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ON THE RATIONAL TRANSFORMATION BETWEEN TWO SPACES.

[From the *Proceedings of the London Mathematical Society*, vol. III. (1869—1871), pp. 127—180. Account of the Paper given at the Meeting 11 March 1869.]

Two figures are rationally transformable each into the other (or, say, there is a rational transformation between the two figures) when to a variable point of each of them there corresponds a single variable point of the other. The figures may be either loci in a space, or *locus in quo* of any number of dimensions; or they may be such spaces themselves. Thus the figures may be each a line (or space of one dimension), each a plane (or space of two dimensions), or each a space of three dimensions; these last are the cases intended to be considered in the present Memoir, which is accordingly entitled, "On the Rational Transformation between Two Spaces." I observe in explanation (to fix the ideas, attending to the case of two planes), that any rational transformation between two planes gives rise to a rational transformation between curves in these planes respectively (one of these curves being any curve whatever): but *non constat*, and it is not in fact the case, that every rational transformation between two plane curves thus arises out of a rational transformation between two planes. The problem of the rational transformation between two planes (or generally between two spaces) is thus a distinct problem from that of the rational transformation between two plane curves (or loci in the two spaces respectively).

I consider in the Memoir, (1) the rational transformation between two lines; this is simply the homographic transformation: (2) the rational transformation between two planes; and here there is little added to what has been done by Prof. Cremona in his memoirs, "Sulle Trasformazioni Geometriche delle Figure Piane," (*Mem. di Bologna*, t. II., 1863, and t. V., 1865; see also "On the Geometrical Transformation of Plane Curves," *British Assoc. Report*, 1864): (3) the rational transformation between two spaces; in regard hereto I examine the general theory, but attend mainly to what I call the lineo-linear transformation; viz., it is assumed that the coordinates

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of a point in the one space, and the coordinates of the corresponding point in the other space are connected by three lineo-linear equations (that is, each equation is linear in the two sets of coordinates respectively). The lineo-linear transformation presents itself in the preceding two cases; viz., between two lines, the homographic transformation (which, as already mentioned, is the only rational transformation) is lineo-linear; and between two planes, the lineo-linear transformation is in fact the well-known inverse transformation $\left(x' : y' : z' = \frac{1}{x} : \frac{1}{y} : \frac{1}{z}\right)$. As regards two spaces, the lineo-linear transformation has not, I think, been discussed in a general manner, and it gives rise to a theory of some complexity, and of great interest.

The General Principle of the Rational Transformation between Two Spaces.

1. In all that follows, the two spaces (lines, planes, or three dimensional spaces, as the case may be), or any corresponding loci in the two spaces respectively, are referred to as the first and second figures respectively. The two figures are in general considered, not as superimposed or situate in a common space, but as existing, each independently of the other, as a separate *locus in quo* or figure in such locus. The unaccented coordinates (x, y), (x, y, z), or (x, y, z, w), as the case may be, refer throughout to a point of the first figure; the accented coordinates refer in like manner to the corresponding point of the second figure [1]. Moreover X, Y,... are used to denote functions of the same order, say n, of the coordinates $(x, y, \ldots)$; viz., (X, Y) are each of them of the form $(*\between x, y)^n$; (X, Y, Z) each of the form $(*\between x, y, z)^n$, (X, Y, Z, W) each of the form $(*\between x, y, z, w)^n$, as the case may be; and in like manner X', Y',... are used to denote functions of the same order, say n', of the coordinates $(x', y', \ldots)$. This being so:

The condition of a rational transformation is that we have simultaneously

$$x' : y', \ldots = X : Y, \ldots ; \quad x : y, \ldots = X' : Y', \ldots$$

viz., these equations must be such that either set shall imply the other set.

2. If, to fix the ideas, we attend to the case of two planes, or take the sets to be

$$x' : y' : z' = X : Y : Z ; \quad x : y : z = X' : Y' : Z',$$

[1] The coordinates (x, y) of a point in a line may be conceived as proportional to given multiples (α times, β times) of the distances of the point from two fixed points on the line; similarly the coordinates (x, y, z) of a point in a plane as proportional to given multiples (α times, β times, γ times) of the perpendicular distances of the point from three fixed lines in the plane; and the coordinates (x, y, z, w) of a point in a space as proportional to given multiples (α times, β times, γ times, δ times) of the perpendicular distances of the point from four fixed planes in the space. Observe that even if the coordinates (x, y) and (x', y') refer to the same line, and to the same two fixed points in this line, they are not of necessity the same coordinates; viz., the factors for x, y may be α, β, and those for x', y' may be α', β'. If these are proportional (viz., if $\alpha : \beta = \alpha' : \beta'$), then (x', y') will be the same coordinates of P' that (x, y) are of P; and in this case, but not otherwise, the equation $xy' - x'y = 0$ will imply the coincidence of the points P, P'. The like remarks apply to the coordinates (x, y, z) and (x, y, z, w).

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then starting with the set $x' : y' : z' = X : Y : Z$, for any given point (x, y, z) whatever in the first figure, we have a single corresponding point (x', y', z') in the second figure; but for any given point (x', y', z') in the second figure, we have *primâ facie* a system of n^2 points in the second figure, viz., these are the common points of intersection of the curves $x' : y' : z' = X : Y : Z$ (in which equations x', y', z' are regarded as given parameters, x, y, z as current coordinates, and the equations therefore represent curves of the order n in the first figure). The curves may however have only a single variable point of intersection; viz., this will be the case if each of the curves passes through the same $n^2 - 1$ fixed points (points, that is, the positions of which are independent of x', y', z'); and in order that the curves in question may each pass through the $n^2 - 1$ points, it is necessary and sufficient that these shall be common points of intersection of the curves $X = 0$, $Y = 0$, $Z = 0$. {Observe that the condition thus imposed upon the curves $X = 0$, $Y = 0$, $Z = 0$ will in certain cases imply that the curves have n^2 common intersections; or, what is the same thing, that the functions X, Y, Z are connected by an identical equation, or syzygy, $\alpha X + \beta Y + \gamma Z = 0$. This must not happen; for if it did, not only there will be no variable point of intersection, and the transformation will on this account fail; but there would also arise a relation $\alpha x' + \beta y' + \gamma z' = 0$ between (x', y', z'), contrary to the hypothesis that (x', y', z') are the coordinates of any point whatever of the second figure. It thus becomes necessary to show that there exist curves $X = 0$, $Y = 0$, $Z = 0$, satisfying the required condition of the $n^2 - 1$ common intersections, but without a remaining common intersection, or, what is the same thing, without any syzygy $\alpha X + \beta Y + \gamma Z = 0$.}

3. The curves $x' : y' : z' = X : Y : Z$ having then a single variable point of intersection, if we take (x, y, z) to be the coordinates of this point, the ratios $x : y : z$ will be determined rationally; that is, as a consequence of the first set of equations, we obtain a second set $x : y : z = X' : Y' : Z'$, where X', Y', Z' will be rational and integral functions of the same order, say n', of the coordinates (x', y', z'); that is, we have a second set of equations, and consequently a rational transformation, as mentioned above.

4. It is easy to see that we have $n' = n$; in fact, consider in the first figure a curve $\alpha X + \beta Y + \gamma Z = 0$, and an arbitrary line $ax + by + cz = 0$; to these respectively correspond, in the second figure, the line $\alpha x' + \beta y' + \gamma z' = 0$, and the curve $aX' + bY' + cZ' = 0$; the curves are of the orders n, n' respectively, or the curve and line of the first figure intersect in n points, and the line and curve of the second figure intersect in n' points; which two systems of points must correspond point to point to each other; that is, we must have $n' = n$. It will presently appear how different the analogous relation is in the transformation between two spaces.

5. Ascending to the case of two spaces, we have here the two sets

$$x' : y' : z' : w' = X : Y : Z : W; \qquad x : y : z : w = X' : Y' : Z' : W',$$

the theory is analogous; the surfaces $x' : y' : z' : w' = X : Y : Z : W$ (surfaces of the order n in the first figure) must have a single variable point of intersection, and they must therefore have a common fixed intersection equivalent to $n^3 - 1$ points of inter-

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section: I say *equivalent* to n^3-1 points, for this fixed intersection need not be n^3-1 points, but it may be or include a curve of intersection(¹). The surfaces $X=0$, $Y=0$, $Z=0$, $W=0$ must consequently have a common intersection equivalent to n^3-1 points; there is (as in the preceding case) a cause of failure to be guarded against, viz., the condition as to the intersection must not be such as to imply one more point of intersection, that is, to imply an identical equation or syzygy $\alpha X+\beta Y+\gamma Z+\delta W=0$ between the functions X, Y, Z, W; but it is assumed that they are not thus connected. There is, then, a single variable point of intersection of the surfaces $x' : y' : z' : w' = X : Y : Z : W$; or taking the coordinates of this point to be (x, y, z, w), we have the ratios $x : y : z : w$ rationally determined; that is, we have a second set of equations $x : y : z : w = X' : Y' : Z' : W'$, where X', Y', Z', W' are rational and integral functions of the same order, say n', in the coordinates (x', y', z', w'); viz., we have the rational transformation, as above, between the two spaces.

6. Suppose that the common intersection of the surfaces $X=0$, $Y=0$, $Z=0$, $W=0$ is or includes a curve of the order ν; and consider in the first figure the two surfaces

$$\alpha X+\beta Y+\gamma Z+\delta W=0,\quad \alpha_1 X+\beta_1 Y+\gamma_1 Z+\delta_1 W=0,$$

and the arbitrary plane $ax+by+cz+dw=0$. The two surfaces intersect in the fixed curve ν, and in a residual curve of the order $n^2-\nu$; hence the two surfaces and the plane meet in ν points on the fixed curve, and in $n^2-\nu$ other points. Corresponding to the surfaces and plane in the first figure, we have in the second figure the two planes

$$\alpha x'+\beta y'+\gamma z'+\delta w'=0,\quad \alpha_1 x'+\beta_1 y'+\gamma_1 z'+\delta_1 w'=0,$$

and the surface $aX'+bY'+cZ'+dW'=0$ of the order n': these intersect in n' points, being a system corresponding point to point with the $n^2-\nu$ points of the first figure; that is, we must have $n'=n^2-\nu$. And conversely, it follows that in the second figure the common intersection of the surfaces $X'=0$, $Y'=0$, $Z'=0$, $W'=0$ will be or include a curve of the order ν'; and that we shall have $n=n'^2-\nu$. Hence also

$$\nu-\nu'=(n-n')(n+n'+1).$$

7. The principle of the rational transformation comes out more clearly in the foregoing two cases than in the case of two lines, which from its very simplicity fails to exhibit the principle so well; and I have accordingly postponed the consideration of it: but the theory is similar to that of the foregoing cases. We must have the two sets (each a single equation) $x' : y' = X : Y$, and $x : y = X' : Y'$. The equation $x' : y' = X : Y$ must give for the ratio $x : y$ a single variable value; viz., there must be $n-1$ constant values (values, that is, independent of x', y'); this can only be the case by reason of the functions having a common factor M of the order $n-1$; but this being so, the common factor divides out, and the equation assumes the form $x' : y' = X : Y$, where X, Y are linear functions of (x, y): and we have then reciprocally

¹ The curve of intersection may consist of distinct curves, each or any of which may be a singular curve of any kind in regard to the several surfaces.

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$x : y = X' : Y'$, where X', Y' are linear functions of (x', y'). Thus in the present case, instead of an infinity of transformations for different values of n, n', we have only the well-known homographic transformation wherein $n = n' = 1$.

8. In the discussion of the foregoing cases of the transformation between two planes and two spaces, it was tacitly assumed that n was greater than 1, and the transformations considered were thus different from the homographic transformation; but it is hardly necessary to remark that the homographic transformation applies to these cases also; viz., for two planes we may have $x' : y' : z' = X : Y : Z$, and $x : y : z = X' : Y' : Z'$, where (X, Y, Z), (X', Y', Z') are linear functions of the two sets of coordinates respectively; and similarly for two spaces $x' : y' : z' : w' = X : Y : Z : W$ and $x : y : z : w = X' : Y' : Z' : W'$, where (X, Y, Z, W), (X', Y', Z', W') are linear functions of the two sets of coordinates respectively. We may, if we please, separate off the homographic transformation (as between two lines, planes, and spaces respectively), and restrict the notion of the rational transformation to the higher or non-linear transformations; in this point of view, the case of two lines would not be considered at all, but the theory of the rational transformation would begin with the case of the two planes. Such severance of the theory is, however, somewhat arbitrary; and moreover the homographic transformation between two lines (being, as mentioned, the only rational transformation) is analogous not only to the homographic transformation between two planes, and to the homographic transformation between two spaces, but it is also analogous to the lineo-linear (or quadric) transformation between two planes, and to the lineo-linear (which is a cubic) transformation between two spaces.

9. For the sake of bringing out this analogy, I shall consider in some detail the homographic transformation between two lines; but as regards the homographic transformations between two planes and between two spaces respectively (although there is room for a like discussion) the theories may be considered as substantially known, and I do not propose to go into them.

The Homographic Transformation between Two Lines.

10. By what precedes, it appears that we have $x' : y' = X : Y$, where (X, Y) are linear functions of (x, y); and conversely, $x : y = X' : Y'$, where X', Y' are linear functions of (x', y'); or what is the same thing, the relation is expressed by a single equation

$$(ax + by)\,x' + (cx + dy)\,y' = 0\,;$$

or, as this may conveniently be written,

$$\left(\begin{matrix} a, & b \\ c, & d \end{matrix}\right)\!\!\Big(x,\ y\Big)(x',\ y') = 0,$$

or, when the expression of the actual values of the coefficients is unnecessary,

$$(*\,\big)(x,\ y)(x',\ y') = 0.$$

We thus see that the rational transformation between two lines is in fact the homographic transformation; and also that it is the lineo-linear transformation.

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11. A special case is when

$$ad - bc = 0.$$

Writing here

$$\frac{c}{a} = \frac{d}{b} = m, = \frac{b'}{a'},$$

the equation is

$$(ax + by)(\;x' + my') = 0,$$

that is

$$(ax + by)(a'x' + b'y') = 0;$$

viz., either $ax + by = 0$, without any relation between x', y'; or else $a'x' + b'y' = 0$, without any relation between x, y; that is, to the single point $ax + by = 0$ of the first figure there corresponds any point whatever of the second figure; and to the single point $a'x' + b'y' = 0$ of the second figure there corresponds any point whatever of the first figure.

12. In the general case where $ad - bc \neq 0$, we may either by a linear transformation ($ax + by$, $cx + dy$ into y, $-x$ or into x, $-y$) of the coordinates of a point of the first figure, or by a linear transformation ($ax' + cy'$, $bx' + dy'$ into y', $-x'$ or into x', $-y'$) of the coordinates of a point in the second figure (or in a variety of ways by simultaneous linear transformations of the two sets of coordinates) transform the relation indifferently into either of the forms $xy' - x'y = 0$, $xx' - yy' = 0$; the former of these, or $x' : y' = x : y$, is the most simple expression of the homographic transformation; the latter, or $x' : y' = \frac{1}{x} : \frac{1}{y}$, is its expression as an inverse transformation.

