\\
\lambda 50 & +\lambda_1 51 & +\lambda_2 52 & +\lambda_3 53 & +\lambda_4 54 & . & =0,
\end{array}$$

which lead to

$$\begin{vmatrix} . & 01, & 02, & 03, & 04, & 05 \\ 10, & . & 12, & 13, & 14, & 15 \\ 20, & 21, & . & 23, & 24, & 25 \\ 30, & 31, & 32, & . & 34, & 35 \\ 40, & 41, & 42, & 43, & . & 45 \\ 50, & 51, & 52, & 53, & 54, & . \end{vmatrix} = 0,$$

a relation between the moments equivalent to the original onefold relation, and consequently expressing that the six lines are in involution. I have not obtained a symmetrical system of values for the ratios $\lambda : \lambda_1 : \lambda_2 : \lambda_3 : \lambda_4 : \lambda_5$.

51. Reverting to the relation which exists between the point O and the plane Ω, it is proper to remark that, since to any given point O there corresponds a single plane Ω, and to any given plane Ω a single point O, it follows that the point O and plane Ω are reciprocal figures; viz. they are reciprocals of the particular kind treated of by Möbius, wherein the reciprocal of a point is a plane through the point, and the reciprocal of a plane a point in the plane; and of which the analytical character is that the reciprocal of the point $(\alpha, \beta, \gamma, \delta)$ is the plane

$$\begin{aligned} &(\quad . \quad h\beta - g\gamma + l\delta)\,x \\ &+(-h\alpha \quad . \quad +f\gamma + m\delta)\,y \\ &+(\quad g\alpha - f\beta \quad . \quad + n\delta)\,z \\ &+(-l\alpha - m\beta - n\gamma \quad . \quad)\,w = 0. \end{aligned}$$

Article No. 52. *A geometrical property of an involution of six lines.*

52. The figure of six lines in involution is connected in various ways with the theory of cubic curves in space, for instance, considering a point A of the curve, this determines with any given line l a plane meeting the curve in two other points, and the line λ which joins these two points may be called the projection of the line l. This being so, if in any osculating plane of the cubic we have six lines, $l, l_1, l_2, l_3, l_4, l_5$, tangents of a conic in that plane, the six projections $\lambda, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$ of these tangents will be a set of lines in involution. I do not stop to prove this theorem or to develope any of its consequences.

Article No. 53. *To find the condition that four given lines may have a twofold tractor.*

53. Taking the coordinates of the given lines to be

$$(a, b, c, f, g, h),\ (a_1, b_1, c_1, f_1, g_1, h_1),\ (a_2, b_2, c_2, f_2, g_2, h_2),\ (a_3, b_3, c_3, f_3, g_3, h_3)$$

then if (A, B, C, F, G, H) be the coordinates of a tractor of these lines, we have

$$\begin{aligned}
&(F,\ G,\ H,\ A,\ B,\ C \between a\ ,\ b\ ,\ c\ ,\ f\ ,\ g\ ,\ h\ ) = 0,\\
&(F,\ G,\ H,\ A,\ B,\ C \between a_1,\ b_1,\ c_1,\ f_1,\ g_1,\ h_1) = 0,\\
&(F,\ G,\ H,\ A,\ B,\ C \between a_2,\ b_2,\ c_2,\ f_2,\ g_2,\ h_2) = 0,\\
&(F,\ G,\ H,\ A,\ B,\ C \between a_3,\ b_3,\ c_3,\ f_3,\ g_3,\ h_3) = 0.
\end{aligned}$$

In virtue of these relations the ratios $A : B : C : F : G : H$ are given linear functions of any one of these ratios or of an arbitrary ratio $u : v$; and we then have $AF + BG + CH = 0$, a quadric equation for determining the unknown ratio. In the case of a twofold tractor, this equation must have equal roots; whence employing as usual the method of indeterminate multipliers, we find

$$\begin{aligned}
A + \lambda a + \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 &= 0,\\
B + \lambda b + \lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3 &= 0,\\
C + \lambda c + \lambda_1 c_1 + \lambda_2 c_2 + \lambda_3 c_3 &= 0,\\
F + \lambda f + \lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 &= 0,\\
G + \lambda g + \lambda_1 g_1 + \lambda_2 g_2 + \lambda_3 g_3 &= 0,\\
H + \lambda h + \lambda_1 h_1 + \lambda_2 h_2 + \lambda_3 h_3 &= 0.
\end{aligned}$$

Hence representing as before the moments of the pairs of lines by 01, 02, &c., we deduce

$$\begin{aligned}
.\quad\ \ + \lambda_1 01 + \lambda_2 02 + \lambda_3 03 &= 0,\\
\lambda 10 + \quad .\quad + \lambda_2 12 + \lambda_3 13 &= 0,\\
\lambda 20 + \lambda_1 21 \quad .\quad + \lambda_3 23 &= 0,\\
\lambda 30 + \lambda_1 31 + \lambda_2 32 \quad . \quad &= 0,
\end{aligned}$$

so that, as already mentioned, we have

$$\left|\begin{array}{cccc} . & 01, & 02, & 03 \\ 10, & . & 12, & 13 \\ 20, & 21, & . & 23 \\ 30, & 31, & 32, & . \end{array}\right| = 0,$$

as the condition that the four given lines may have a twofold tractor.

