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NOTE ON THE SOLUTION OF THE QUARTIC EQUATION $\alpha U + 6\beta H = 0$.

[From the *Mathematische Annalen*, vol. I. (1869), pp. 54, 55.]

IF U denote the quartic function $(a, b, c, d, e \between x, y)^4$, H its Hessian

$$= (ac - b^2,\ 2(ad - bc),\ ae + 2bd - 3c^2,\ 2(be - cd),\ ce - d^2 \between x, y)^4,$$

α and β constants, then we may find the linear factors of the function $\alpha U + 6\beta H$ (or what is the same thing solve the equation $\alpha U + 6\beta H = 0$) by a formula almost identical with that given by me (Fifth Memoir on Quantics, *Phil. Trans.* vol. CXLVIII. (1858), see p. 446, [156]) in regard to the original quartic function U.

In fact (reproducing the investigation) if I, J are the two invariants, $M = \dfrac{I^3}{4J^2}$, Φ the cubicovariant

$$= (-a^2d + 3abc - 2b^3,\ \&c \between x, y)^6,$$

then the identical equation $JU^3 - IU^2H + 4H^3 = -\Phi^2$, may be written $(1, 0, -M, M \between IH, JU)^3 = -\frac{1}{4}I^3\Phi^2$, whence if ω_1, ω_2, ω_3 are the roots of the equation $(1, 0, -M, M \between \omega, 1)^3 = 0$, or what is the same thing $\omega^3 - M(\omega - 1) = 0$; then the functions

$$IH - \omega_1 JU,\ IH - \omega_2 JU,\ IH - \omega_3 JU$$

are each of them a square: writing

$$(\omega_2 - \omega_3)(IH - \omega_1 JU) = X^2,$$
$$(\omega_3 - \omega_1)(IH - \omega_2 JU) = Y^2,$$
$$(\omega_1 - \omega_2)(IH - \omega_3 JU) = Z^2,$$

so that identically $X^2+Y^2+Z^3=0$, the expression $\alpha X+\beta Y+\gamma Z$ will be a square if only $\alpha^2+\beta^2+\gamma^2=0$. (To see this observe that in virtue of the equation $X^2+Y^2+Z^2=0$, we have $X+iY$, $X-iY$ each of them a square, and thence

$$\alpha X+\beta Y+\gamma Z, \ = \tfrac{1}{2}(\alpha+i\beta)(X-iY)+\tfrac{1}{2}(\alpha-i\beta)(X-iY)-\gamma i\sqrt{X^2+Y^2},$$

is a square if the condition in question be satisfied.)

Hence in particular writing

$$\sqrt{\omega_2-\omega_3}\sqrt{\alpha I+6\beta\omega_1 J}, \ldots, \quad \sqrt{\omega_1-\omega_2}\sqrt{\alpha I+6\beta\omega_3 J},$$

for α, β, γ, we have

$$(\omega_2-\omega_3)\sqrt{\alpha I+6\beta\omega_1 J}\sqrt{IH+\omega_1 JU}+\ldots+(\omega_1-\omega_2)\sqrt{\alpha I+6\beta\omega_3 J}\sqrt{IH+\omega_3 JU}$$

a perfect square, and since the product of the four different values is a multiple of $(\alpha U+6\beta H)^2$ (this is most readily seen by observing that for $\alpha U+6\beta H=0$, the irrational expression omitting a factor is $(\omega_2-\omega_3)(\alpha I+6\beta\omega_1 J)+\ldots+(\omega_1-\omega_2)(\alpha I+6\beta\omega_3 J)$, which vanishes identically) it follows that the expression in question is the square of a linear factor of $\alpha U+6\beta H$.

It thus appears that the radicals (other than those arising from the solution of $U=0$) contained in the solution of the equation $\alpha U+6\beta H=0$ are the three roots

$$\sqrt{\alpha I+6\beta\omega_1 J}, \quad \sqrt{\alpha I+6\beta\omega_2 J}, \quad \sqrt{\alpha I+6\beta\omega_3 J}.$$

Cambridge, September 2, 1868.