13. If, to fix the precise signification of the coordinates (x, y), we employ the distances from a fixed point O in the line; taking the distances of the two fixed points (say A, B) to be α, β, and that of the variable point P to be ρ, then we have x, y proportional to given multiples $p(\rho - \alpha)$, $q(\rho - \beta)$ of the distances from the two fixed points; or writing $\frac{p}{q} = n$, we may say that the coordinate $\frac{x}{y}$ of the point P is $= n\frac{\rho - \alpha}{\rho - \beta}$; or in particular, if $n = 1$, then the coordinate is $= \frac{\rho - \alpha}{\rho - \beta}$. If for $n\frac{\rho - \alpha}{\rho - \beta}$ we write $\frac{-\beta\rho + \lambda}{\rho - \beta}$, and then take $\beta = \infty$, we see that in a particular system of coordinates, A at O, B at ∞, the coordinate $\frac{x}{y}$ is $= \rho$. Proceeding in the same manner in regard to the coordinates (x', y'), for a particular system of coordinates, A' at O', B' at ∞, the coordinate $\frac{x'}{y'}$ of P' will be $= \rho'$. And the correspondence of the points P, P' will be given by an equation

$$a\rho\rho' + b\rho' + c\rho + d = 0.$$

14. The equation just mentioned is often convenient for obtaining a precise statement of theorems. Thus taking A, B at pleasure on the first line, A', B' the corresponding points on the second line, we have

$$\rho' = -\frac{c\rho + d}{a\rho + b},$$

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and thence

$$\alpha' = -\frac{c\alpha + d}{a\alpha + b},$$

$$\beta' = -\frac{c\beta + d}{a\beta + b},$$

$$\rho' - \alpha' = \frac{(ad - bc)(\rho - \alpha)}{(a\alpha + b)(a\rho + b)},$$

$$\rho' - \beta' = \frac{(ad - bc)(\rho - \beta)}{(a\beta + b)(a\rho + b)};$$

and consequently

$$\frac{\rho' - \alpha'}{\rho' - \beta'} = \frac{a\beta + b}{a\alpha + b}\,\frac{\rho - \alpha}{\rho - \beta},$$

which is of the form

$$\frac{\rho' - \alpha'}{\rho' - \beta'} = m\,\frac{\rho - \alpha}{\rho - \beta},$$

where (the correspondence $a\rho\rho' + b\rho' + c\rho + d = 0$ being given, and also the fixed points A, B) m has a determinate value not assumable at pleasure. If, however, the fixed points A, B be not given, then we may determine a relation between them, such that m shall have any given value not being $=1$; we have in fact only to write

$$a\beta + b = m(a\alpha + b),$$

that is

$$a(\beta - m\alpha) + b(1 - m) = 0,$$

($m = 1$ would give $\alpha = \beta$ and the transformation would fail). In particular we may write $m = -1$, we have then

$$a(\alpha + \beta) + 2b = 0;$$

or the sum of the two distances OA, OB has a given value $= -\dfrac{2b}{a}$ dependent on the transformation; one of these points being assumed at pleasure, the other is known: the points A', B' are also known, and the equation of correspondence is

$$\frac{\rho' - \alpha'}{\rho' - \beta'} + \frac{\rho - \alpha}{\rho - \beta} = 0;$$

it is moreover easy to show that we have

$$a(\alpha' + \beta') + 2c = 0.$$

15. In what precedes, the two lines are considered as distinct lines, not of necessity existing in a common space. But they may be considered, not only as existing in the common space, but as superimposed the one on the other. Suppose this is so, and moreover that the fixed points A', B' coincide with A, B respectively, and that the coordinates (x, y) and (x', y') are the same coordinates; so that the equation $xy' - x'y = 0$ will imply the coincidence of the points P, P'.

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16. If $ad-bc=0$, the equation of correspondence becomes

$$(ax+by)(a'x'+b'y')=0,$$

and as before, to a single given point $ax+by=0$, considered as belonging to the first figure, there corresponds every point whatever of the line, or second figure: to a single given point $a'x'+b'y'=0$ (the same as, or different from, the first point), considered as belonging to the second figure, there corresponds every point whatever of the line, or first figure.

17. Excluding the foregoing case, or assuming $ad-bc \neq 0$, there are in general on the line two points such that to each of them considered as belonging to either figure there corresponds the same point considered as belonging to the other figure, or say there are two united points: in fact, writing $x' : y' = x : y$, we find $ax^2+(b+c)xy+dy^2=0$, a quadric equation for the determination of the points in question. Unless $4ad-(b+c)^2=0$, this equation will have two unequal roots; and taking the two points so determined for the fixed points $A=A'$, $B=B'$, the equation of correspondence will assume the form $xy'-kx'y=0$. In this equation k cannot be $=1$; for if it were so, the equation would be $xy'-x'y=0$; that is, the points P, P' would be always one and the same point. The equation may, however, be $xy'+x'y=0$; the points P, P' are then harmonics in regard to the fixed points A, B. It is to be observed, that if the equation $xy'-kx'y=0$ be unaltered by the interchange of (x, y) and (x', y') we must have $k^2-1=0$, or since $=1$ is excluded, we must have $k=-1$.

18. The original equation $(ax+by)x'+(cx+dy)y'=0$ is unaltered by the interchange, only if $b-c=0$; the equation $4ad-(b+c)^2=0$ becomes in this case $ad-bc=0$, which by hypothesis is not satisfied; the two distinct points $A=A'$, $B=B'$ consequently exist. That is, if the correspondence between the two points P, P' is such that whether P be considered as belonging to the first figure or to the second figure, there corresponds to it in the other figure the same point P'—or say if the correspondence between the points P, P' is a symmetrical correspondence—then as united points in the superimposed figures we have the two distinct points A, B: and the correspondence of the points P, P' is given by the condition that these are harmonics in regard to the points A, B.

19. There is still the case to be considered where $4ad-(b+c)^2=0$; the equation $ax^2+(b+c)xy+dy^2=0$ has here equal roots, or the two united points coincide together, or form a single point. Taking this point to be the point A, the coordinate whereof is $x : y = 0 : 1$, we must, it is clear, have $d=0$, and therefore also $b+c=0$: the relation between the coordinates (x, y) and (x', y') is then $axx'+b(xy'-x'y)=0$; viz., this is the form assumed by the equation of correspondence when instead of two united points there is a double united point, and this is taken to be the fixed point A.

20. It is to be observed, that we cannot have either $b=0$, for this would give $xx'=0$, which belongs to the excluded case $ad-bc=0$; nor $a=0$, for this would give $xy'-x'y=0$: excluding these cases, the equation is of necessity altered by the inter-

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change of (x, y) and (x', y'); that is, in the case of a double united point, the transformation is essentially unsymmetrical.

By what precedes, if the other fixed point be taken to be at infinity, the coordinates $x : y$ and $x' : y'$ may be taken to be ρ, ρ' respectively; viz., ρ, ρ' will denote the distances of the points P, P' from the double united point A; and the equation of correspondence then becomes $\rho\rho' + b(\rho - \rho') = 0$; that is, $(\rho - b)(\rho' + b) + b^2 = 0$.

21. The original equation $axx' + byx' + cxy' + dyy' = 0$ can be reduced to the inverse form $xx' - yy' = 0$ only (it is clear) in the symmetrical case $b = c$; here, transforming to the united points, the equation is, by what precedes (*ante*, No. 17) $xy' + x'y = 0$. This equation can be written $(lx + my)(lx' + my') - (lx - my)(lx' - my') = 0$, where $l : m$ is arbitrary; viz., we have thus an equation of the required form.

22. In further explanation, start from the equation $a\rho\rho' + b(\rho + \rho') + d = 0$; that is, $(a\rho + b)(a\rho' + b) + ad - b^2 = 0$, or say $(\rho - \alpha)(\rho' - \alpha) - k^2 = 0$; this may be reduced to $\rho\rho' - 1 = 0$; viz., the point O from which are measured the distances ρ, ρ' is here the mid-point between the two united points A, B; and the unit of distance is $\frac{1}{2}AB$; the equation expresses that the points P, P', harmonics in regard to the two points A, B, are the images one of the other in regard to the circle described upon AB as diameter. Take any two corresponding points L, L'; if the distances of these be λ, λ', we have $\lambda\lambda' = 1$; and hence

$$(\rho - \lambda)(\rho' - \lambda) = 1 - \lambda(\rho + \rho') + \lambda^2 = \lambda(\lambda + \lambda' - \rho - \rho'),$$
$$(\rho - \lambda')(\rho' - \lambda') = 1 - \lambda'(\rho + \rho') + \lambda'^2 = \lambda'(\lambda + \lambda' - \rho - \rho');$$

and consequently

$$\frac{\rho - \lambda}{\rho - \lambda'} \cdot \frac{\rho' - \lambda}{\rho' - \lambda'} = \frac{\lambda}{\lambda'};$$

which, writing

$$\frac{x}{y} = \frac{k(\rho - \lambda)}{\rho - \lambda'}, \quad \frac{x'}{y'} = \frac{k(\rho' - \lambda)}{\rho' - \lambda'},$$

$$k^2 = \frac{\lambda}{\lambda'} \text{ (so that } k^2 \neq 1\text{); or, } k = \lambda = \frac{1}{\lambda'},$$

becomes $xx' - yy' = 0$; that is, the correspondence of the points P, P' being symmetrical, if the coordinate $\frac{x}{y}$ of P be taken to be a multiple of the ratio of the distances PL, PL' of P from any two corresponding points L, L' (and of course the coordinate $\frac{x'}{y'}$ of P' to be the same multiple of the ratio of the distances $P'L$, $P'L'$), the equation of correspondence is obtained in the inverse form $xx' - yy' = 0$.

The Rational Transformation between Two Planes.

23. Starting from the equations $x' : y' : z' = X : Y : Z$, where $X = 0$, $Y = 0$, $Z = 0$ are curves in the first plane, of the same order n, it has been seen that in order that we may thence have a rational transformation between the two planes, the curves

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$X=0$, $Y=0$, $Z=0$ must have a common intersection of n^2-1 points, and no more; that is, they must not have a complete common intersection of n^2 points. In the case $n=2$, taking the n^2-1 points in the first plane to be any three points whatever, the condition that the curves shall be conics passing through the three points does not in anywise imply that the conics shall have a common fourth point of intersection; and we have thus a rational transformation as required; viz., the first set of equations is $x' : y' : z' = X : Y : Z$, where $X=0$, $Y=0$, $Z=0$ are conics passing through the same three points of the first plane; and as it is easy to see (but which will be subsequently shown more in detail), the second set is the similar one $x : y : z = X' : Y' : Z'$, where $X'=0$, $Y'=0$, $Z'=0$ are conics passing through the same three points in the second plane; this may be called the quadric transformation between the two planes.

24. But the like theory would not apply to the case $n=3$; if the n^2-1 points in the first plane were any eight points whatever, the cubics $X=0$, $Y=0$, $Z=0$, intersecting in these eight points, would have a common ninth point of intersection, and the transformation would fail; and so for any higher value of n, taking at pleasure any $\frac{1}{2}n(n+3)-1$ of the n^2-1 points of the first plane, the curves $X=0$, $Y=0$, $Z=0$ of the order n passing through these $\frac{1}{2}n(n+3)-1$ points, would have in common all their remaining points of intersection, and the transformation would fail. A transformation can only be obtained by taking the n^2-1 points in such wise that these can be made to be the common intersection of the curves, and at the same time that the number of conditions imposed upon each of the curves $X=0$, $Y=0$, $Z=0$ shall be at most $=\frac{1}{2}n(n+3)-1$.

25. And this requirement may be satisfied; viz., the number of conditions may be made to be $=\frac{1}{2}n(n+3)-1$, by assuming that certain of the n^2-1 points of intersections are multiple intersections of the curves. For if we have a given point which is an α-tuple point on each of the curves $X=0$, $Y=0$, $Z=0$, then this counts for α^2 points of intersection of any two of the curves, and thus for α^2 points of the n^2-1 points: but the condition that the given point shall be on any one of the curves, say the curve $X=0$, an α-tuple point, imposes on the curve, not α^2, but only $\frac{1}{2}\alpha(\alpha+1)$ conditions: and we have in this way a reduction whereby the number of conditions for passing through the n^2-1 points can be lowered from n^2-1 to the required number $\frac{1}{2}n(n+3)-1$.

26. In particular, for $n=3$, we may for the n^2-1 points of the first plane take a point as a double point on each of the cubic curves $X=0$, $Y=0$, $Z=0$ (which therefore reckons as four points), and take any other four points. Each of the curves is determined by the conditions of having a given point for double point, and of passing through the same four other given points; that is, by $3+4=7$ conditions; and the three cubic curves $X=0$, $Y=0$, $Z=0$ have for the common intersection the double point reckoning as four points, and the given other four points; that is, they have a common intersection of $4+4=8$ points; but this does not imply that they have a common ninth point of intersection; we have therefore a rational transformation as required; viz., the first set of equations is $x' : y' : z' = X : Y : Z$; where $X=0$, $Y=0$, $Z=0$ are cubics in the first plane having each of them a double point at the same given point and

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also each passing through the same four given points; the second set of equations is $x : y : z = X' : Y' : Z'$, where $X' = 0$, $Y' = 0$, $Z' = 0$ are like cubics in the second plane.

27. Generally suppose that the $n^2 - 1$ points in the first plane are made up of α_1 points, which are simple points; α_2 points, which are double points; α_3 points, which are triple points, ... α_{n-1} points, which are $(n-1)$tuple points ($\alpha_{n-1} = 1$ or 0), on each of the three curves; these will represent a system of $n^2 - 1$ points if only

$$\alpha_1 + 4\alpha_2 + 9\alpha_3 \ldots \quad + \quad (n-1)^2 \alpha_{n-1} = n^2 - 1.$$

The number of conditions imposed on each of the curves $X = 0$, $Y = 0$, $Z = 0$ will be $\alpha_1 + 3\alpha_2 + 6\alpha_3 \ldots + \frac{1}{2}n(n-1)\alpha_{n-1}$; for the reason presently appearing, I exclude the case of this being $< \frac{1}{2}n(n+3) - 2$; and therefore assume it to be $= \frac{1}{2}n(n+3) - 2$. In fact, writing

$$\alpha_1 + 3\alpha_2 + 6\alpha_3 \ldots \quad + \tfrac{1}{2}n(n-1) \ \alpha_{n-1} = \tfrac{1}{2}n(n+3) - 2,$$

this combined with the former equation gives

$$\alpha_2 + 3\alpha_3 \ldots + \tfrac{1}{2}(n-1)(n-2) \ \alpha_{n-1} = \tfrac{1}{2}(n-1)(n-2);$$

viz., the singularities are equivalent to $\frac{1}{2}(n-1)(n-2)$ double points, that is, to the maximum number of double points of a curve of the order n; or say each of the curves $X = 0$, $Y = 0$, $Z = 0$ is a curve of the order n having a deficiency $= 0$; that is, it is a unicursal curve of the order n. Hence also, taking (a, b, c) any constant factors whatever, the curve $aX + bY + cZ = 0$ is unicursal.

28. It is important to remark that the conclusion follows directly from the general notion of the rational transformation; in fact, the equation $aX + bY + cZ = 0$ is satisfied if $x : y : z = X' : Y' : Z'$; $ax' + by' + cz' = 0$. The last of these equations determines the ratios $x' : y' : z'$ in terms of a single parameter (e.g. the ratio $x' : y'$), and we have then $x : y : z$ expressed as rational functions of this parameter; that is, the curve is unicursal.

29. Suppose for a moment that it was possible to have

$$\alpha_1 + 3\alpha_2 + 6\alpha_3 \ldots + \qquad \tfrac{1}{2}n(n-1)\alpha_{n-1} < \tfrac{1}{2}n(n+3) - 2.$$

Combining in the same way with the first equation, it would follow that

$$\alpha_2 + 3\alpha_3 \ldots + \tfrac{1}{2}(n-1)(n-2) \ \alpha_{n-1} > \tfrac{1}{2}(n-1)(n-2),$$

which would imply that the curves $X = 0$, $Y = 0$, $Z = 0$ break up each of them into inferior curves: but more than this, the coefficients a, b, c being arbitrary, it would imply that the curve $aX + bY + cZ = 0$ breaks up into inferior curves; this can only be the case if X, Y, Z have a common factor, say M; that is, if $X, Y, Z = MX_1, MY_1, MZ_1$; but we could then omit the common factor, and in place of $x' : y' : z' = X : Y : Z$ write $x' : y' : z' = X_1 : Y_1 : Z_1$, where $X_1 = 0$, $Y_1 = 0$, $Z_1 = 0$, are proper curves, not breaking up; the above supposition may therefore be excluded from consideration.