Article Nos. 54 to 56. *Hyperboloid passing through three given lines.*

54. The direct investigation is somewhat tedious; but I write down, and will afterwards verify, the equation of the hyperboloid passing through the three given lines

$$(a_1,\ b_1,\ c_1,\ f_1,\ g_1,\ h_1),\ (a_2,\ b_2,\ c_2,\ f_2,\ g_2,\ h_2),\ (a_3,\ b_3,\ c_3,\ f_3,\ g_3,\ h_3).$$

Writing for shortness (agh), &c. to denote the determinants

$$\left|\begin{array}{ccc} a_1, & g_1, & h_1 \\ a_2, & g_2, & h_2 \\ a_3, & g_3, & h_3 \end{array}\right|, \text{ \&c.}$$

the equation of the hyperboloid is

$$\begin{aligned}(agh)\,x^2+(bhf)\,y^2\ +\ (cfg)\,z^2+(abc)\ w^2&\\ +[(abg)\ -(cah)]\,xw+[(bfg)\ \ +(chf)]\,yz&\\ +[(bch)\ \ -(abf)]\,yw+[(cgh)\ \ +(afg)]\,zx&\\ +[(caf)\ \ -(bcg)]\,zw+[(ahf)\ \ +(bgh)]\,xy=0.&\end{aligned}$$

In fact, we have

$$\begin{aligned}(agh)&=a_1(g_2h_3-g_3h_2)+g_1(h_2a_3-h_3a_2)+h_1(a_2g_3-a_3g_2)\\ &=a\,.\,gh\ +\qquad g\,.\,ha\qquad +h\,.\,ag,\end{aligned}$$

where a, &c. stand for a_1, &c. and gh, &c. for $g_2h_3-g_3h_2$, &c. Hence the foregoing equation may be written

$$\begin{aligned}&x^2\,(a\,.\,gh+g\,.\,ha+h\,.\,ag)\\ &+y^2\,(b\,.\,hf+h\,.\,fb+f\,.\,bh)\\ &+z^2\,(c\,.\,fg+f\,.\,gc+g\,.\,cf)\\ &+w^2\,(a\,.\,bc+b\,.\,ca+c\,.\,ab)\\ &+xw\begin{pmatrix}a\,.\,bg+b\,.\,ga+g\,.\,ab\\ -c\,.\,ah-a\,.\,hc-h\,.\,ca\end{pmatrix}+yz\begin{pmatrix}b\,.\,fg+f\,.\,gb+g\,.\,bf\\ +c\,.\,hf+h\,.\,fc+f\,.\,ch\end{pmatrix}\\ &+yw\begin{pmatrix}b\,.\,ch+c\,.\,hb-h\,.\,bc\\ -a\,.\,bf-b\,.\,fa-f\,.\,ab\end{pmatrix}+zx\begin{pmatrix}c\,.\,gh+g\,.\,hc+h\,.\,cg\\ +a\,.\,fg+f\,.\,ga+g\,.\,af\end{pmatrix}\\ &+zw\begin{pmatrix}c\,.\,af+a\,.\,fc+f\,.\,ca\\ -b\,.\,cg-c\,.\,gb-g\,.\,bc\end{pmatrix}+xy\begin{pmatrix}a\,.\,hf+h\,.\,fa+f\,.\,ch\\ +b\,.\,gh+g\,.\,hb+h\,.\,bg\end{pmatrix}=0.\end{aligned}$$

55. This is

$$\begin{aligned}&bc\,.\,w\,(\qquad hy-gz+aw)\\ &+ca\,.\,w\,(-hx\qquad +fz+bw)\\ &+ab\,.\,w\,(\ gx-fy\qquad +cw)\\ &+gh\,.\,x\ (ax+by+cz)\\ &+hf\,.\,y\ (ax+by+cz)\\ &+fg\,.\,z\ (ax+by+cz)\\ &+af\,\{w\,(ax+by+cz)-x\,(\qquad hy-gz+aw)\}\\ &+bg\,\{w\,(ax+by+cz)-y\,(-hx\qquad +fz+bw)\}\\ &+ch\,\{w\,(ax+by+cz)-z\,(\ gx-fy\qquad +cw)\}\\ &-bf\,.\,y\,(\qquad hy-gz+aw)\\ &-cf\,.\,z\,(\qquad hy-gz+aw)\\ &-cg\,.\,z\,(-hx\qquad +fz+bw)\\ &-ag\,.\,x\,(-hx\qquad +fz+bw)\\ &-ah\,.\,x\,(gx-fy\qquad +cw)\\ &-bh\,.\,y\,(gx-fy\qquad +cw)=0.\end{aligned}$$

56. Hence writing

$$(X, Y, Z, W) = \begin{vmatrix} . , & h, & -g, & a, \\ -h, & . , & f, & b, \\ g, & -f, & . , & c, \\ -a, & -b, & -c, & . \end{vmatrix} (x, y, z, w),$$

the foregoing equation is

$$\begin{aligned} &bc \,.\, wX + ca \,.\, wY + ab \,.\, wZ - gh \,.\, xW - hf \,.\, yW - fg \,.\, zW \\ &- af(wW + xX) \quad - bg\,(wW + yY) \quad - ch\,(wW + zZ) \\ &- bf \,.\, yX - cg \,.\, zY - ah \,.\, xZ - cf \,.\, zX - ag \,.\, xY - bh \,.\, yZ = 0\,; \end{aligned}$$

or, collecting and arranging, this is

$$\begin{aligned} &X\ \{-af \,.\, x - bf \,.\, y - cf \,.\, z + bc \,.\, w\} \\ &+ Y\ \{-ag \,.\, x - bg \,.\, y - cg \,.\, z + ca \,.\, w\} \\ &+ Z\ \{-ah \,.\, x - bh \,.\, y - ch \,.\, z + ab \,.\, w\} \\ &+ W\{-gh \,.\, x - hf \,.\, y - fg \,.\, z + (af \,. + bg \,. + ch \,.)\, w\} = 0, \end{aligned}$$