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30. We have thus a transformation in which the first set of equations is $x' : y' : z' = X : Y : Z$, where $X=0$, $Y=0$, $Z=0$ are curves in the first plane, of the same order n, having in common $\alpha_1, \alpha_2 \ldots \alpha_{n-1}$ points which are simple points, double points,... $(n-1)$tuple points respectively on each of the curves; these numbers satisfy the conditions

$$\alpha_1 + 4\alpha_2 + 9\alpha_3 \ldots + \quad (n-1)^2 \alpha_{n-1} = n^2 - 1,$$

$$\alpha_1 + 3\alpha_2 + 6\alpha_3 \ldots + \quad \tfrac{1}{2}n(n-1)\ \alpha_{n-1} = \tfrac{1}{2}(n^2 + 3n) - 2\,;$$

conditions which give, as above,

$$\alpha_2 + 3\alpha_3 \ldots + \tfrac{1}{2}(n-1)(n-2)\,\alpha_{n-1} = \tfrac{1}{2}(n-1)(n-2),$$

and also

$$\alpha_1 + 2\alpha_2 + 3\alpha_3 \ldots + \quad (n-1)\,\alpha_{n-1} = 3n - 3\,;$$

so that the relations between $\alpha_1, \alpha_2 \ldots \alpha_{n-1}$ are given by any two of these four equations.

31. The second set of equations then is $x : y : z = X' : Y' : Z'$, where $X'=0$, $Y'=0$, $Z'=0$ are curves in the second plane, of the same order n; and it is clear that these must be curves such as those in the first plane; viz., they must have in common $\alpha_1', \alpha_2', \ldots \alpha'_{n-1}$ points, which are simple points, double points, ... $(n-1)$tuple points respectively on each of the curves, the relations between these numbers being expressed by any two of the four equations

$$\alpha_1' + 4\alpha_2' + 9\alpha_3' \ldots + \quad (n-1)^2 \alpha'_{n-1} = n^2 - 1,$$

$$\alpha_1' + 3\alpha_2' + 6\alpha_3' \ldots + \quad \tfrac{1}{2}n(n-1)\ \alpha'_{n-1} = \tfrac{1}{2}n(n+3) - 2,$$

$$\alpha'_2 + 3\alpha_3' \ldots + \tfrac{1}{2}(n-1)(n-2)\,\alpha'_{n-1} = \tfrac{1}{2}(n-1)(n-2),$$

$$\alpha_1' + 2\alpha_2' + 3\alpha_3' \ldots + \quad (n-1)\ \alpha'_{n-1} = 3n - 3.$$

32. To any line $ax' + by' + cz' = 0$ in the second plane there corresponds in the first plane a curve $aX + bY + cZ$ of the order n; and to any line $a'x + b'y + c'z = 0$ in the first plane there corresponds in the second plane a curve $a'X' + b'Y' + c'Z' = 0$ of the same order n; the curves $aX + bY + cZ = 0$ in the first plane are, it is clear, a system, and the entire system, of curves each satisfying the conditions which have been stated in regard to the individual curves $X=0$, $Y=0$, $Z=0$, and being as already mentioned unicursal; and similarly the curves $a'X' + b'Y' + c'Z' = 0$ in the second plane are a system, and the entire system, of curves each satisfying the conditions which have been stated in regard to the individual curves $X'=0$, $Y'=0$, $Z'=0$; and being also unicursal. We may say that to the lines of the second plane there corresponds in the first plane the *réseau* of curves $aX + bY + cZ = 0$; and to the lines of the first plane there corresponds in the second plane the *réseau* of curves $a'X' + b'Y' + c'Z' = 0$; these *réseau* being systems satisfying respectively the conditions just referred to.

33. We have next to enquire what are the curves in the second plane which correspond to the $\alpha_1 + \alpha_2 \ldots + \alpha_{n-1}$ points of the first plane. I remark that the $\alpha_1 + \alpha_2 \ldots + \alpha_{n-1}$ points are termed by Cremona the *principal points* of the first plane, and the corresponding curves the *principal curves* of the second plane. But it will be

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more convenient to say that the $\alpha_1 + \alpha_2 \ldots + \alpha_{n-1}$ points are the *principal system* of the first plane, and the corresponding curves the *principal counter-system* of the second plane. And of course the $\alpha_1' + \alpha_2' \ldots + \alpha'_{n-1}$ points will be the principal system of the second plane, and the corresponding curves the principal counter-system of the first plane.

34. The Jacobian (curve) of the curves $X=0$, $Y=0$, $Z=0$ is, of course, the Jacobian of any three curves $aX+bY+cZ=0$ of the first plane, or it may be called the Jacobian of the reseau of the first plane; and similarly, the Jacobian of the curves $X'=0$, $Y'=0$, $Z'=0$ is the Jacobian of the reseau of the second plane.

35. I say that to each point α_1 of the first figure there corresponds in the second figure a line; to each point α_2 a conic; to each point α_3 a nodal cubic; ... and generally, to each point α_r a unicursal r-thic curve; the entire system of the curves corresponding to the $\alpha_1 + \alpha_2 + \alpha_3 \ldots + \alpha_{n-1}$ points, that is, the principal counter-system of the second plane, is thus made up of α_1 lines, α_2 conics, α_3 nodal cubics, ... α_r unicursal r-thics, ... α_{n-1} unicursal $(n-1)$thics. It is thus a curve of the aggregate order $\alpha_1 + 2\alpha_2 + 3\alpha_3 \ldots + (n-1)\alpha_{n-1}, = 3n-3$; and it is in fact the Jacobian of the reseau of the second plane; as such, it passes through each point α_1' two times, each point α_2' five times, ... each point α_r' $3r-1$ times, ... each point α'_{n-1} $3n-4$ times.

36. The reciprocal theorem is of course true. The Jacobian of the reseau of the first plane is thus made up of α_1' lines, α_2' conics, α_3' nodal cubics, ... α_r' unicursal r-thics, ... α'_{n-1} unicursal $(n-1)$thics. Calculating the Jacobian of the reseau of the first plane, we have thus the numbers $\alpha_1', \alpha_2', \ldots \alpha'_{n-1}$, which determine the nature of the principal system of the second plane.

37. I indicate as follows the analytical proof of the theorem that to a principal point α_r of the first plane there corresponds in the second plane a unicursal r-thic. Consider the simplest case, $r=1$; if in the equations $x' : y' : z' = X : Y : Z$ the coordinates (x, y, z) are considered as belonging to a point α_1, these values give identically $X=0$, $Y=0$, $Z=0$; hence for the consecutive point $x+\delta x$, $y+\delta y$, $z+\delta z$, if (A, B, C) denote the derived functions of X, (A_1, B_1, C_1) those of Y, (A_2, B_2, C_2) those of Z, we have

$$\begin{aligned} x' : y' : z' = \; & A\,\delta x + B\,\delta y + C\,\delta z \\ : \; & A_1\delta x + B_1\delta y + C_1\delta z \\ : \; & A_2\delta x + B_2\delta y + C_2\delta z. \end{aligned}$$

We have $\begin{vmatrix} A, & B, & C \\ A_1, & B_1, & C_1 \\ A_2, & B_2, & C_2 \end{vmatrix} = 0$, for the determinant is the value, at the point α_1 in question, of the Jacobian of the reseau of the first plane; and the Jacobian curve passing through α_1 (in fact, having there a double point), the value is $= 0$.

38. Hence x', y', z', considered as corresponding to a point indefinitely near to α_1, are connected by a linear equation. Corresponding to α_1 we have in the second figure

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a line. But it is to be observed, further, that the equation of the line is that obtained by writing in the foregoing equations, say $\delta z = 0$, and eliminating the remaining quantities δx, δy: or, what is the same thing, we may consider the equation of the line as given by the equations

$$\begin{aligned} x' : y' : z' = \ & A\,\delta x + B\,\delta y \\ & : A_1\delta x + B_1\delta y \\ & : A_2\delta x + B_2\delta y, \end{aligned}$$

where δx, δy are indeterminate parameters to be eliminated.

39. In the case of a point α_r we have in like manner

$$\begin{aligned} x' : y' : z' = \ & (a\ , \ldots \between \delta x,\ \delta y,\ \delta z)^r \\ & : (a_1, \ldots \between \delta x,\ \delta y,\ \delta z)^r \\ & : (a_2, \ldots \between \delta x,\ \delta y,\ \delta z)^r, \end{aligned}$$

where $(a, \ldots)$, $(a_1, \ldots)$, $(a_2, \ldots)$ are the r-th derived functions of X, Y, Z respectively. In virtue of the relation of the point α_r to the curves $X = 0$, $Y = 0$, $Z = 0$, the coefficients will be such as to allow of the simultaneous elimination from these equations of the three quantities δx, δy, δz. The result of the elimination will be the same as if, writing say $\delta z = 0$, we eliminate δx, δy; or, what is the same thing, the relation of x', y', z' may be regarded as given by the equations

$$\begin{aligned} x' : y' : z' = \ & (a\ , \ldots \between \delta x,\ \delta y)^r \\ & : (a_1, \ldots \between \delta x,\ \delta y)^r \\ & : (a_2, \ldots \between \delta x,\ \delta y)^r, \end{aligned}$$

where δx, δy are indeterminate parameters. These equations obviously express that the point (x', y', z') is situate on a unicursal curve of the order r.

40. It is further to be shown that the r-thic curve thus corresponding to α_r is part of the Jacobian of the reseau of the second plane. The Jacobian in question is the locus of the new double point of those curves of the reseau which have a new double point; that is, a double point not included among the $\alpha_2' + \alpha_3' \ldots + \alpha'_{n-1}$ singular points of the principal system of the second plane. But a curve of the reseau being unicursal, can only acquire a new double point by breaking up into inferior curves. Consider, in the first figure, any line through α_r, the corresponding curve in the second figure is made up of the unicursal r-thic curve, which corresponds to the point α_r, together with a residual curve variable with the line through α_r; this is a unicursal curve of the order $n - r$. The aggregate curve of the order $r + (n - r)$ has singular points equivalent to $\frac{1}{2}(n-1)(n-2)+1$ double points[1]; that is, the singularities are those belonging to the principal system of the second plane, together with a new double

[1] In general, if $r + r' = n$, and the curves r, r' are each unicursal, then the aggregate singularity arising from the singularities of the two curves and from their intersections, is equivalent to $\frac{1}{2}(r-1)(r-2) + \frac{1}{2}(r'-1)(r'-2) + rr'$, that is, to $\frac{1}{2}(r+r'-1)(r+r'-2)+1$, or $\frac{1}{2}(n-1)(n-2)+1$ double points.

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point constituted by an intersection of the curves r, $n-r$. {Observe that the two curves have only this single intersection; viz., the remaining $r(n-r)-1$ intersections are at points $\alpha_2'+\alpha_3' \ldots + \alpha'_{n-1}$ of the principal system of the second plane.} We have thus, in the second plane, a series of curves, each of them having a new double point; viz., these are the several curves which correspond to the lines through α_r in the first figure. Each of the curves is a fixed curve r together with a variable curve $n-r$. The new double point is an intersection of the two curves; that is, it is a variable point on the curve r. The locus of the new double point is thus the curve r; therefore the curve r is part of the Jacobian of the reseau of the second plane. Since each point α_r gives a curve r, the curves in question form an aggregate curve of the order $\alpha_1 + 2\alpha_2 \ldots + (n-1)\alpha_{n-1}, = 3n-3$; viz., this is the order of the Jacobian; or, as stated, the curves r (that is, the principal counter-system of the second plane) constitute the Jacobian of the reseau of this plane.

41. The numerical systems $(\alpha_1, \alpha_2 \ldots \alpha_{n-1})$ and $(\alpha_1', \alpha_2' \ldots \alpha'_{n-1})$ are each of them a solution of the same two indeterminate equations

$$\Sigma r^2 \alpha_r = n^2 - 1, \quad \Sigma r \alpha_r = 3n - 3,$$

but not every solution of these equations is admissible; for instance, if $r > \frac{1}{2}n$, then α_r is $=0$ or 1, for $\alpha_r = 2$ would imply a curve of the order n with two r-tuple points, and the line joining these would meet the curve in more than r points; similarly, $r > \frac{2}{5}n$, α_r is $=4$ at most, for $\alpha_r = 5$ would imply a curve of the order n with five r-tuple points, and the conic through these would meet the curve in more than $2n$ points; and there are of course other like restrictions. The different admissible systems up to $n=10$ are tabulated in Cremona's Memoir; and he has also given systems belonging to certain specified forms of n: these results are as follows:

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n	2	3	4		5			6				7				
a_1	3	4	6	3	8	3	0	10	1	4	3	12	2	0	5	3
a_2		1	0	3	0	3	6	0	4	1	4	0	3	3	0	5
a_3			1	0	0	1	0	0	2	3	0	0	2	4	3	0
a_4					1	0	0	0	0	0	1	0	1	0	1	0
a_5								1	0	0	0	0	0	0	0	1
a_6												1	0	0	0	0
a_7																
a_8																
a_9																
	1	1	1	2	1	2	3	1	2	3	4	1	2	3	4	5

n	8									9									
a_1	14	3	1	0	3	6	0	2	3	16	4	2	0	3	7	1	3	0	1
a_2	0	2	3	0	6	0	5	0	3	0	1	3	4	7	0	4	0	3	1
a_3	0	3	2	7	0	1	2	5	0	0	4	1	0	0	0	3	4	3	3
a_4	0	0	2	0	0	3	0	1	3	0	0	2	4	0	3	0	1	1	3
a_5	0	1	0	0	0	0	1	0	0	0	0	1	0	0	1	0	1	1	0
a_6	0	0	0	0	1	0	0	0	0	0	1	0	0	0	0	1	0	0	0
a_7	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
a_8										1	0	0	0	0	0	0	0	0	0
a_9																			
	1	2	3	4	5	6	7	8	9(1)	1	2	3	4	5	6	7	8	9	10

n	10																
a_1	18	5	1	0	0	3	8	2	4	1	2	3	3	3	0	0	1
a_2	0	0	4	2	0	8	0	3	0	3	1	3	3	0	6	1	0
a_3	0	5	0	2	7	0	0	4	3	2	3	0	1	0	0	5	2
a_4	0	0	2	3	0	0	1	0	2	2	1	3	0	6	0	0	5
a_5	0	0	2	1	0	0	3	0	0	0	2	0	3	0	3	2	0
a_6	0	0	0	0	1	0	0	0	1	1	0	1	0	0	0	0	0
a_7	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
a_8	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
a_9	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17

[1] Omitted by Cremona.

$n = p$

α_1	$=$	$2p-2$
α_{p-1}	$=$	1

$n = 2p$

α_1	$=$	3	$2p-2$
α_2	$=$	$2p-2$	0
α_{p-1}	$=$	0	1
α_p	$=$	0	3
α_{2p-2}	$=$	1	0

$n = 2p+1$

α_1	$=$	3	$2p-1$
α_2	$=$	$2p-1$	0
α_p	$=$	0	3
α_{p+1}	$=$	0	1
α_{2p-1}	$=$	1	0

$n = 3p$

α_1	$=$	1	$2p-3$
α_2	$=$	4	0
α_3	$=$	$2p-3$	0
α_p	$=$	0	4
α_{p+1}	$=$	0	1
α_{2p-1}	$=$	0	1
α_{3p-3}	$=$	1	0

α_1	$=$	4	$2p-2$
α_2	$=$	1	0
α_3	$=$	$2p-2$	0
α_{p-1}	$=$	0	1
α_p	$=$	0	4
α_{2p}	$=$	0	1
α_{3p-3}	$=$	1	0

$n = 3p+1$

α_1	$=$	2	$2p-2$
α_2	$=$	3	0
α_3	$=$	$2p-2$	0
α_p	$=$	0	3
α_{p+1}	$=$	0	2
α_{2p}	$=$	0	1
α_{3p-2}	$=$	1	0

α_1	$=$	5	$2p-1$
α_3	$=$	$2p-1$	0
α_p	$=$	0	5
α_{2p+1}	$=$	0	1
α_{3p-2}	$=$	1	0

$n = 3p+2$

α_1	$=$	3	$2p-1$
α_2	$=$	2	0
α_3	$=$	$2p-1$	0
α_p	$=$	0	2
α_{p+1}	$=$	0	3
α_{2p+1}	$=$	0	1
α_{3p-1}	$=$	1	0

α_1	$=$	0	$2p-2$
α_2	$=$	5	0
α_3	$=$	$2p-2$	0
α_{p+1}	$=$	0	5
α_{2p}	$=$	0	1
α_{3p-1}	$=$	1	0

$n = 4p$

α_1	$=$	1	$2p-3$
α_2	$=$	3	0
α_3	$=$	2	0
α_4	$=$	$2p-3$	0
α_p	$=$	0	3
α_{p+1}	$=$	0	1
α_{2p-1}	$=$	0	1
α_{2p}	$=$	0	2
α_{4p-1}	$=$	1	0

α_1	$=$	2	$2p-4$
α_3	$=$	5	0
α_4	$=$	$2p-4$	0
α_p	$=$	0	5
α_{p+1}	$=$	0	2
α_{3p-1}	$=$	0	1
α_{4p-4}	$=$	1	0

α_1	$=$	3	$2p-2$
α_2	$=$	3	0
α_4	$=$	$2p-2$	0
α_{p-1}	$=$	0	1
α_p	$=$	0	3
α_{2p}	$=$	0	3
α_{4p-4}	$=$	1	0

α_1	$=$	6	$2p-2$
α_3	$=$	1	0
α_4	$=$	$2p-2$	0
α_{p-1}	$=$	0	1
α_p	$=$	0	6
α_{3p}	$=$	0	1
α_{4p-4}	$=$	1	0

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$n = 4p + 1$

$a_1 =$	0	$2p-3$
$a_2 =$	3	0
$a_3 =$	3	0
$a_4 =$	$2p-3$	0
$a_p =$	0	1
$a_{p+1} =$	0	3
$a_{2p} =$	0	3
$a_{4p-3} =$	1	0

$a_1 =$	2	$2p-2$
$a_2 =$	3	0
$a_3 =$	1	0
$a_4 =$	$2p-2$	0
$a_p =$	0	3
$a_{p+1} =$	0	1
$a_{2p} =$	0	2
$a_{2p+1} =$	0	1
$a_{4p-3} =$	1	0

$a_1 =$	3	$2p-3$
$a_3 =$	4	0
$a_4 =$	$2p-3$	0
$a_p =$	0	4
$a_{p+1} =$	0	3
$a_{3p} =$	0	1
$a_{4p-3} =$	1	0

$a_1 =$	7	$2p-1$
$a_4 =$	$2p-1$	0
$a_p =$	0	7
$a_{3p+1} =$	0	1
$a_{4p-3} =$	1	0

$n = 4p + 2$

$a_1 =$	0	$2p-4$
$a_3 =$	7	0
$a_4 =$	$2p-4$	0
$a_{p+1} =$	0	7
$a_{3p} =$	0	1
$a_{4p-2} =$	1	0

$a_1 =$	1	$2p-2$
$a_2 =$	3	0
$a_3 =$	2	0
$a_4 =$	$2p-2$	0
$a_p =$	0	1
$a_{p+1} =$	0	3
$a_{2p} =$	0	1
$a_{2p+1} =$	0	2
$a_{4p-2} =$	1	0

$a_1 =$	3	$2p-1$
$a_2 =$	3	0
$a_4 =$	$2p-1$	0
$a_p =$	0	3
$a_{p+1} =$	0	1
$a_{2p+1} =$	0	3
$a_{4p-2} =$	1	0

$a_1 =$	4	$2p-2$
$a_3 =$	3	0
$a_4 =$	$2p-2$	0
$a_p =$	0	3
$a_{p+1} =$	0	4
$a_{3p} =$	0	1
$a_{4p-2} =$	1	0

$n = 4p + 3$

$a_1 =$	0	$2p-2$
$a_2 =$	3	0
$a_3 =$	3	0
$a_4 =$	$2p-2$	0
$a_{p+1} =$	0	3
$a_{p+2} =$	0	1
$a_{2p+1} =$	0	3
$a_{4p-1} =$	1	0

$a_1 =$	9	$2p-3$
$a_3 =$	6	0
$a_4 =$	$2p-3$	0
$a_{p+1} =$	0	6
$a_{p+2} =$	0	1
$a_{3p+1} =$	0	1
$a_{4p-1} =$	1	0

$a_1 =$	2	$2p-1$
$a_2 =$	3	0
$a_3 =$	1	0
$a_4 =$	$2p-1$	0
$a_p =$	0	1
$a_{p+1} =$	0	3
$a_{2p+1} =$	0	2
$a_{2p+2} =$	0	1
$a_{4p-1} =$	1	0

$a_1 =$	5	$2p-1$
$a_3 =$	2	0
$a_4 =$	$2p-1$	0
$a_p =$	0	2
$a_{p+1} =$	0	5
$a_{3p+2} =$	0	1
$a_{4p-1} =$	1	0

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42. The system $(\alpha_1, \alpha_2 \ldots \alpha_{n-1})$ geometrically determines completely the system $(\alpha_1', \alpha_2' \ldots \alpha'_{n-1})$; it ought therefore to determine it arithmetically; that is, given the one series of numbers, we ought to be able to determine, or at least to select from the table, the other series of numbers. Cremona has shown that the two series *consist of the same numbers in the same or a different order.* By examination of the tables, it appears that there are certain columns which are single (that is, no other column contains in a different order the same numbers), others that occur in pairs, the two columns of a pair containing the same numbers in a different order. Where the column is single, it is clear that this must give as well the values of $(\alpha_1', \alpha_2' \ldots \alpha'_{n-1})$ as of $(\alpha_1, \alpha_2 \ldots \alpha_{n-1})$. Where there is a pair of columns, as far as Cremona has examined, if the one column is taken to be $(\alpha_1, \alpha_2 \ldots \alpha_{n-1})$ the other column is $(\alpha_1', \alpha_2' \ldots \alpha'_{n-1})$; it appears, however, not to be shown that this is universally the case; viz., it is not shown but that the two columns, instead of being reckoned as a pair, might be reckoned as two separate columns, each by itself representing the values as well of $(\alpha_1, \alpha_2 \ldots \alpha_{n-1})$ as of $(\alpha_1', \alpha_2' \ldots \alpha'_{n-1})$; neither is it shown that there are not, in any case, more than two columns having the same numbers in different orders. It seems, however, natural to suppose that the law, as exhibited in the tables, holds good generally; viz., that the tables contain only single columns, each giving the values as well of $(\alpha_1, \alpha_2 \ldots \alpha_{n-1})$ as of $(\alpha_1', \alpha_2' \ldots \alpha'_{n-1})$; or else pairs of columns, one giving the values of $(\alpha_1, \alpha_2 \ldots \alpha_{n-1})$, and the other those of $(\alpha_1', \alpha_2' \ldots \alpha'_{n-1})$; or, say, that the partitions are either *sibi-reciprocal*, or else *conjugate*.

43. Assuming that the two systems $(\alpha_1, \alpha_2 \ldots \alpha_{n-1})$ and $(\alpha_1', \alpha_2' \ldots \alpha'_{n-1})$ are each known, there is still a question of grouping to be settled; viz., the Jacobian of the first plane consists of α_1' lines, α_2' conics, ... α'_{n-1} unicursal $(n-1)$-thics; each line, each conic, &c., passes a certain number of times through *certain* of the points $\alpha_1, \alpha_2 \ldots \alpha_{n-1}$: but through which of them? For instance, each of the α_1' lines will pass through two of the points $\alpha_1, \alpha_2, \ldots \alpha_{n-1}$: will these be points α_1, or points α_2, &c., or a point α_1 and a point α_2, &c.? The mere symmetry of the different groups of points determines certain conditions of the solution[1]; for instance, if any particular one of the α_1' lines passes through two points α_r, then each of the α_1' lines must pass through two points α_r; and since the points α_r are symmetrical, we must in this way use all the pairs of points α_r; that is, if $\alpha_1' = \frac{1}{2}\alpha_r(\alpha_r+1)$, but not otherwise, *it may be* that each of the α_1' lines passes through two of the points α_r. In the case of an equality $\alpha_r = \alpha_s$ we could not hereby decide whether the line passed through two points α_r or through two points α_s. So, again, if any one of the α_1' lines pass through a point α_r and a point α_s, then each of the α_1' lines must do so likewise, and we must hereby exhaust the combinations of a point α_r with a point α_s; viz., the assumed relation can only hold good if $\alpha_1' = \alpha_r\alpha_s$. Similarly, each of the α_2' conics will pass through five of the points $\alpha_1, \alpha_2 \ldots \alpha_{n-1}$; each of the α_3' nodal cubics will pass twice through one (have a double point there) and through six others of the points $\alpha_1, \alpha_2 \ldots \alpha_{n-1}$: which are the points so passed through? I do not know how a general solution is to be obtained, but most of the cases within the limits of the foregoing table have

[1] It is by such considerations of symmetry that Cremona has demonstrated the before mentioned theorem of the identity of the numbers $(\alpha_1, \alpha_2 \ldots \alpha_{n-1})$ and $(\alpha_1', \alpha_2' \ldots \alpha'_{n-1})$.

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been investigated by Cremona. The results may conveniently be stated in a tabular form; the tables exhibit in the outside upper line the values of $\alpha_1, \alpha_2 \ldots \alpha_{n-1}$, and in the outside left-hand line the values of $\alpha_1', \alpha_2' \ldots \alpha'_{n-1}$, and they are to be read as follows: Each of the $\left\{\begin{array}{l}\alpha_1' \text{ lines} \\ \alpha_2' \text{ conics} \\ \&\text{c.}\end{array}\right\}$ passes ( ) times through ( ) of the points $\alpha_1, \alpha_2 \ldots \alpha_{n-1}$ respectively; the numbers in the table being those of the points passed through, and the indices in the table (index $= 1$ when no index is expressed) showing the number of times of passage, that is, showing whether the point is a simple, double, triple, &c., point on the curve referred to.

44. Thus (in the tables which follow) the last of the tables $n = 6$ gives the constitution of the Jacobian of the first plane, where the principal system is (3, 4, 0, 1, 0); and it is to be read:

Each of the 4 lines passes through 1 of the points α_1 and through the point α_4;
The 1 conic „ „ 4 of the points α_2 and through the point α_4;
Each of the 3 cubics „ „ 2 of the points α_1, 4 of the points α_2, and twice through the point α_4 (that is, α_4 is a double point on each cubic).

It is hardly necessary to remark that the tables are sibi-reciprocal, or else conjugate, as appears by the outer lines of each table.

TABLE $n = 2$.

	$\alpha_1 = 3$
$\alpha_1' = 3$	2

[was originally printed, 3.]

TABLE $n = 3$.

	$\alpha_1 = 4$	$\alpha_2 = 1$
$\alpha_1' = 4$	1	1
$\alpha_2' = 1$	4	1

TABLES $n = 4$.

	$\alpha_1 = 6$	$\alpha_2 = 0$	$\alpha_3 = 1$
$\alpha_1' = 6$	1		1
$\alpha_2' = 0$			
$\alpha_3' = 1$	6		1^2

	$\alpha_1 = 3$	$\alpha_2 = 3$	$\alpha_3 = 0$
3		2	
3	2	3	
0			

TABLES $n = 5$.

	$a_1 = 8$	$a_2 = 0$	$a_3 = 0$	$a_4 = 1$
$a_1' = 8$	1			1
$a_2' = 0$				
$a_3' = 0$				
$a_4' = 1$	8			1^3

	$a_1 = 3$	$a_2 = 3$	$a_3 = 1$	$a_4 = 0$
3		1	1	
3	1	3	1	
1	3	3	1^2	
0				

	$a_1 = 0$	$a_2 = 6$	$a_3 = 0$	$a_4 = 0$
0				
6		5		
0				
0				

TABLES $n = 6$.

	$a_1 = 10$	$a_2 = 0$	$a_3 = 0$	$a_4 = 0$	$a_5 = 1$
$a_1' = 10$	1				1
$a_2' = 0$					
$a_3' = 0$					
$a_4' = 0$					
$a_5' = 1$	10				1^4

	$a_1 = 1$	$a_2 = 4$	$a_3 = 2$	$a_4 = 0$	$a_5 = 0$
1			2		
4		3	2		
2	1	4	$1^2, 1$*		
0					
0					

	$a_1 = 4$	$a_2 = 1$	$a_3 = 3$	$a_4 = 0$	$a_5 = 0$
$a_1' = 3$			2		
$a_2' = 4$	1	1	3		
$a_3' = 0$					
$a_4' = 1$	4	1	3^2		
$a_5' = 0$					

	$a_1 = 3$	$a_2 = 4$	$a_3 = 0$	$a_4 = 1$	$a_5 = 0$
4	1			1	
1		4		1	
3	2	4		1^2	
0					
0					

* Read, "Each of the two cubics passes through the point a_1, the four points a_2, and, (1^2, 1), twice through one and once through the other of the points a_2."

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45. It is to be remarked upon the tables—first, as regards the lines: if we add the numbers in each line, reckoning m^p as mp, (that is, each multiple point, according to the number of branches through it,) the sums for the successive lines are 2, 5, 8, 11, 14, &c.; that is, each line passes through 2 points, each conic through 5 points, each cubic through 8 points, each quartic through 11 points, &c. But if we add the numbers reckoning m^p as $m \cdot \frac{1}{2}p(p+1)$, (that is, each multiple point according to its effect in the determination of the curve,) then the sums are 2, 5, 9, 14, 20, &c., that is, all the curves are completely determined, viz., the line by 2 conditions, the conic by 5 conditions, the cubic by 9 conditions, &c. Secondly, as regards the columns, if for any column, reckoning m^p as mp, we multiply each number by the corresponding outside left-hand number, add, and divide the sum by the outside number at the head of the column, the successive results are 2, 5, 8, 11, 14, &c.; this merely expresses the known circumstance that the Jacobian passes $3r-1$ times through each point α_r.

46. The analogous tables showing the passage of the Jacobian through the principal system, in the solutions belonging to certain special forms of n, are

TABLE $n = p$.

	$\alpha_1 = 2p-2$	$\alpha_{p-1} = 1$
$\alpha_1' = 2p-2$	1	1
$\alpha'_{p-1} = 1$	$2p-2$	1^{p-2}

TABLES $n = 2p$.

	$\alpha_1 = 3$	$\alpha_2 = 2p-2$	$\alpha_{p-1} = 0$	$\alpha_p = 0$	$\alpha_{2p-2} = 1$
$\alpha_1' = 2p-2$	1				1
$\alpha_2' = 0$					
$\alpha'_{p-1} = 1$		$2p-2$			1^{p-2}
$\alpha_p' = 3$	2	$2p-2$			1^{p-1}
$\alpha'_{2p-2} = 0$					

	$\alpha_1 = 2p-2$	$\alpha_2 = 0$	$\alpha_{p-1} = 1$	$\alpha_p = 3$	$\alpha_{2p-2} = 0$
$\alpha_1' = 3$				2	
$\alpha_2' = 2p-2$	1		1	3	
$\alpha'_{p-1} = 0$					
$\alpha_p' = 0$					
$\alpha'_{2p-2} = 1$	$2p-2$		1^{p-2}	3^{p-1}	

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TABLES $n = 2p+1$.