which is satisfied by $X = 0$, $Y = 0$, $Z = 0$, $W = 0$; that is, since (a, b, c, f, g, h) have been written in place of $(a_1, b_1, c_1, f_1, g_1, h_1)$, by $X_1 = 0$, $Y_1 = 0$, $Z_1 = 0$, $W_1 = 0$ (if we thus denote the corresponding functions of $(a_1, b_1, c_1, f_1, g_1, h_1)$), that is, the hyperboloid passes through the line $(a_1, b_1, c_1, f_1, g_1, h_1)$; and similarly it passes through the other two lines.

Article Nos. 57 and 58. *The six coordinates defined as to their absolute magnitudes.*

57. In all that precedes, the absolute magnitudes of the coordinates have been left indeterminate, only the ratios being attended to. But the magnitudes of the six coordinates may be fixed in a very simple manner as follows; viz. using ordinary rectangular coordinates, then for any line, if x_0, y_0, z_0 are the coordinates of a particular point on this line, and α, β, γ the inclinations of the line to the axes, the coordinates of another point on the line are

$$x_0 + r\cos\alpha,\ y_0 + r\cos\beta,\ z_0 + r\cos\gamma\,;$$

and hence writing

$$\begin{matrix} x_0 + r\cos\alpha, & y_0 + r\cos\beta, & z_0 + r\cos\gamma, & 1, \\ x_0 & , y_0 & , z_0 & , 1, \end{matrix}$$

we have

$$a : b : c : f : g : h = z_0\cos\beta - y_0\cos\gamma : x_0\cos\gamma - z_0\cos\alpha : y_0\cos\beta - x_0\cos\alpha : \cos\alpha : \cos\beta : \cos\gamma.$$

Or we may take

$$\begin{aligned} a &= z_0\cos\beta - y_0\cos\gamma, & f &= \cos\alpha, \\ b &= x_0\cos\gamma - z_0\cos\alpha, & g &= \cos\beta, \\ c &= y_0\cos\alpha - x_0\cos\beta, & h &= \cos\gamma, \end{aligned}$$

values which of course satisfy, as they should do, the relation $af+bg+ch=0$. It is hardly necessary to remark, that the values of a, b, c are not altered on substituting for x_0, y_0, z_0 the coordinates $x_0+s\cos\alpha$, $y_0+s\cos\beta$, $z_0+s\cos\gamma$ of any other point on the line.

58. Considering any two lines (a, b, c, f, g, h), $(a_1, b_1, c_1, f_1, g_1, h_1)$, if we define the moment of the two lines to be the product of the perpendicular distance into the sine of the inclination of the two lines, then we have, Moment

$$= af_1+bg_1+ch_1+fa_1+gb_1+hc_1,$$

viz. we have now a quantitative definition of the function of the coordinates previously called the moment of the two lines.

For the demonstration of this formula it is to be remarked, that taking on the first line a segment of the length r, the coordinates of its extremities being (x_0, y_0, z_0) and $(x_0+r\cos\alpha,\ y_0+r\cos\beta,\ z_0+r\cos\gamma)$,

and on the second line a segment of the length r_1 the coordinates of its extremities being (x_0', y_0', z_0') and $(x_0'+r_1\cos\alpha_1, y_0'+r_1\cos\beta_1, z_0'+r_1\cos\gamma_1)$ and joining the extremities of these segments so as to form a tetrahedron, the volume of the tetrahedron is

$$= \tfrac{1}{6}rr_1(af_1+bg_1+ch_1+fa_1+gb_1+hc_1).$$

But the volume of the tetrahedron is also equal to $\frac{1}{6}$ of the product of the opposite edges into their perpendicular distance into the sine of the inclination of the two edges [1]; that is, it is $=\frac{1}{6}rr_1$ into the moment of the two lines, and we have thus the formula in question.

Article Nos. 59 to 75. *Statical and Kinematical Applications.*

The coordinates (a, b, c, f, g, h), as last defined, are peculiarly convenient in kinematical and mechanical questions, as will appear from the following investigations.

59. Using the term rotation to denote an infinitesimal rotation, I say first that a rotation λ round the line (a, b, c, f, g, h) produces in the point (x, y, z) rigidly connected with this line the displacements

$$\begin{aligned}\delta x &= \lambda(\ \ .\ \ -hy+gz-a),\\ \delta y &= \lambda(\ \ hx\ \ .\ \ -fz-b),\\ \delta z &= \lambda(-gx+fy\ \ .\ \ -c\ ).\end{aligned}$$

[1] I take the opportunity of mentioning a very simple demonstration of this formula: taking the opposite edges to be r, r_1, their inclination $=\theta$, and perpendicular distance $=h$; the section of the tetrahedron by a plane parallel to the two edges at the distances z, $h-z$ from the two edges respectively is a parallelogram, the sides of which are $\frac{r(h-z)}{h}$ and $\frac{r_1 z}{h}$ respectively, and their inclination is $=\theta$; the area of the section is therefore $\frac{rr_1}{h^2}\sin\theta \,.\, z(h-z)$ and the volume of the tetrahedron is $=\frac{rr_1}{h^2}\sin\theta\int_0^h z(h-z)\,dz,\ =\frac{1}{6}rr_1 h\sin\theta$. The same result is however obtained still more simply by drawing a plane through one of the two edges perpendicular to the other edge; the volume is then equal to the sum or the difference of the volumes of two tetrahedra standing on a common triangular base; and the required result at once follows.