	$\alpha_1 = 3$	$\alpha_2 = 2p-1$	$\alpha_p = 0$	$\alpha_{p+1} = 0$	$\alpha_{2p-1} = 1$
$\alpha_1' = 2p-1$		1			1
$\alpha_2' = 0$					
$\alpha_p' = 3$	1	$2p-1$			1^{p-1}
$\alpha'_{p+1} = 1$	3	$2p-1$			1^p
$\alpha'_{2p-1} = 0$					

	$\alpha_1 = 2p-1$	$\alpha_2 = 0$	$\alpha_p = 3$	$\alpha_{p+1} = 1$	$\alpha_{2p-1} = 0$
$\alpha_1' = 3$	1			1	
$\alpha_2' = 2p-1$	1		3	1	
$\alpha_p' = 0$					
$\alpha'_{p+1} = 0$					
$\alpha'_{2p-1} = 1$	$2p-1$		3^{p-1}	1^p	

TABLES $n = 3p$.

	$\alpha_1 = 1$	$\alpha_2 = 4$	$\alpha_3 = 2p-3$	$\alpha_p = 0$	$\alpha_{p+1} = 0$	$\alpha_{2p-1} = 0$	$\alpha_{3p-3} = 1$
$\alpha_1' = 2p-3$			1				1
$\alpha_2' = 0$							
$\alpha_3' = 0$							
$\alpha_p' = 4$		3	$2p-3$				1^{p-1}
$\alpha'_{p+1} = 1$	1	4	$2p-3$				1^p
$\alpha'_{2p-1} = 1$	1	4	$(2p-3)^2$				1^{2p-3}
$\alpha'_{3p-3} = 0$							

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	$\alpha_1 = 2p-3$	$\alpha_2 = 0$	$\alpha_3 = 0$	$\alpha_p = 4$	$\alpha_{p+1} = 1$	$\alpha_{2p-1} = 1$	$\alpha_{3p-3} = 0$
$\alpha_1' = 1$					1	1	
$\alpha_2' = 4$				3	1	1	
$\alpha_3' = 2p-3$	1			4	1	1^2	
$\alpha_p' = 0$							
$\alpha'_{p+1} = 0$							
$\alpha'_{2p-1} = 0$							
$\alpha'_{3p-3} = 1$	$2p-3$			4^{p-1}	1^p	1^{2p-3}	

	$\alpha_1 = 4$	$\alpha_2 = 1$	$\alpha_3 = 2p-2$	$\alpha_{p-1} = 0$	$\alpha_p = 0$	$\alpha_{2p} = 0$	$\alpha_{3p-3} = 1$
$\alpha_1' = 2p-2$			1				1
$\alpha_2' = 0$							
$\alpha_3' = 0$							
$\alpha'_{p-1} = 1$			$2p-2$				1^{p-2}
$\alpha_p' = 4$	1	1	$2p-2$				1^{p-1}
$\alpha'_{2p} = 1$	4	1	$(2p-2)^2$				1^{2p-2}
$\alpha'_{3p-3} = 0$							

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	$\alpha_1 = 2p-2$	$\alpha_2 = 0$	$\alpha_3 = 0$	$\alpha_{p-1} = 1$	$\alpha_p = 4$	$\alpha_{2p} = 1$	$\alpha_{3p-3} = 0$
$\alpha_1' = 4$					1	1	
$\alpha_2' = 1$					4	1	
$\alpha_3' = 2p-2$	1			1	4	1^2	
$\alpha'_{p-1} = 0$							
$\alpha_p' = 0$							
$\alpha'_{2p} = 0$							
$\alpha'_{3p-3} = 1$	$2p-2$			1^{p-2}	4^{p-1}	1^{2p-2}	

47. The before mentioned theorem, that $(\alpha_1, \alpha_2 \ldots \alpha_{n-1})$ and $(\alpha_1', \alpha_2' \ldots \alpha'_{n-1})$ are the same series of numbers, of course implies $\Sigma\alpha_r = \Sigma\alpha_r'$; this relation Cremona demonstrates independently, by consideration of the pencil of curves $(aX + bY + cZ) + \theta(a_1X + b_1Y + c_1Z) = 0$, ($\theta$ a variable parameter,) which corresponds in the first plane to the pencil of lines $(ax' + by' + cz') + \theta(a_1x' + b_1y' + c_1z') = 0$, which pass through a fixed point $(ax' + by' + cz' = 0$, $a_1x' + b_1y' + c_1z' = 0)$ in the second figure. In general, in the pencil $U + \theta V = 0$ (U, V given functions of the order n) there are $3(n-1)^2$ values of θ, each giving a nodal curve. But in the present case each of the curves $U = 0$, $V = 0$ has multiple points at the principal points α_r of the first plane: the question is to obtain the number of values which give a curve having one new double point; and this is found to be $= 3(n-1)^2 - \Sigma(r-1)(3r+1)\alpha_r$. We have $\Sigma r^2\alpha_r = n^2 - 1$, $\Sigma r\alpha_r = 3n - 3$; or, substituting, the value of θ is $= \Sigma\alpha_r$. But the curves which have an additional double point are those which correspond to the lines which in the second figure pass through one of the principal points α_r'; viz., these are the lines drawn from the point $(ax' + by' + cz' = 0$, $a_1x' + b_1y' + c_1z' = 0)$ to the several principal points α_r'; and the number of them is $= \Sigma\alpha_r'$. We have thus the required relation $\Sigma\alpha_r = \Sigma\alpha_r'$.

The Quadric Transformation between Two Planes.

48. This is of course given by what precedes. The principal system in each plane is a set of three points; and the Jacobian of the same plane is the set of three lines joining each pair of points; that is, the three lines of either plane are the principal counter-system of the other plane. But to give the analytical investigation

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directly: taking the coordinates (x, y, z) to refer to the principal system of the first plane (viz., taking the three points to be the vertices of the triangle formed by the lines $x=0$, $y=0$, $z=0$), then $X=0$, $Y=0$, $Z=0$ being conics through the three points, the functions X, Y, Z will be each of them of the form $fyz+gzx+hxy$; x', y', z' being proportional to three such functions, there will be linear functions of x', y', z' proportional to yz, zx, xy; or taking these linear functions of the original (x', y', z') for the coordinates (x', y', z') of a point in the second plane, the formulæ of transformation will be $x' : y' : z' = yz : zx : xy$, and we have then conversely $x : y : z = y'z' : z'x' : x'y'$; that is, the formulæ for the transformation in question are

$$x' : y' : z' = yz : zx : xy, \text{ and } x : y : z = y'z' : z'x' : x'y'.$$

We at once verify *à posteriori* that the Jacobian in the first plane is $xyz=0$, and that in the second plane is $x'y'z'=0$.

The equations may be written

$$x' : y' : z' = \frac{1}{x} : \frac{1}{y} : \frac{1}{z}, \text{ and } x : y : z = \frac{1}{x'} : \frac{1}{y'} : \frac{1}{z'},$$

or, if we please, $xx'=yy'=zz'$; the transformation is thus given as an inverse transformation.

{49. With respect to the metrical interpretation and actual construction of the transformation, it is to be observed that if x, y, z be taken to be proportional (not to given multiples of the perpendicular distances, but) to the perpendicular distances of P from the sides of the triangle in the first plane, and similarly x', y', z' to be proportional to the perpendicular distances of P' from the sides of the triangle in the second plane, then in general the equations of transformation must be written, not as above, but in the form $\frac{xx'}{f}=\frac{yy'}{g}=\frac{zz'}{h}$, involving arbitrary multipliers $f : g : h$. We may imagine in the second plane a point P'' determined by coordinates (x'', y'', z''),—the same coordinates as (x', y', z'), that is, proportional to the perpendicular distances of P'' from the sides of the triangle in the second plane,—which point P'' corresponds homographically to P in such wise that $\frac{x}{f} : \frac{y}{g} : \frac{z}{h} = x'' : y'' : z''$. We have then, in the second plane, the two points P', P'' corresponding to each other in such wise that $x'x''=y'y''=z'z''$; and either of these points being given, the other can at once be constructed; viz., it is obvious that, joining P', P'' with any vertex, say A', of the triangle $A'B'C'$, the lines $A'P'$, $A'P''$ are equally inclined to the bisectors of the angle A'; and consequently, P' being given, we have the three lines $A'P''$, $B'P''$, $C'P''$ intersecting in a common point P'', which is therefore determined by means of any two of these lines. We have thus a geometrical construction of the transformation between P and P'.}

50. The analysis assumes that the principal points A, B, C of the first figure are three distinct points; but they may two of them, or all three, coincide. In the first case, say if B, C coincide, the line BC is still to be regarded as having a definite direction; and taking $x=0$ for this line, $y=0$ for the line joining A with

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(BC), and $z=0$ an arbitrary line through A, the functions X, Y, Z will be each of them of the form $by^2+2gzx+2hxy$; and replacing, as before, the original x', y', z' by linear functions of these quantities, these linear functions being taken for the coordinates (x', y', z'), we may write $x' : y' : z' = y^2 : xy : xz$. Forming the converse system, the equations for the transformation are

$$x' : y' : z' = y^2 : xy : xz, \text{ and } x : y : z = y'^2 : x'y' : x'z',$$

so that the points A', B', C' in the second plane are related as the points in the first plane; viz., B', C' coincide, the line $B'C'$ being definite.

It is easy to verify that the Jacobian in the first plane is $xy^2=0$, and the Jacobian in the second plane is $x'y'^2=0$.

51. Secondly, if A, B, C all coincide, these being however consecutive points on a curve of finite curvature, or say on a conic; then, taking $x=0$ for the tangent at (ABC), $z=0$ for any other tangent, and $y=0$ for the chord of contact, the functions X, Y, Z will be of the form $ax^2+b(y^2-zx)+2hxy$; whence we may write $x' : y' : z' = x^2 : xy : y^2-xz$. Forming the converse equations, the equations of transformation are

$$x' : y' : z' = x^2 : xy : y^2-xz, \text{ and } x : y : z = x'^2 : x'y' : y'^2-x'z';$$

so that the points A', B', C' in the second plane are related as those of the first plane; viz., they are the consecutive points of a curve of continuous curvature.

We may verify that the Jacobian of the first plane is $x^3=0$, and the Jacobian of the second plane $x'^3=0$.

The Lineo-linear Transformation between Two Planes.

52. We have two equations of the form

$$(a, \ldots \between x, y, z \between x', y', z') = 0,$$
$$(a_1, \ldots \between x, y, z \between x', y', z') = 0;$$

writing these in the form

$$P\,x' + Q\,y' + R\,z' = 0,$$
$$P_1x' + Q_1y' + R_1z' = 0,$$

where (P, Q, R, P_1, Q_1, R_1) are linear functions of (x, y, z), we have

$$x', y', z' \text{ proportional to } \left\| \begin{matrix} P, & Q, & R \\ P_1, & Q_1, & R_1 \end{matrix} \right\|,$$

that is to $X : Y : Z$, where $X=0$, $Y=0$, $Z=0$ are conics each passing through the same three points in the first plane.

And conversely, writing the equations in the form

$$P'\,x + Q'\,y + R'\,z = 0,$$
$$P_1'x + Q_1'y + R_1'z = 0,$$

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where $(P', Q', R', P_1', Q_1', R_1')$ are linear functions of (x', y', z'), we have

$$x,\ y,\ z \text{ proportional to } \left\| \begin{matrix} P', & Q', & R' \\ P_1', & Q_1', & R_1' \end{matrix} \right\|,$$

that is to X', Y', Z', where $X'=0$, $Y'=0$, $Z'=0$ are conics each passing through the same three points in the second plane.

53. The lineo-linear transformation is thus the same thing as the quadric transformation. It is, moreover, clear that the equations must, by linear transformations on the two sets of variables respectively, and by linear combination of the two equations, be reducible into forms giving the before-mentioned values of $x : y : z$ and $x' : y' : z'$ respectively. Thus, in the general case, where in each plane the three points are distinct points, the lineo-linear equations will be reducible to

$$xx' - yy' = 0, \quad xx' - zz' = 0\,;$$

in the case where B, C in the first plane, and B', C' in the second plane respectively coincide, the forms will be

$$xx' - yy' = 0, \quad yz' - y'z = 0\,;$$

and in the case where A, B, C in the first plane, and A', B', C' in the second plane respectively coincide, the forms will be

$$xy' - yx' = 0, \quad xz' - yy' + zx' = 0.$$

The determination of the actual formulæ for these reductions would, it is probable, give rise to investigations of considerable interest.

The General Rational Transformation between Two Planes (resumed).

54. Consider, as above, the first plane or figure with a principal system $(\alpha_1, \alpha_2 \ldots \alpha_{n-1})$, and the second plane or figure with a principal system $(\alpha_1', \alpha_2' \ldots \alpha'_{n-1})$. To a line in the second plane there corresponds in the first plane a curve of the order n passing 1 time through each of the points α_1, 2 times through each of the points α_2, 3 times through each of the points α_3, &c.; or, as we may write this:

$$\begin{array}{l}\text{First figure.}\\ \begin{array}{lcccc} \text{Points} & \alpha_1 & \alpha_2 & \alpha_3 \ldots & \alpha_{n-1} \\ \hline & 1 & 2 & 3 & n-1 \\ & 1 & 2 & 3 & n-1 \\ & 1 & \vdots & 3 & n-1 \\ & \vdots & & \vdots & \vdots \end{array} \left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\} \text{curve order } n \end{array} \qquad \begin{array}{l}\text{Second figure.}\\ \begin{array}{lcccc} \text{Points} & \alpha_1' & \alpha_2' & \alpha_3' \ldots & \alpha'_{n-1} \\ \hline & 0 & 0 & 0 & n-1 \\ & 0 & 0 & 0 & n-1 \\ & \vdots & \vdots & 0 & \vdots \\ & & & \vdots & \end{array} \left.\vphantom{\begin{array}{c}1\\1\\1\\1\end{array}}\right\} \text{curve order } 1 \end{array}$$

viz., the 1's denote the number of times which the curve of the order n passes through the several points α_1 respectively; the 2's the number of times which the curve passes through the several points α_2 respectively; and so on.

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55. We may, in the second figure, in the place of a line consider a curve of the order k'. If the equation hereof is $(*\between x', y', z')^{k'}=0$, then the corresponding curve in the first figure is $(*\between X, Y, Z)^{k'}=0$; viz., this is a curve of the order $k=nk'$. If, however, the curve in the second figure passes once or more times through all or any of the points $\alpha_1', \alpha_2', \ldots \alpha'_{n-1}$, then there will be a depression in the order of the corresponding curve in the first figure; and, moreover, this curve will pass a certain number of times through all or some of the points $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_{n-1}$. The diagram of the correspondence will be:

First figure.		Second figure.	
$\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_{n-1}$		$\alpha_1', \alpha_2', \alpha_3', \ldots \alpha'_{n-1}$	
$\left.\begin{matrix} a_1 & a_2 & a_3 & a_{n-1} \\ b_1 & b_2 & b_3 & b_{n-1} \\ c_1 & c_2 & \vdots & \vdots \\ \vdots & \vdots & & \end{matrix}\right\}$	curve order k	$\left.\begin{matrix} a_1' & a_2' & a_3' & a'_{n-1} \\ b_1' & b_2' & b_3' & b'_{n-1} \\ c_1' & \vdots & \vdots & c'_{n-1} \\ \vdots & & & \vdots \end{matrix}\right\}$	curve order k'

where $a_1, b_1, c_1 \ldots$ denote the number of times that the curve of the order k passes through the several points α_1 respectively, (viz., the number of the letters $a_1, b_1, c_1 \ldots$ is $=\alpha_1$, any or all of them being zeros,) $a_2, b_2, c_2 \ldots$ the number of times that the curve passes through the several points α_2 respectively, (viz., the number of the letters $a_2, b_2, c_2 \ldots$ is $=\alpha_2$, any or all of them being zeros,) and so on; and the like for the curve in the second figure.

56. By what precedes, it is easy to see that, if the curve k' passes through a point α_1', then the curve k throws off a line, and the depression of order is $=1$; so, if the curve passes 2 times, 3 times, ... or a_1' times through the point in question, then the curve throws off the line repeated 2 times, 3 times, ... a_1' times, or the depression of order is $=2, 3, \ldots$ or a_1'; and the like for each of the points α_1'; so that, writing for shortness $a_1'+b_1'+c_1'+\ldots=\Sigma a_1'$, the depression of order on account of the passages through the several points α_1 is $=\Sigma a_1'$. Similarly, for each time of passage through a point α_2', there is thrown off a conic; or if $a_2'+b_2'+\ldots=\Sigma a_2'$, then the depression of order is $=2\Sigma a_2'$, and so on; and the like for the figure in the other plane; and we thus arrive at the equations

$$k = k'n - \Sigma\,(a_1' + 2a_2' + 3a_3' \ldots + \overline{n-1}\,a'_{n-1})$$
$$k' = kn - \Sigma\,(a_1 + 2a_2 + 3a_3 \ldots + \overline{n-1}\,a_{n-1}).$$

57. The simplest case is when the curve k' does not pass through any of the points $\alpha_1', \alpha_2', \ldots \alpha'_{n-1}$. We have then

$$a_1'=b_1'=c_1'\ldots=0,\ a_2'=b_2'\ldots=0,\ \ldots\ldots\ a'_{n-1}=b'_{n-1}\ldots=0;$$

consequently $k=k'n$. And, moreover, it is easy to see that

$$a_1=b_1\ldots=k',\quad a_2=b_2\ldots=2k',\ \ldots\ldots\ a_{n-1}=b_{n-1}\ldots=(n-1)\,k';$$

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so that the correspondence is:

$$
\begin{array}{cccc|c}
\multicolumn{4}{c}{\text{First Figure.}} & \\
\alpha_1, & \alpha_2, & \alpha_3, \ldots & \alpha_{n-1} & \\
\hline
k' & 2k' & 3k' & (n-1)k' & \\
k' & 2k' & 3k' & (n-1)k' & \text{curve order } k = nk' \\
\vdots & \vdots & \vdots & \vdots &
\end{array}
\qquad
\begin{array}{cccc|c}
\multicolumn{4}{c}{\text{Second Figure.}} & \\
\alpha_1', & \alpha_2', & \alpha_3', \ldots & \alpha'_{n-1} & \\
\hline
0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 0 & \text{curve order } k'. \\
\vdots & \vdots & \vdots & &
\end{array}
$$

We have

$$\Sigma a_1 = k'\alpha_1, \quad \Sigma a_2 = 2k'\alpha_2, \; \ldots\ldots \; \Sigma a_1(a_1 - 1) = k'(k'-1)\alpha_1, \; \&c.,$$

and the formulæ for k, k' become

$$k = k'n, \quad k' = kn - k'\{\alpha_1 + 4\alpha_2 \ldots + (n-1)^2 \alpha_{n-1}\};$$

viz., the second equation is here $k' = kn - k'(n^2 - 1)$; that is, $k'n^2 = kn$, agreeing, as it should do, with the first equation.

58. Moreover, the deficiency-relation is

$$\tfrac{1}{2}(k-1)(k-2) - \Sigma\,\tfrac{1}{2}\{k'(k'-1) + 2k'(2k'-1) \ldots + \overline{n-1}\,k'(\overline{n-1}\,k'-1)\} = \tfrac{1}{2}(k'-1)(k'-2);$$

or, what is the same thing, this is

$$
\begin{aligned}
(nk'-1)(nk'-2) - (k'-1)(k'-2) = \;& k'^2\{\alpha_1 + 4\alpha_2 \ldots + (n-1)^2 \alpha_{n-1}\}. \\
& - k'\;\{\alpha_1 + 2\alpha_2 \ldots + (n-1)\;\alpha_{n-1}\}.
\end{aligned}
$$

The right-hand side is

$$k'^2(n^2-1) - k'(3n-3) = (n-1)\{(\overline{n+1}\,k'^2 - 3k')\},$$

and we have thus the identical equation

$$(nk'-1)(nk'-2) - (k'-1)(k'-2) = (n-1)k'\{(n+1)k' - 3\}.$$

59. It should be possible, when the nature of the correspondence between the two planes is completely given, to express each of the numbers $a_1, b_1, c_1, \ldots a_{n-1}, b_{n-1}, \ldots$ in terms of $k', a_1', b_1', c_1', \ldots a'_{n-1}, b'_{n-1}, \ldots$; and reciprocally each of the numbers $a_1', b_1', c_1', \ldots a'_{n-1}, b'_{n-1}, \ldots$ in terms of $k, a_1, b_1, c_1, \ldots a_{n-1}, b_{n-1}, \ldots$; thus completing a system of relations between the two sets

$$(k, a_1, b_1, \ldots a_{n-1}, b_{n-1}, \ldots), \qquad (k', a_1', b_1', \ldots a'_{n-1}, b'_{n-1}, \ldots);$$

but even if the theory was known, there would be considerable difficulty in forming a proper algorithm for the expression of these relations.

60. The two curves must have each of them the same deficiency. It is to be noticed, that if the curve in the first plane passes any number of times through a point P, which is not one of the points $\alpha_1, \alpha_2, \alpha_3, \ldots$ or α_{n-1}, then the corresponding curve in the second plane will pass the same number of times through the corresponding point P', which point will not be one of the points $\alpha_1', \alpha_2', \ldots \alpha'_{n-1}$. The

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points P, P' will therefore contribute equal values to the deficiencies of the two curves respectively; so that, in equating the two deficiencies, we may disregard P, P', and attend only to the points $\alpha_1, \alpha_2, \ldots \alpha_{n-1}$ of the first plane, and $\alpha_1', \alpha_2', \ldots \alpha'_{n-1}$ of the second plane. The required relation thus is

$$\tfrac{1}{2}(k-1)(k-2) - \Sigma\,\tfrac{1}{2}\{a_1(a_1-1) + a_2(a_2-1) \ldots + a_{n-1}(a_{n-1}-1)\}$$
$$= \tfrac{1}{2}(k'-1)(k'-2) - \Sigma\,\tfrac{1}{2}\{a_1'(a_1'-1) + a_2'(a_2'-1) \ldots + a'_{n-1}(a'_{n-1}-1)\}.$$

61. In the case of the quadric transformation $n=2$, we have in the first plane the three points α_1, say these are A, B, C; and in the second plane the three points α_1', say these are A', B', C'. And if in the first plane the curve of the order k passes a, b, c times through the three points respectively, and in the second plane the corresponding curve of the order k' passes a', b', c' times through the three points respectively, then it is easy to obtain

$$\begin{array}{ll|ll} k' = 2k - a - b - c, & & & k = 2k' - a' - b' - c', \\ a' = \;\; k - b - c, & & & a = \;\; k' - b' - c', \\ b' = \;\; k - c - a, & & & b = \;\; k' - c' - a', \\ c' = \;\; k - a - b. & & & c = \;\; k' - a' - b'. \end{array}$$

The Quadric Transformation any number of times repeated.

62. We may successively repeat the quadric transformation according to the type:

First Fig.	Second Fig.	Third Fig.	Fourth Fig.
A, B, C	A', B', C'		
	D', E', F'	D'', E'', F''	
		G'', H'', I''	G''', H''', I'''

viz., in the transformation between the first and second figures, the principal systems are ABC and $A'B'C'$ respectively; in that between the second and third figures, they are $D'E'F'$ and $D''E''F''$ respectively; in that between the third and fourth figures, they are $G''H''I''$ and $G'''H'''I'''$; and so on. And it is then easy to see that between the first and any subsequent figure we have a rational transformation of the order 2 for the second figure, 4 for the third figure, 8 for the fourth figure, and so on.

63. But to further explain the relation, we may complete the diagram, by taking, in the transformation between the second and third figures, A'', B'', C'' to correspond to A', B', C'; similarly, in that between the third and fourth, A''', B''', C''' to correspond to A'', B'', C''; and D''', E''', F''' to correspond to D'', E'', F''. And so in the transformation between the second and third figure, we may make G', H', I'

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correspond to G'', H'', I'', and between the first and second figures make D, E, F correspond to D', E', F', and G, H, I to G', H', I', the diagram being thus:

First Fig.	Second Fig.	Third Fig.	Fourth Fig.
A, B, C	A', B', C'	A'', B'', C''	A''', B''', C'''
D, E, F	D', E', F'	D'', E'', F''	D''', E''', F'''
G, H, I	G', H', I'	G'', H'', I''	G''', H''', I'''

Observe that in the principal systems (for instance, A, B, C and A', B', C') the points A, B, C correspond, not to the points A', B', C', but to the lines $B'C'$, $C'A'$, $A'B'$ respectively; and so in the other case.

64. Consider now a line in the first figure: there corresponds hereto in the second figure a conic through the points A', B', C'; and to this conic there corresponds in the third figure a quartic curve passing through each of the points A'', B'', C'' once, and through each of the points D'', E'', F'' twice. And conversely, to a line in the third figure corresponds in the second figure a conic through the points D', E', F'; and hereto in the first figure a quartic through the points D, E, F, once and through the points A, B, C twice; that is, we have between the first and third figures a quartic transformation wherein $\alpha_1 = \alpha_2 = 3$ and $\alpha_1' = \alpha_2' = 3$, or say a quartic transformation $3_1 3_2$ and $3_1 3_2$. In like manner, passing to the fourth figure, to a line in the first figure corresponds in the fourth figure an octic curve passing through A''', B''', C''' once, through D''', E''', F''' twice, and through G''', H''', I''' four times; and conversely, to a line in the fourth figure there corresponds in the first figure an octic curve passing through the points G, H, I once, the points D, E, F twice, and the points A, B, C four times; that is, between the first and fourth figures we have an octic transformation, wherein $\alpha_1 = \alpha_2 = \alpha_4 = 3$, $\alpha_1' = \alpha_2' = \alpha_4' = 3$, or say a transformation, order 8, of the form $3_1 3_2 3_4$ and $3_1 3_2 3_4$. And so between the first and fifth figures there is a transformation, order 16, of the form $3_1 3_2 3_4 3_8$ and $3_1 3_2 3_4 3_8$.

65. It is, moreover, easy to find the Jacobians or counter-systems in the several transformations respectively. Thus, in the transformation between the first and second figures, in the second figure the Jacobian consists of 3 lines such as $B'C'$ (viz., these are, of course, the lines $B'C'$, $C'A'$, $A'B'$). Hence, in the transformation between the first and third figures, the Jacobian in the third figure consists of

$$\begin{array}{ll} 3 \text{ conics} & B''C''\,(D''E''F''), \\ 3 \text{ lines} & D''E''; \end{array}$$

viz., one of the conics is that through the five points B'', C'', D'', E'', F'', one of the lines that through the two points D'', E''. And so in the fourth figure, the Jacobian consists of

$$\begin{array}{lll} 3 \text{ quartics} & B'''C'''\,(D'''E'''F''')_1\,(G'''H'''I''')_2, & \\ 3 \text{ conics} & D'''E''' & (G'''H'''I''')_1, \\ 3 \text{ lines} & & G'''H'''; \end{array}$$

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viz., one of the quartics passes through B''', C'''; through D''', E''', F''' each once; and through G''', H''', I''' each twice. And so in the fifth figure the Jacobian consists of

$$\begin{array}{lll} 3 \text{ octics} & B''''C'''' & (D''''E''''F'''')_1 (G''''H''''I'''')_2 (J''''K''''L'''')_4, \\ 3 \text{ quartics} & D''''E'''' & (G''''H''''I'''')_1 (J''''K''''L'''')_2, \\ 3 \text{ conics} & G''''H'''' & (J''''K''''L'''')_1, \\ 3 \text{ lines} & & J''''K'''', \end{array}$$

and so on.

66. The conditions are in each case sufficient for the determination of the curve. This depends on the numerical relation

$$4 + 3 \{1 \cdot 2 + 2 \cdot 3 + 4 \cdot 5 + 8 \cdot 9 \ldots + 2^\theta (2^\theta + 1)\} = 2^{\theta+1} (2^{\theta+1} + 3).$$

The term in { } is

$$\begin{aligned} & 1 + 4 + 16 \ldots + 2^{2\theta} \\ & + 1 + 2 + 4 \ldots + 2^\theta, \end{aligned}$$

that is

$$\frac{2^{2\theta+2} - 1}{2^2 - 1} + \frac{2^{\theta+1} - 1}{2 - 1},$$

which is

$$\begin{aligned} &= \tfrac{1}{3} [2^{2\theta+2} - 1 + 3 (2^{\theta+1} - 1)], \\ &= \tfrac{1}{3} [2^{2\theta+2} + 3 \cdot 2^{\theta+1} - 4]; \end{aligned}$$

and the relation is thus identically true.

67. Conversely, in the transformation between the first figure and the several other figures respectively, the Jacobian of the first figure is

$$3 \text{ lines} \quad AB\text{; and so} \qquad \text{for order 2, between first and second figures;}$$

$$\left.\begin{array}{ll} 3 \text{ conics} & DE(ABC)_1 \\ 3 \text{ lines} & AB \end{array}\right\} \text{ for order 4, between first and third figures;}$$

$$\left.\begin{array}{ll} 3 \text{ quartics} & GH(DEF)_1 (ABC)_2 \\ 3 \text{ conics} & DE \quad (ABC)_1 \\ 3 \text{ lines} & AB \end{array}\right\} \text{ for order 8, between first and fourth figures;}$$

$$\left.\begin{array}{ll} 3 \text{ octics} & JK (GHI)_1 (DEF)_2 (ABC)_4 \\ 3 \text{ quartics} & GH \quad (DEF)_1 (ABC)_2 \\ 3 \text{ conics} & DE \quad (ABC)_1 \\ 3 \text{ lines} & AB \end{array}\right\} \text{ for order 16, between first and fifth figures;}$$

and so on.

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Special Cases—Reduction of the General Rational Transformation to a Series of Quadric Transformations.

68. It was remarked by Mr Clifford that any Cremona-transformation whatever may be obtained by this method of repeated quadric transformations, if only the principal systems, instead of being completely arbitrary, are properly related to each other. To take the simplest instance; suppose that we have

First figure.	Second figure.	Third figure.
A, B, C	A', B', C'	B'', C''
	$\Vert$	
E, F	D', E', F'	D'', E'', F''

viz., in the transformation between the first and second figures, we have the principal systems ABC and $A'B'C'$ (arbitrary as before); but in the transformation between the second and third figures, the principal systems are $D'E'F'$ and $D''E''F''$, where D', instead of being arbitrary, coincides with A'. And we then have B'', C'' in the third figure corresponding to B', C' in the second figure, and E, F in the first figure corresponding to E', F' in the second figure. This being so, to a line in the first figure corresponds in the second figure a conic through A', B', C'. But $A'=D'$; viz., this conic passes through a point D' of the principal system of the second figure, in regard to the transformation between the second and third figures. That is, (k, a, b, c referring to the second figure, and k', a', b', c' to the third figure, $k=2$, $a=1$, $b=0$, $c=0$, and therefore $k'=3$, $a'=2$, $b'=1$, $c'=1$,) corresponding to the conic we have in the third figure a curve, order 3 (cubic curve), passing twice through D'', but once through E'' and F'' respectively; this cubic curve passes also through the points B'', C'' which correspond to B', C' respectively; that is,

cubic passes through E'', F'', B'', C'' each 1 time

" " D'' 2 times;

or, corresponding to a line in the first figure, we have in the third figure a curve, order 3, passing through four fixed points each 1 time, and through one fixed point 2 times. That is, we have $n=3$, $\alpha_1'=4$, $\alpha_2'=1$. And in the same manner, to a line in the third figure there corresponds in the first figure a cubic through four fixed points (viz., B, C, E, F) each 1 time, and through one fixed point, A, 2 times; so that also $\alpha_1=4$, $\alpha_2=1$. The transformation is thus of the order 3, and the form $4_1\,1_2$ and $4_1\,1_2$ (this is in fact the only cubic transformation; see the Tables, *ante*, No. 41).