In fact assuming for a moment that the axis of rotation passes through the origin, then for the point P coordinates x, y, z, the square of the perpendicular distance from the axis is

$$\begin{aligned} &(\quad . \quad -y\cos\gamma + z\cos\beta)^2 \\ +&(\; x\cos\gamma \quad . \quad -z\cos\alpha\,)^2 \\ +&(-x\cos\beta + y\cos\alpha \quad . \quad)^2, \end{aligned}$$

and the expressions which enter into this formula denote as follows; viz. if through the point P at right angles to the plane through P and the axis of rotation we draw a line PQ, $=$ perpendicular distance of P from the axis of rotation, then the coordinates of Q referred to P as origin are

$$\begin{array}{c} . \quad -y\cos\gamma + z\cos\beta, \\ x\cos\gamma \quad . \quad -z\cos\alpha, \\ -x\cos\beta + y\cos\alpha \quad . \quad , \end{array}$$

respectively. Hence the foregoing quantities each multiplied by λ are the displacements of the point P in the directions of the axes, produced by the rotation λ.

60. Suppose that the axis of rotation (instead of passing through the origin) pass through the point (x_0, y_0, z_0); the only difference is that we must in the formula write $(x-x_0, y-y_0, z-z_0)$ in place of (x, y, z): and attending to the significations of the six coordinates, it thus appears that the displacements produced by the rotation are equal to λ into the expressions

$$\begin{array}{c} . \quad -hy + gz - a, \\ hx \quad . \quad -fz - b, \\ -gx + fy \quad . \quad -c, \end{array}$$

respectively; which is the theorem in question.

61. I say secondly that considering in a solid body the point (x, y, z) situate in the line (a, b, c, f, g, h), and writing

$$a, b, c, f, g, h = z\cos\beta - y\cos\gamma,\ x\cos\gamma - z\cos\alpha,\ y\cos\alpha - x\cos\beta,\ \cos\alpha,\ \cos\beta,\ \cos\gamma,$$

then for any infinitesimal motion of the solid body the displacement of the point in the direction of the line is

$$= ap + bq + cr + fl + gm + hn,$$

where p, q, r, l, m, n are constants depending on the infinitesimal motion.

In fact for any infinitesimal motion of a solid body the displacements of the point (x, y, z) are

$$\begin{aligned} \delta x &= l \quad . \quad + ry - qz, \\ \delta y &= m - rx \quad . \quad + pz, \\ \delta z &= n + qx - py \quad . \quad , \end{aligned}$$

and hence the displacement in the direction of the line is

$$= \cos\alpha\,\delta x + \cos\beta\,\delta y + \cos\gamma\,\delta z,$$

which attending to the significations of (a, b, c, f, g, h) is

$$= ap + bq + cr + fl + gm + hn,$$

and we have thus the theorem in question.

62. It thus appears that for a system of rotations

$$\begin{array}{lll} \lambda_1 & \text{about the line} & (a_1, b_1, c_1, f_1, g_1, h_1), \\ \lambda_2 & \quad ,, & (a_2, b_2, c_2, f_2, g_2, h_2), \\ \&c. & \quad ,, & \&c. \end{array}$$

the displacements of the point (x, y, z) rigidly connected with the several lines are

$$\begin{aligned} \delta x &= \quad . \quad - y\Sigma h\lambda + z\Sigma g\lambda - \Sigma a\lambda, \\ \delta y &= \quad x\Sigma h\lambda \quad . \quad - z\Sigma f\lambda - \Sigma b\Sigma, \\ \delta z &= -x\Sigma g\lambda + y\Sigma f\lambda \quad . \quad - \Sigma c\lambda, \end{aligned}$$

and when the rotations are in equilibrium then the displacements $(\delta x, \delta y, \delta z)$ of any point (x, y, z) whatever must each of them vanish; that is, we must have

$$\Sigma\lambda a = 0, \quad \Sigma\lambda b = 0, \quad \Sigma\lambda c = 0, \quad \Sigma\lambda f = 0, \quad \Sigma\lambda g = 0, \quad \Sigma\lambda h = 0,$$

which are therefore the conditions for the equilibrium of the system of rotations λ_1, λ_2, &c.

63. And it further appears that for a system of forces acting on a rigid body,

$$\begin{array}{lll} \lambda_1 & \text{along the line} & (a_1, b_1, c_1, f_1, g_1, h_1), \\ \lambda_2 & \quad ,, & (a_2, b_2, c_2, f_2, g_2, h_2), \\ \&c. & & \end{array}$$

the conditions of equilibrium as given by the Principle of Virtual Velocities is

$$\Sigma\lambda\,(ap + bq + cr + fl + gm + hn) = 0,$$

or what is the same thing, that we have

$$\Sigma\lambda a = 0, \quad \Sigma\lambda b = 0, \quad \Sigma\lambda c = 0, \quad \Sigma\lambda f = 0, \quad \Sigma\lambda g = 0, \quad \Sigma\lambda h = 0,$$

for the conditions of equilibrium of the system of forces λ_1, λ_2, &c. The conditions of equilibrium are thus precisely the same in the case of a system of rotations (infinitesimal rotations) and in that of a system of forces.