69. Mr Clifford has also devised a very convenient algorithm for this decomposition of a transformation of any order into quadric transformations. The quadric transformation is denoted by [3], the cubic transformation by [41], the quartic transformations by [601], [330], the quintic ones by [8001], [3310], [0600], and so on; see the Tables just referred to. (This is substantially the same as a notation employed above, the zeros enabling the omission of the suffixes; viz., $[8001]=8_1\,1_4$; and so in other cases.)

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70. The foregoing result is represented thus [4, 1] = [3)(0, 0, 1), which I proceed to explain. Consider in the first figure a line; the symbol [3] denotes that in the second figure we have a conic with three points (α_1'). We are about to apply to this a quadric transformation; (0, 0, 0) would denote that the three points of the principal system in the second figure were all of them arbitrary; (0, 0, 1) that one of these points was a point α_1'; (0, 1, 1) that two of them were points α_1'; (1, 1, 1) that all three of them were points α_1; (0, 0, 2) would denote that one of the points was a point α_2'; only in the present case we can have no such symbol, by reason that there are no points α_2'. Hence [3)(001) denotes that the conic has applied to it a quadric transformation such that, in the transformation thereof, one point of the principal system coincides with one of the points (α_1') on the conic. To [3], *quà* quadric transformation, belongs the number 2: and from 2, (001) we derive 3, (112), {in general k, (a, b, c) gives k', (a', b', c'), where $k' = 2k - a - b - c$, $a' = k - b - c$, $b' = k - c - a$, $c' = k - a - b$}. $k = 2$ corresponds to a symbol [3] of one number, $k' = 3$ to a symbol of two numbers; viz., we change [3] into [30]; we then, in the symbols (112) and (001), consider the frequencies of the several numbers 1, 2, ... taking those in the first symbol as positive, and those in the second symbol as negative; or, what is the same thing, representing the frequency as an index, we have $1^2\,2^1$, 1^{-1}; or, combining, $1^{2-1}\,2^1$; these indices are then added on to the numbers of [30]; viz., the index of 1 to the first number, the index of 2 to the second number (and, in the case of more numbers, so on): [30] is thus converted into [41], and we have the required equation

[41] = [3)(001),

where the *rationale* of this algorithmic process appears by the explanation, *ante,* No. 68.

71. As another example take

[8001] = [601)(003).

To [601], *quà* quartic transformation, belongs the number 4; and from 4, (003) we form 5, (114); where the 5 indicates that [601] is to be changed into [6010]; then (114), (003), writing them in the form $1^2 2^0 3^{-1} 4^1$, show that to the numbers of [6010] we are to add 2, 0, -1, 1; thus changing the symbol into [8001], so that we have the required relation.

72. Mr Clifford calculated in this way the following table, showing how any transformation of an order not exceeding 8 can be expressed by means of a series of quadric transformations; the symbols Cr. 3, Cr. 4 . 1; 4 . 2, &c., refer to the order and number of Cremona's tables, *ante,* No. 41.

Cr. 3 . = [41] = [3)(001),

Cr. 4 . 1 = [601] = [41)(002) = [3)(001)(002),

4 . 2 = [330] = [3)(000) = [41)(011) = [3)(001)(011),

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Cr. 5 . 1 = [8001] = [601≬003) = [3≬001≬002≬003),

5 . 2 = [3310] = [41≬001) = [3≬001≬001),

5 . 3 = [0600] = [330≬111) = [3≬000≬111) = [3≬001≬011≬111),

Cr. 6 . 1 = [10,0001] = [8001≬004) = [3≬001≬002≬003≬004),

6 . 2 = [14200] = [330≬011) = [3≬000≬011) = [3≬001≬011≬011),

6 . 3 = [41300] = [601≬011) = [3≬001≬002≬011)

= [3310≬022) = [3≬001≬001≬022) } = [3≬001≬000),

6 . 4 = [34010] = [330≬002) = [3≬000≬002),

= [3310≬013) = [3≬001≬001≬013),

Cr. 7 . 1 = [12,00001] = [10,0001≬005) = [3≬001≬002≬003≬004≬005),

7 . 2 = [330≬001) = [3≬000≬001) = [232100],

7 . 3 = [034000] = [3310≬111) = [3≬001≬001≬111),

7 . 4 = [503100] = [601≬001) = [3≬001≬002≬001),

7 . 5 = [350010] = [3310≬003) = [3≬001≬001≬003),

Cr. 8 . 1 = [14,000001] = [3≬001≬002≬003≬004≬005≬006),

8 . 2 = [3230100] = [3310≬002) = [3≬001≬001≬002),

8 . 3 = [1322000] = [3310≬011) = [3≬001≬001≬011),

8 . 4 = [0070000] = [034000≬222) = [3≬001≬001≬111≬222),

8 . 5 = [3600010] = [34010≬004) = [330≬002≬004) = [3≬000≬002≬004),

8 . 6 = [6013000] = [601≬000) = [3≬001≬002≬000),

8 . 7 = [0520100] = [0600≬002) = [3≬000≬111≬222),

8 . 8 = [2051000] = [41300≬112) = [3≬001≬000≬112),

8 . 9 = [3303000] = [3≬000≬000≬000).

73. The reduction as above of a transformation to a series of quadric transformations, enables the determination of the reciprocal transformation; or, what is the same thing, the determination of the Jacobian of the first figure; see the example, *ante*, No. 67, where it appears that the reciprocal transformation of [41] is [41]. But I do not see any easy algorithmic process for the determination of the reciprocal transformation, or still less any general form in which the result can be expressed; and I do not at present pursue the inquiry.

The Rational Transformation between Two Spaces.

74. The general principles have been already explained: the two systems $x' : y' : z' : w' = X : Y : Z : W$ and $x : y : z : w = X' : Y' : Z' : W'$ must be derivable the one from the other; and starting with the first system, this will be the case if

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only the surfaces $X=0$, $Y=0$, $Z=0$, $W=0$ have a common intersection equivalent to n^3-1 points of intersection, but not equivalent to a complete common intersection of n^3 points. The last-mentioned circumstance would arise, if the condition of the common intersection should impose upon the surface more than $\frac{1}{6}(n+1)(n+2)(n+3)-4$ conditions; viz., the surfaces would then be connected by an identical equation or syzygy $\alpha X+\beta Y+\gamma Z+\delta W=0$. The common intersection is a figure composed of points and curves: say it is the principal system in the first space; the problem is, to determine a principal system equivalent to n^3-1 points of intersection but such that the number of conditions to be satisfied by a surface passing through it is not more than

$$\tfrac{1}{6}(n+1)(n+2)(n+3)-4.$$

75. The following locutions are convenient. We may say that the number of conditions imposed upon a surface of the order n which passes through the common intersection is the *Postulation* of this intersection; and that the number of points represented by the common intersection (in regard to the points of intersection of any three surfaces each of the order n which pass through it) is the *Equivalence* of this intersection. The conditions above referred to are thus

$$\text{Equivalence}=n^3-1,$$

$$\text{Postulation} \ngtr \tfrac{1}{6}(n+1)(n+2)(n+3)-4.$$

76. It would appear by the analogy of the rational transformation between two planes, that the only cases to be considered are those for which

$$\text{Postulation}=\tfrac{1}{6}(n+1)(n+2)(n+3)-4\,;$$

but I cannot say whether this is so.

77. In the transformation between two planes, the two conditions lead, as was seen, to the result that the curve $aX+bY+cZ=0$ is unicursal. I do not see that in the present case of two spaces, the two conditions lead to the corresponding result that the surface $aX+bY+cZ+dW=0$ is unicursal; that this is so, appears, however, at once from the general notion of the rational transformation. In fact, the equation in question $aX+bY+cZ+dW=0$ is satisfied by $x:y:z:w=X':Y':Z':W'$ and $ax'+by'+cz'+dw'=0$; the last equation determines the ratios $x':y':z':w'$ in terms of two arbitrary parameters (say these are $x':y'$ and $x':z'$), and we have then $x:y:z:w$ proportional to rational functions of these two parameters; that is, the surface $aX+bY+cZ+dW=0$ is unicursal. And similarly the surface $aX'+bY'+cZ'+dW'=0$ is unicursal.

78. In the most general point of view, the principal system will contain a given number of points which are simple points, a given number which are quadriconical points, a given number which are cubiconical points, &c. &c., on the surfaces; and similarly a given number of curves which are simple curves, a given number which are double curves, &c. &c., on the surfaces. But, to simplify, I will consider that it includes only points which are simple points, and a curve which is a simple curve

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on the surfaces: this curve may, however, break up into separate curves, and we thus, in fact, admit the case where there are any number of separate curves each of them a simple curve on the surfaces. It is right to remark that we cannot assert *à priori*—and it is not in fact the case—that the principal system in the second space will be subject to the like restrictions: starting with such a principal system in the first space, we may be led in the second space to a principal system including a curve which is a double curve on the surfaces; an instance of this will in fact occur.

79. It is shown (Salmon's *Solid Geometry*, 2nd ed., p. 283, [Ed. 4, p. 321]), that in the intersection of three surfaces of the orders μ, ν, ρ respectively, a curve of intersection of the order m and class r counts as $m(\mu+\nu+\rho-2)-r$ points of intersection. For a curve without actual double points or stationary points, we have $r=m(m-1)-2h$, where h is the number of apparent double points; or, substituting, we have the curve counting for $m(\mu+\nu+\rho-2)-m(m-1)+2h$ points of intersection; this is in fact a more general form of the formula, inasmuch as it extends to the case of a curve with actual double points and stationary points. Or, what is the same thing, the three surfaces intersecting in the curve of the order m with h apparent double points, will besides intersect in $\mu\nu\rho-m(\mu+\nu+\rho-2)+m(m-1)-2h$ points; viz., the curve may, besides the apparent double points, have actual double points and stationary points; but these do not affect the formula.

80. Some caution is necessary in the application of the theorem. For instance, to consider cases that will present themselves in the sequel: let the surfaces be cubics $(\mu=\nu=\rho=3)$; the number of remaining intersections is given as $=27-7m+m(m-1)-2h$. Suppose that the curve consists of four non-intersecting lines, $m=4$, $h=6$, the number is given as $=-1$. But observe in this case there are two lines each meeting the four given lines; that is, any cubic surface passing through the four given lines meets these two lines each of them in four points, that is, the cubic passes also through each of the two lines; the complete *curve*-intersection of the surfaces is made up of the six lines $m=6$, $h=7$ (since each of the two lines, as intersecting the four lines, gives actual double points, but the two lines together give one apparent double point), and the expression for the number of the remaining points of intersection becomes $=27-42+30-14=1$, which is correct.

81. Similarly, if the given curve of intersection be a conic and two non-intersecting lines, there is here in the plane of the conic a line meeting each of the two given lines, and therefore meeting the cubic surface, in four points, that is, lying wholly in the cubic surface: the complete *curve*-intersection consists of the conic, the two given lines, and the last-mentioned line, $m=5$, $h=5$, and the number of points of intersection is $=27-35+20-10$, $=2$, which is correct. Again, if the given curve of intersection be two conics, here the line of intersection of the planes of the conics lies in the cubic surface; or, for the complete *curve*-intersection we have $m=5$, $h=4$; and the number of points is $27-35+20-8$, $=4$. If in this last case each or either of the conics become a pair of intersecting lines, or if in the preceding case the conic becomes a pair of intersecting lines, the results remain unaltered.

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82. If a surface of the order μ pass through a curve of the order m and class r without stationary points or actual double points, this imposes on the surface a number of conditions $=(\mu+1)m-\frac{1}{2}r$. In the case in question, the value of r is $=m(m-1)-2h$; or, substituting, the number of conditions is $=(\mu+1)m-\frac{1}{2}m(m-1)+h$; and the formula in this form holds good even in the case where the curve has stationary points and actual double points. Thus $\mu=3$, the number of conditions is $=4m-\frac{1}{2}m(m-1)+h$. If the curve be a line, $m=1$, $h=0$, number of conditions is $=4$; if the curve be a pair of non-intersecting lines, $m=2$, $h=1$, number of conditions is $=8$. And so in general, if the curve consist of k non-intersecting lines ($k=4$ at most), then $m=k$, $h=\frac{1}{2}k(k-1)$, and the number of conditions is $=4k$. If the curve be a conic, or a pair of intersecting lines, $m=2$, $h=1$, and the number of conditions is $=7$. If the curve consist of k lines, such that there are θ pairs of intersecting lines, then $m=k$, $h=\frac{1}{2}k(k-1)-\theta$, and the number of conditions is $=4k-\theta$. It is obvious that, the number of conditions for a line being $=4$, that for the k lines with θ intersecting pairs must have the foregoing value $4k-\theta$. In fact, when the lines do not intersect, we take on each line 4 points, and the cubic surface passing through any such 4 points will contain the line; but for two lines which intersect, taking this point, and on each of the intersecting lines 3 other points, the cubic surface through the 7 points will pass through the two lines; and so in other cases.

83. The formula must, in some instances, be applied with caution. Thus, given five non-intersecting lines $k=5$, $\theta=0$, and the number of conditions is $=20$; and a cubic surface cannot be, in general, made to pass through the lines. But if the five lines are met by any other line, then a cubic surface, if it pass through the five lines, will pass through this sixth line; for the six lines $k=6$, $\theta=5$, and the number of conditions is $24-5=19$; so that there is a determinate cubic surface through the six lines, and therefore through the five lines related in the manner just referred to.

84. Recurring to the problem of transformation, it appears by what precedes, that if the principal system in the first plane consists of α_1 points, and of a curve of the order m_1 with h_1 apparent double points (the α_1 points being simple points, and the curve a simple curve on the surfaces), then the conditions for a transformation are

$$(3n-2)\,m_1-\ \ m_1(m_1-1)+2h_1+\alpha_1=n^3-1,$$
$$(\ n+1)\,m_1-\tfrac{1}{2}m_1(m_1-1)+\ \ h_1+\alpha_1=\tfrac{1}{6}(n+1)(n+2)(n+3)-4,$$

where, in the second line, instead of $\not>$ I have written $=$. I remark, in passing, that I have ascertained that an actual triple point counts as an apparent double point; or, what is the same thing, that if the curve has t_1 actual triple points, then we may, instead of h_1, write h_1+t_1. The equations give

$$m_1(4n-5-m_1)=\tfrac{1}{3}(n-1)(5n^2-\ n-12)-2h_1,$$
$$(n-4)\,m_1-\alpha=\tfrac{1}{3}(n-1)(2n^2-4n-15),$$

to which may be joined

$$(3n+8)\,m_1-2m_1(m_1-1)+4h_1+5\alpha_1=(n-1)(6n+17).$$

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The first two equations for the successive values of n give

$$\begin{array}{lll} n=2, & m_1(3-m_1)=2-2h_1, & 2m_1+\alpha_1=5; \\ n=3, & m_1(7-m_1)=20-2h_1, & m_1+\alpha_1=6; \\ n=4, & m_1(11-m_1)=64-2h_1, & \alpha_1=-1; \\ n=5, & m_1(15-m_1)=144-2h_1, & -m_1+\alpha_1=-20; \\ n=6, & m_1(19-m_1)=270-2h_1, & -2m_1+\alpha_1=-55; \\ \&c. & \&c. & \&c. \end{array}$$

85. It is remarkable that for $n=4$ there is no solution of the geometrical problem; in fact, $\alpha_1=-1$, a negative value of α_1, shows that this is so. For the higher values of n, there seem to be solutions with large values of m_1, h_1, α_1; for example, $n=5$, we have $m_1=20+\alpha_1$, is $=20$ at least. Writing $m_1=20$, we have $-100=144-2h_1$, or $2h_1=244$. The highest value of $2h_1$ is $=(m_1-1)(m_1-2)$, which for $m_1=20$ is $=342$; or the foregoing value $2h_1=244$ is admissible. Thus $m_1=20$, $h_1=122$, $\alpha_1=0$ gives a solution; and, moreover, any larger value of m_1, say $m_1=20+\alpha$, gives an admissible solution, $m_1=20+\alpha$, $h_1=122+\frac{1}{2}\alpha(\alpha+25)$, $\alpha_1=\alpha$. And so for $n=6$, &c.; but I have not further examined any of these cases, and do not understand them.