64. It now appears that the greater portion of the investigations in the first part of the present paper are applicable, and may be considered as relating, to the equilibrium of forces (or of rotations; but as the two theories are identical, it is sufficient to attend to one of them), and that we have in effect solved the following

question, "Given any system of two, three, four, five or six lines considered as belonging to a solid body, to determine the relations between these lines in order that there may exist along them forces which are in equilibrium;" but for greater clearness I will consider the several cases in order; it is hardly necessary to remark that when the forces exist the equilibrium will depend on the ratios only, and that the absolute magnitude of any one of the forces may be assumed at pleasure.

65. The condition in the case of two lines is of course that these shall coincide together, or form one and the same line; and the forces are then equal and opposite forces.

66. In the case of three lines, these must meet in a point and lie in a plane; and the force along each line must then be as the sine of the angle between the other two lines.

67. Supposing that the forces are λ along the line (a, b, c, f, g, h), λ_1 along the line $(a_1, b_1, c_1, f_1, g_1, h_1)$, and λ_2 along the line $(a_2, b_2, c_2, f_2, g_2, h_2)$, the conditions of equilibrium are $\lambda a + \lambda_1 a_1 + \lambda_2 a_2 = 0, \ldots\ldots \lambda h + \lambda_1 h_1 + \lambda_2 h_2 = 0$, any two of which determine the ratios $\lambda : \lambda_1 : \lambda_2$; these ratios were not worked out *ante* No. 38 for the reason that with the coordinates there made use of, a symmetrical solution was not obtainable; but in the present case, selecting the last three equations, these are

$$\begin{aligned} \lambda \cos \alpha + \lambda_1 \cos \alpha_1 + \lambda_2 \cos \alpha_2 &= 0, \\ \lambda \cos \beta + \lambda_1 \cos \beta_1 + \lambda_2 \cos \beta_2 &= 0, \\ \lambda \cos \gamma + \lambda_1 \cos \gamma_1 + \lambda_2 \cos \gamma_2 &= 0, \end{aligned}$$

giving in the first instance an equation which expresses that the three lines (assumed to meet in a point) lie in the same plane: and then if 01, 02, 12 be the angles between the pairs of lines respectively, giving by an easy transformation

$$\begin{aligned} \lambda \quad\quad + \lambda_1 \cos 01 + \lambda_2 \cos 02 &= 0, \\ \lambda \cos 10 + \lambda_1 \quad\quad + \lambda_2 \cos 12 &= 0, \\ \lambda \cos 20 + \lambda_1 \cos 21 + \lambda_2 \quad\quad &= 0. \end{aligned}$$

68. Putting for shortness A, B, C in the place of 12, 20, 01 respectively, we thence find

$$\begin{vmatrix} 1 , & \cos C , & \cos B \\ \cos C , & 1 , & \cos A \\ \cos B , & \cos A , & 1 \end{vmatrix} = 0,$$

that is

$$1 - \cos^2 A - \cos^2 B - \cos^2 C + 2 \cos A \cos B \cos C = 0,$$

equivalent to $A + B + C = 2\pi$; and then from the first and second equations

$$\begin{aligned} \lambda : \lambda_1 : \lambda_2 &= \cos A \cos C - \cos B : \cos B \cos C - \cos A : 1 - \cos^2 C, \\ &= \sin A \sin C : \sin B \sin C : \sin^2 C, \\ &= \sin A : \sin B : \sin C, \end{aligned}$$

which is the required formula.

69. In the case of four given lines the condition (as noticed by Möbius) is that the four lines shall be generating lines of the same hyperboloid. In fact every line which meets three of the four lines must also meet the fourth line; for otherwise the moment of the system about such line would not be $=0$. Calling the lines 0, 1, 2, 3 and writing as before 01, 02, &c. for the moments of the several pairs of lines, then taking the moments of the system about the four lines respectively, we obtain directly the before-mentioned system of equations

$$\begin{array}{llll} \cdot & \lambda_1 01 & + \lambda_2 02 & + \lambda_3 03 = 0, \\ \lambda 10 & \cdot & + \lambda_2 12 & + \lambda_3 13 = 0, \\ \lambda 20 & + \lambda_1 21 & \cdot & + \lambda_3 23 = 0, \\ \lambda 30 & + \lambda_1 31 & + \lambda_1 41 & \cdot \quad = 0, \end{array}$$

leading as before to the relation

$$\sqrt{01}\,\sqrt{23} + \sqrt{02}\,\sqrt{31} + \sqrt{03}\,\sqrt{12} = 0,$$

and to the values

$$\lambda : \lambda_1 : \lambda_2 : \lambda_3 = \sqrt{12}\,\sqrt{23}\,\sqrt{31} : \sqrt{23}\,\sqrt{30}\,\sqrt{02} : \sqrt{30}\,\sqrt{01}\,\sqrt{13} : \sqrt{01}\,\sqrt{12}\,\sqrt{20}$$

for the proportional magnitudes of the forces. These last equations give

$$\lambda\lambda_1\, 01 = \lambda_2\lambda_3\, 23,$$

which, representing each force by a segment on the line along which the force acts, denotes that the tetrahedron of any two of the forces is equal to the tetrahedron of the other two forces; this is in fact equivalent to the theorem of M. Chasles, that if a system of forces be in any manner whatever reduced to two forces, the tetrahedron formed by these two forces has a constant volume.

70. In the case of five given lines, the lines must have a pair of tractors. Any four of the lines have in fact two tractors; and each of these tractors must meet the fifth line, for otherwise the moment of the system about the tractor would not be $=0$. In the case where the four lines have a twofold tractor, the foregoing consideration shows only that the fifth line meets the twofold tractor, but it fails to show that the twofold tractor is a twofold tractor in regard to the fifth line.

71. I stop to consider this particular case under the present statical point of view. Taking the twofold tractor for the axis of z; let the line 0 meet this line in the point $(0, 0, c)$, the coordinates (a, b, c, f, g, h) of this line being consequently

$$(c\cos\beta,\quad -c\cos\alpha,\quad 0,\quad \cos\alpha,\quad \cos\beta,\quad \cos\gamma)$$

and the like for the other four lines 1, 2, 3, 4. Using the sign Σ to refer to the last-mentioned four lines the equations of equilibrium become

$$\begin{aligned} \lambda c\cos\beta + \Sigma\lambda_1 c_1 \cos\beta_1 &= 0, \\ \lambda c\cos\alpha + \Sigma\lambda_1 c_1 \cos\alpha_1 &= 0, \\ \lambda\ \cos\alpha + \Sigma\lambda_1\ \cos\alpha_1 &= 0, \\ \lambda\ \cos\beta + \Sigma\lambda_1\ \cos\beta_1 &= 0, \\ \lambda\ \cos\gamma + \Sigma\lambda_1\ \cos\gamma_1 &= 0. \end{aligned}$$

These equations give

$$\frac{\Sigma\lambda_1 c_1 \cos\beta_1}{\Sigma\lambda_1 \cos\alpha_1} = \frac{c\cos\beta}{\cos\alpha};$$

we may without loss of generality take the homographic conditions which express that the axis of z is a twofold tractor of the four lines to be

$$\frac{c_1\cos\beta_1}{\cos\alpha_1} = \frac{c_2\cos\beta_2}{\cos\alpha_2} = \frac{c_3\cos\beta_3}{\cos\alpha_3} = \frac{c_4\cos\beta_4}{\cos\alpha_4} = k,$$

and this being so, the last-mentioned equation becomes

$$\frac{c\cos\beta}{\cos\alpha} = k;$$

and it thus appears that the axis of z is a twofold tractor in regard also to the line 0.

72. In the case of six lines such that there exist along them forces which are in equilibrium, taking this as a definition of the involution of six lines, we may very readily obtain from statical considerations the before-mentioned construction of the sixth line; viz. it may be shown that given any five of the lines, say the lines 1, 2, 3, 4, 5 and a point O, we can through the point O determine a plane Ω, such that any line whatever through the point O and in the plane Ω is in involution with the five given lines. Consider the tractors of any four of the lines, say the lines 2, 3, 4, 5; we may through the point O draw a line OA meeting the two tractors; that is, the lines 2, 3, 4, 5 and the line OA will have a pair of common tractors. There consequently exist along these lines forces which are in equilibrium; and since only the ratios are material, the absolute magnitude of the force along the line OA may be anything whatever. Similarly, considering the tractors of the lines 1, 3, 4, 5, and through O a line OB meeting these tractors, then there exist along the lines 1, 3, 4, 5 and the line OB forces which are in equilibrium, and the absolute magnitude of the force along the line OB may be anything whatever. Hence, combining the two sets of forces, we have, along a line through O in the plane OA, OB, but otherwise indeterminate in its direction, a force in equilibrium with forces along the lines 1, 2, 3, 4, 5; that is, the line found as above is a line in involution with the lines 1, 2, 3, 4, 5.

73. It is to be added, that through O we cannot, out of the plane OA, OB, draw a line in involution with the lines 1, 2, 3, 4, 5; for if any such line OK existed, then we should have along each of the lines OA, OB, OK forces in equilibrium with forces along the lines 1, 2, 3, 4, 5; and the magnitudes of the three forces being each of them anything whatever, it would follow that along any line whatever through the point O there would exist a force in equilibrium with forces along the lines 1, 2, 3, 4, 5; that is, any line whatever through the point O would be a line in involution with these lines.

74. It hence appears, that drawing OA to meet the tractors of 2, 3, 4, 5; OB to meet those of 3, 4, 5, 1; OC to meet those of 4, 5, 1, 2; OD to meet those of 5, 1, 2, 3; and OE to meet those of 1, 2, 3, 4; the lines OA, OB, OC, OD, OE will be in one plane, say the plane Ω: and that any line through O in the plane Ω will be a line in involution with the lines 1, 2, 3, 4, 5.

75. There is another statical representation of the involution of six lines. If a system of forces act on a solid body, then taking six lines at random, the system will be in equilibrium if the sum of the moments be $=0$ in regard to each of the six lines. But if the six lines be in involution; then, for the very reason that a rotation about one of these lines is resolvable into rotations about the other five lines, if the sum of the moments be $=0$ for each of the five lines, it will also be $=0$ for the sixth line: that is, it is *not* sufficient for the equilibrium of the forces that the sum of the moments shall be $=0$ for each of the six lines. And we thus see that six lines in involution are lines such that the equilibrium of a system of forces about each of the six lines as axes does *not* insure the equilibrium of the system.

Article Nos. 76 and 77. *Transformation of Coordinates.*

76. Reverting to the general definition of the six coordinates (a, b, c, f, g, h) of a line by means of the points $(\alpha, \beta, \gamma, \delta)$ and $(\alpha', \beta', \gamma', \delta')$ on the line; suppose that instead of the original coordinate planes $x=0$, $y=0$, $z=0$, $w=0$ (forming a tetrahedron $ABCD$) we have new coordinate planes $x_0=0$, $y_0=0$, $z_0=0$, $w_0=0$ (forming a tetrahedron $A_0B_0C_0D_0$); and that the relations between the two sets of current coordinates are given by the equations

$$\begin{aligned} x : y : z : w = \ & (\lambda_1, \mu_1, \nu_1, \rho_1 \between x_0, y_0, z_0, w_0) \\ : \ & (\lambda_2, \mu_2, \nu_2, \rho_2 \between x_0, y_0, z_0, w_0) \\ : \ & (\lambda_3, \mu_3, \nu_3, \rho_3 \between x_0, y_0, z_0, w_0) \\ : \ & (\lambda_4, \mu_4, \nu_4, \rho_4 \between x_0, y_0, z_0, w_0), \end{aligned}$$

with, of course, the like relations between the original coordinates $(\alpha, \beta, \gamma, \delta)$ and new coordinates $(\alpha_0, \beta_0, \gamma_0, \delta_0)$, and between the original coordinates $(\alpha', \beta', \gamma', \delta')$ and the new coordinates $(\alpha_0', \beta_0', \gamma_0', \delta_0')$, of the two points on the line (a, b, c, f, g, h); then taking $(a_0, b_0, c_0, f_0, g_0, h_0)$ as the new values of the six coordinates of the line, viz. writing

$$\begin{array}{ccccccccccc} a_0 & : & b_0 & : & c_0 & : & f_0 & : & g_0 & : & h_0 \\ =\beta_0\gamma_0'-\beta_0'\gamma_0 & : & \gamma_0\alpha_0'-\gamma_0'\alpha_0 & : & \alpha_0\beta_0'-\alpha_0'\beta_0 & : & \alpha_0\delta_0'-\alpha_0'\delta_0 & : & \beta_0\delta_0'-\beta_0'\delta_0 & : & \gamma_0\delta_0'-\gamma_0'\delta_0, \end{array}$$

we obtain a system of formulæ which may be conveniently written as follows:

$$\begin{aligned} a : b : c : f : g : h \\ = \overset{\mu\nu}{23} a_0 + \overset{\nu\lambda}{23} b_0 + \overset{\lambda\mu}{23} c_0 + \overset{\lambda\rho}{23} f_0 + \overset{\mu\rho}{23} g_0 + \overset{\nu\rho}{23} h_0 \\ : 31 \\ : 12 \\ : 14 \\ : 24 \\ : 34 \end{aligned}$$

viz. the top line stands for $(\mu_2\nu_3-\mu_3\nu_2)\,a_0+(\nu_2\lambda_3-\nu_3\lambda_2)\,b_0+\&c.$, and the other lines are obtained from this by mere alterations of the suffixes.

77. As to the interpretation of these formulæ, taking

$$ABCD \text{ as the fundamental tetrahedron for } (x, y, z, w),$$

$$A_0B_0C_0D_0 \quad ,, \qquad ,, \quad (x_0, y_0, z_0, w_0),$$

then

$$(\lambda_1, \mu_1, \nu_1, \rho_1 \between x_0, y_0, z_0, w_0) = 0 \text{ is the equation of plane } BCD,$$

$$(\lambda_2, \mu_2, \nu_2, \rho_2 \between \quad\quad) = 0 \quad ,, \quad CDA,$$

$$(\lambda_3, \mu_3, \nu_3, \rho_3 \between \quad ,, \quad) = 0 \quad ,, \quad DAB,$$

$$(\lambda_4, \mu_4, \nu_4, \rho_4 \between \quad ,, \quad) = 0 \quad ,, \quad ABC,$$

whence, observing that the second and third equations belong to two planes each passing through the line DA, it appears that the coefficients

$$\frac{\mu\nu}{23}, \frac{\nu\lambda}{23}, \frac{\lambda\mu}{23}, \frac{\lambda\rho}{23}, \frac{\mu\rho}{23}, \frac{\nu\rho}{23}$$

are the six coordinates of the line DA, expressed in regard to the tetrahedron $(A_0B_0C_0D_0)$; and similarly that the coefficients in the six expressions of the transformation formula are the six coordinates of the lines AD, BD, CD, BC, CA, AB respectively in regard to the tetrahedron $(A_0B_0C_0D_0)$.

In the preceding formulæ for the transformation of coordinates the ratios only have been attended to, no determinate absolute magnitudes have been assigned to the coordinates (a, b, c, f, g, h). But I will nevertheless show how we may attribute absolute magnitudes to these coordinates.

Article Nos. 78 to 80. *New definition of the six coordinates as to their absolute magnitudes.*

78. I assume (x, y, z, w) to be "volume" coordinates; viz. taking as before $ABCD$ for the fundamental tetrahedron, and denoting the point (x, y, z, w) by P, I assume that we have

$$x : y : z : w : 1 = PBCD : APCD : ABPD : ABCP : ABCD,$$

where $PBCD$, &c. denote the volumes of the several tetrahedra $PBCD$, &c. It is to be noticed that the volume is in every case taken with a determinate sign: analytically the sign may be fixed by taking (x_a, y_a, z_a), &c., as the Cartesian coordinates of the points A, &c. and writing

$$PBCD = \begin{vmatrix} x_p, & x_b, & x_c, & x_d \\ y_p, & y_b, & y_c, & y_d \\ z_p, & z_b, & z_c, & z_d \\ 1, & 1, & 1, & 1 \end{vmatrix}, \text{ \&c.}$$

(whence of course $PBCD = PCDA = -PCBD$, &c. according to the rule of signs): or we may in an equivalent manner, but less easily, determine the sign, by considering the sense of the rotation about CD (considered as an axis drawn from C to D) which would be produced by a force along PB (from P to B).

79. It is to be observed that the foregoing values give identically $x+y+z+w=1$, so that the equation of the plane infinity is $x+y+z+w=0$. The values of the coordinates (x, y, z, w) may be written

$$x : y : z : w : 1 = PBCD : PCAD : PABD : PCBA : ABCD;$$

or in the original form

$$x : y : z : w : 1 = PBCD : APCD : ABPD : ABCP : ABCD,$$

as may be most convenient.

80. Denoting the points $(\alpha, \beta, \gamma, \delta)$ and $(\alpha', \beta', \gamma', \delta')$ by Q, Q' respectively, we have

$$\alpha : \beta : \gamma : \delta : 1 = QBCD : AQCD : ABQD : ABCQ : ABCD$$

and

$$\alpha' : \beta' : \gamma' : \delta' : 1 = Q'BCD : AQ'CD : ABQ'D : ABCQ' : ABCD,$$

and writing

$$(a, b, c, f, g, h) = (\beta\gamma' - \beta'\gamma,\ \gamma\alpha' - \gamma'\alpha,\ \alpha\beta' - \alpha'\beta,\ \alpha\delta' - \alpha'\delta,\ \beta\delta' - \beta'\delta,\ \gamma\delta' - \gamma'\delta),$$

viz. the two sets being taken to be *equal*, $a = \beta\gamma' - \beta'\gamma$, &c. instead of merely proportional, then it is easily seen that we obtain

$$\begin{aligned} & a : b : c : f : g : h : 1 \\ &= AQQ'D : Q'BQD : QQ'CD : QBCQ' : AQCQ' : ABQQ' : ABCD, \end{aligned}$$

that is, in order to form the first six combinations we successively replace

$$(B, C), (C, A), (A, B), (A, D), (B, D), (C, D)$$

in $ABCD$ by (Q, Q').

Article No. 81. *Resulting formulæ of Transformation.*

81. For the transformation of coordinates if we assume

$$\begin{aligned} x &= (\lambda_1, \mu_1, \nu_1, \rho_1 \between x_0, y_0, z_0, w_0), \\ y &= (\lambda_2, \mu_2, \nu_2, \rho_2 \between \quad ,, \quad), \\ z &= (\lambda_3, \mu_3, \nu_3, \rho_3 \between \quad ,, \quad), \\ w &= (\lambda_4, \mu_4, \nu_4, \rho_4 \between \quad ,, \quad), \end{aligned}$$

and take also (a, b, c, f, g, h), $(a_0, b_0, c_0, f_0, g_0, h_0)$ respectively *equal*, instead of merely proportional, to the foregoing values, then, observing that for the point A_0 we have $(x_0, y_0, z_0, w_0) = (1, 0, 0, 0)$ we see that $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the $ABCD-$coordinates of A_0; and the like as to the other sets of coefficients; viz. we have

$$\begin{aligned} \lambda_1 : \lambda_2 : \lambda_3 : \lambda_4 : 1 &= A_0BCD : AA_0CD : ABA_0D : ABCA_0 : ABCD \\ \mu_1 : \mu_2 : \mu_3 : \mu_4 : 1 &= B_0 \ ,, \ : ,, B_0 ,, \ : \ ,, B_0 ,, : \ ,, \ B_0 : \ ,, \\ \nu_1 : \nu_2 : \nu_3 : \nu_4 : 1 &= C_0 \ ,, \ : ,, C_0 ,, \ : \ ,, C_0 ,, : \ ,, \ C_0 : \ ,, \\ \rho_1 : \rho_2 : \rho_3 : \rho_4 : 1 &= D_0 \ ,, \ : ,, D_0 ,, \ : \ ,, D_0 ,, : \ ,, \ D_0 : \ ,, \end{aligned}$$

and we hence find

$$\frac{\mu\nu}{23} : \frac{\nu\lambda}{23} : \frac{\lambda\mu}{23} : \frac{\lambda\rho}{23} : \frac{\mu\rho}{23} : \frac{\nu\rho}{23} : 1$$

$$= AB_0C_0D : AC_0A_0D : AA_0B_0D : AA_0D_0D : AB_0D_0D : AC_0D_0D : ABCD$$

viz. multiplying the last-mentioned set of terms by $A_0B_0C_0D_0 \div ABCD$, in order to make the last term equal to unity, we see that the coefficients $\frac{\mu\nu}{23}$, $\frac{\nu\lambda}{23}$ &c. are equal to $\frac{A_0B_0C_0D_0}{ABCD}$ into the six $(ABCD)_0$ – coordinates respectively of the line AD by means of the points A, D thereof. And similarly in the six expressions which enter into the formula of transformation, the coefficients are $= \frac{A_0B_0C_0D_0}{ABCD}$ into the six $(ABCD)_0$ – coordinates of the

line AD	in regard to points	A, D	thereof
„ BD	„	B, D	„
„ CD	„	C, D	„
„ BC	„	B, C	„
„ CA	„	C, A	„
„ AB	„	A, B	„

The foregoing theory of the transformation of coordinates seemed to me interesting for its own sake, and I have developed it in preference to the more simple theory which might easily be established of the case in which the coordinates are quantitatively defined as being equal to

$$(z_0 \cos\beta - y_0 \cos\gamma,\ x_0 \cos\gamma - z_0 \cos\alpha,\ y_0 \cos\beta - x_0 \cos\alpha,\ \cos\alpha,\ \cos\beta,\ \cos\gamma)$$

respectively.