There remain the cases $n=2$, $n=3$. For $n=2$, since $2m_1+\alpha_1=5$, we have $m_1=0$, 1, or 2; $m_1=0$ gives $h_1=0$, which is not admissible. The remaining solutions are $m_1=1$, $h_1=0$, $\alpha_1=3$; and $m_1=2$, $h_1=0$, $\alpha_1=1$.

For $n=3$, since $m_1+\alpha_1=6$, we have $m_1=0$, 1, 2, 3, 4, 5, or 6. $m_1=0$ gives $h_1=10$; $m_1=1$ gives $h_1=7$; $m_1=2$ gives $h_1=5$; $m_1=3$ gives $h_1=4$: these values are not geometrically admissible. The remaining cases are $m_1=4$, $h_1=4$, $\alpha_1=2$; $m_1=5$, $h_1=5$, $\alpha_1=1$; $m_1=6$, $h_1=7$, $\alpha_1=0$.

86. The reciprocal transformation is in every case of the order $n'=n^2-m_1$. Hence considering the quadric transformations:

First, the case $n=2$, $m_1=1$, $h_1=0$, $\alpha_1=3$; the reciprocal transformation is of the order $n'=3$. Suppose for a moment that the principal system in the second space is of the same nature as that above considered in the first space, consisting of α_1' points, and a curve of the order m_1' with h_1' apparent double points (the α_1' points each a simple point, and the curve a simple curve on the surfaces $X'=0$, &c.). Passing back to the original transformation, we should have $2=9-m_1'$, that is, $m_1'=7$. But it has just been seen that, for $n=3$, the only values of m_1 are 4, 5, 6; hence for $n'=3$ we cannot have $m_1'=7$. The explanation is, that the principal system in the second space *is not of the form in question;* it, in fact, consists (as will appear) of three lines each a simple line, and of another line which is a double line on the surfaces $X'=0$, &c. In the intersection of any two of these surfaces, the three lines count each once, the double line four times, and the order of the curve of intersection is thus $3+4=7$, as it should be. The principal system may be characterized $\alpha_1'=0$, $m_1'=3$, $h_1'=3$, $m_2'=1$, $h_2'=0$.

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Next, the case $n=2$, $m_1=2$, $h_1=0$, $\alpha_1=1$: the reciprocal transformation is of the order $n'=2$; it is evidently not of the form above considered (for this would make the original transformation to be of the order 3). Hence, assuming (as it seems allowable to do) that the principal system does not contain any multiple point or curve, the reciprocal transformation will be of the same form as the original one; viz., we shall have $n'=2$, $m_1'=2$, $h_1'=0$, $\alpha_1'=1$.

87. Considering next the cubic transformations, or those belonging to $n=3$; in the case $m_1=4$, $h_1=4$, $\alpha_1=2$, the reciprocal transformation is of the order $9-4$, $=5$; and in the case $m_1=5$, $h_1=5$, $\alpha_1=1$, the reciprocal transformation is of the order $9-5$, $=4$: I do not consider these cases. But $m_1=6$, $h_1=7$, $\alpha_1=0$, the reciprocal transformation is of the order $9-6$, $=3$; and assuming (as seems allowable) that the principal system does not contain any multiple point or curve, it must be of the same form as the original transformation, that is, we must have $n'=3$, $m_1'=6$, $h_1'=7$, $\alpha_1'=0$.

88. The transformations to be studied are thus,—1° The quadri-quadric transformation $n=2$, $m_1=2$, $h_1=0$, $\alpha_1=1$, and $n'=2$, $m_1'=2$, $h_1'=0$, $\alpha_1'=1$; the principal system in each space consists of a point and of a conic (which may be a pair of intersecting lines); and the surfaces are quadrics. 2° The quadri-cubic transformation $n=2$, $m_1=1$, $h_1=0$, $\alpha_1=3$, and $n'=3$, $\alpha_1'=0$, $m_1'=3$, $h_1'=3$, $m_2'=1$, $h_2'=0$: in the first space the principal system consists of three points and a line, and the surfaces are quadrics: in the second figure, the principal system consists of three simple lines and a double line; and the surfaces are cubic surfaces passing through this principal system, that is, they are cubic scrolls. 3° The cubo-cubic transformation $n=3$, $\alpha_1=0$, $m_1=6$, $h_1=7$, and $n'=3$, $\alpha_1'=0$, $m_1'=6$, $h_1'=7$; in each space the principal system is a sextic curve with seven apparent double points (but there are different cases to be considered according as the sextic curve does or does not break up into inferior curves), and the surfaces are cubic surfaces through the sextic curve.

The Quadri-quadric Transformation between Two Spaces.

89. Starting from the equations $x' : y' : z' : w' = X : Y : Z : W$, we have here $X=0$, &c., quadric surfaces passing through a given point and a given conic (which may be a pair of intersecting lines). Take $x=0$, $y=0$, $z=0$ for the coordinates of the given point; $w=0$ for the equation of the plane of the conic; the conic is then given as the intersection of this plane by a cone having the given point for its vertex; or say the equations of the conic are $w=0$, $(a, \ldots \between x, y, z)^2=0$; the general equation of a quadric through the point and conic is $w(\alpha x+\beta y+\gamma z)+\delta(a, \ldots \between x, y, z)^2=0$; and it hence appears that the equations of the transformation may be taken to be

$$x' : y' : z' : w' = xw : yw : zw : (a, \ldots \between x, y, z)^2;$$

these give at once a reciprocal system of the same form; viz., the two sets are

$$x' : y' : z' : w' = xw : yw : zw : (a, \ldots \between x, y, z)^2,$$

and

$$x : y : z : w = x'w' : y'w' : z'w' : (a, \ldots \between x', y', z')^2.$$

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90. The Jacobian of the first space is at once found to be

$$w^2(a, \ldots \between x, y, z)^2 = 0\,;$$

that of the second space is of course

$$w'^2(a, \ldots \between x', y', z')^2 = 0.$$

The two spaces are similar to each other; we may say that there is in each of them a principal point and a principal conic; that the plane of the conic is the principal plane, and the cone having its vertex at the point and passing through the conic is the principal cone. To the principal point of either space corresponds any point whatever in the principal plane of the other space; and conversely. More definitely, the points of the one principal plane and the infinitesimal elements of direction through the principal point of the other space correspond according to the equations $x : y : z = x' : y' : z'$. To any point on the principal conic of either space corresponds in the other space, not a mere element of direction through the principal point of the other space, but a line of the principal cone; that is, to the points of the principal conic of the one space correspond the lines of the principal cone of the other space. The Jacobian of either space, consisting of the principal plane twice, and of the principal cone, is thus the principal counter-system of the other space.

91. {Writing $(a, \ldots \between x, y, z)^2 = x^2 + y^2 + z^2$, $w = w' = 1$, the equations of transformation become

$$x' : y' : z' : 1 = x : y : z : x^2 + y^2 + z^2,$$

and

$$x : y : z : 1 = x' : y' : z' : x'^2 + y'^2 + z'^2,$$

or, what is the same thing, if for shortness

$$x^2 + y^2 + z^2 = r^2, \quad x'^2 + y'^2 + z'^2 = r'^2,$$

the equations are

$$x' = \frac{x}{r^2}, \quad y' = \frac{y}{r^2}, \quad z' = \frac{z}{r^2}\,; \text{ and } x = \frac{x'}{r'^2}, \quad y = \frac{y'}{r'^2}, \quad z = \frac{z'}{r'^2},$$

whence also $rr' = 1$; this is the well known transformation by reciprocal radius vectors.}

92. The principal conic may be a pair of intersecting lines; taking its equations to be $w = 0$, $xy = 0$, the equations of transformation here become

$$x' : y' : z' : w' = xw : yw : zw : xy,$$

and

$$x : y : z : w = x'w' : y'w' : z'w' : x'y'.$$

There is no difficulty in the further development of the theory.

The Quadri-cubic Transformation between Two Spaces.

93. It will be convenient to have the unaccented letters (x, y, z, w) referring to the cubic surfaces. I will therefore take the quadric surfaces in the second figure; viz., I will start from the equations $x : y : z : w = X' : Y' : Z' : W'$, where $X' = 0$,

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$Y'=0$, $Z'=0$, $W'=0$ are quadric surfaces passing through three fixed points (say the principal points) and through a fixed line (say the principal line) in the second figure. Taking $x'=0$, $y'=0$ for the planes passing through the principal line and through two of the principal points respectively; $z'=0$ for the plane passing through the three principal points, $w'=0$ for an arbitrary plane passing through the first mentioned two principal points, the implicit factors of x', y', w' may be so determined that for the third principal point $x'=y'=-w'$. That is, we shall have

$$\begin{array}{ll} \text{for principal line} & x'=0,\ y'=0, \\ \text{for principal points} & (x'=0,\ z'=0,\ w'=0), \\ \quad\text{"} & (y'=0,\ z'=0,\ w'=0), \\ \quad\text{"} & (x'=y'=-w',\ z'=0), \end{array}$$

and this being so, the equation of a quadric surface through the principal points and line will be

$$(\alpha x'+\beta y')\,z'+\gamma x'\,(y'+w')+\delta y'\,(x'+w'),$$

and the equations of transformation may be taken to be

$$x : y : z : w \qquad = x'z' : y'z' : x'\,(y'+w') : y'\,(x'+w').$$

94. Writing these in the extended form

$$x : y : z : w : x-y : z-w = x'z' : y'z' : x'\,(y'+w') : y'\,(x'+w') : z'\,(x'-y') : w'\,(x'-y')$$

and forming also the equation

$$xy : (xw-yz) = z' : x'-y',$$

we at once derive the reciprocal system of equations

$$x' : y' : z' : w' \qquad = x\,(xw-yz) : y\,(xw-yz) : (x-y)\,xy : (z-w)\,xy,$$

so that this is a cubic transformation. And the cubic surface in the first space (corresponding to an arbitrary plane $ax'+by'+cz'+dw'=0$ of the second space) is $(ax+by)(xw-yz)+c\,(x-y)\,xy+d\,(z-w)\,xy=0$; viz., this is a cubic surface having the fixed double line $(x=0,\ y=0)$, the fixed simple lines $(x=0,\ z=0)$, $(y=0,\ w=0)$, and $(x-y=0,\ z-w=0)$; it has also the variable simple line $(dz+cx=0,\ dw+cy=0)$. The principal figure of the first space thus consists of the three simple lines $(x=0,\ z=0)$, $(y=0,\ w=0)$, $(x-y=0,\ z-w=0)$, and of the line $(x=0,\ y=0)$, a double line counting four times in the intersection of two of the cubic surfaces.

95. The cubic surface as having the double line $(x=0,\ y=0)$ is a cubic scroll, and this line is the nodal directrix thereof; the line $(dz+cx=0,\ dw+cy=0)$ is the simple directrix; the lines $(x=0,\ z=0)$, $(y=0,\ w=0)$, $(x-y=0,\ z-w=0)$ are at once seen to be lines meeting each of these directrix lines; and they are generating lines of the scroll. To explain the generation of the scroll, observe that the section by any plane is a cubic curve having a given double point (viz., the intersection of the plane with the nodal directrix); and three other given points (viz., the intersections of

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the plane with the three generating lines respectively); this cubic also passes through the intersection of the plane with the simple directrix. Conversely, if the plane be assumed at pleasure, and if, taking for the simple directrix any line which meets the given generating lines, we draw a cubic as above, then the scroll is the scroll generated by a line which meets each of the directrix lines, and also the cubic.

If the plane be taken to pass through any generating line, then the cubic section breaks up into this line, and a conic; the conic does not meet the simple directrix, but it meets the nodal directrix; and any such conic will serve as a directrix; viz., the scroll is generated by the lines which meet the two directrix lines and the conic.

96. Any two scrolls as above meet in the three fixed generating lines, and in the nodal directrix counting four times; they consequently meet besides in a curve of the second order, which is a conic (one of the conics just referred to). In order to further explain the theory, suppose for a moment that the two scrolls had only a common nodal directrix; they would besides meet in a quintic curve; this curve would meet the nodal directrix in four points, viz., the points at which the two scrolls have a common tangent plane. Now if at any point of the nodal directrix the two scrolls have a common generating line, then the plane through this line and the nodal line is one of the two tangent planes of each scroll; that is, the scrolls have this plane for a common tangent plane. Hence, in the case of the common three generating lines, the points where these meet the nodal line are three of the four points just referred to; there remains therefore one point, which is the point where the conic meets the nodal line; through this point there are for each of the scrolls two generating lines; one of these for the first scroll, and one for the second scroll, lie in a plane with the nodal line; the other two determine the plane of the conic; and the tangent to the conic at its intersection with the nodal line is the intersection of the plane of the conic with the plane of the first-mentioned two generating lines.

97. Analytically we have the two equations

$$c\ (x-y)\,xy + (ax\ + by\ )\,(xw - yz) + d\ \ (z-w)\,xy = 0,$$
$$c'\,(x-y)\,xy + (a'x + b'y)\,(xw - yz) + d'\,(z-w)\,xy = 0\,;$$

or, combining these equations so as to eliminate successively the terms in $x\,(xw-yz)$ and $y\,(xw-yz)$, and for this purpose writing

$$(bc'-b'c,\ ca'-c'a,\ ab'-a'b,\ ad'-a'd,\ bd'-b'd,\ cd'-c'd) = (\mathrm{a},\ \mathrm{b},\ \mathrm{c},\ \mathrm{f},\ \mathrm{g},\ \mathrm{h}),$$

and therefore

$$\mathrm{af} + \mathrm{bg} + \mathrm{ch} = 0,$$

we have

$$\mathrm{b}\,(x-y)\,x - \mathrm{c}\,(xw-yz) - \mathrm{f}\ (z-w)\,x = 0,$$
$$-\,\mathrm{a}\,(x-y)\,y + \mathrm{c}\,(xw-yz) - \mathrm{g}\,(z-w)\,y = 0,$$

and multiplying the first of these by $\mathrm{c}+\mathrm{g}$ and the second by $\mathrm{c}-\mathrm{f}$, and adding, the whole divides by $x-y$, and the final result is

$$(\mathrm{c}+\mathrm{g})\,(\mathrm{b}x - \mathrm{f}z) - (\mathrm{c}-\mathrm{f})\,(\mathrm{a}y + gw) = 0\,;$$

viz., this is the equation of the plane of the conic.

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98. Any two scrolls as above meeting in a conic, a third scroll will meet the conic in six points; but these include the point on the nodal directrix twice, and the points on the three fixed generating lines each once; there is left a single point of intersection, viz., this is the *one variable* point of intersection of the three scrolls; which is in accordance with the theory.

99. For the Jacobian of the second space, we have

$$\begin{vmatrix} z', & 0, & x', & 0 \\ 0, & z', & y', & 0 \\ y'+w', & x', & 0, & 0 \\ y', & x'+w', & 0, & y' \end{vmatrix} = 0;$$

that is, $2x'y'z'(x'-y')=0$; viz., $z'=0$ is the plane containing the three principal points; and $x'=0$, $y'=0$, $x'-y'=0$ are the planes which pass through the principal line and the three principal points respectively.

100. For the Jacobian of the first space, we have

$$\begin{vmatrix} 2xw-yz, & -xz, & -xy, & x^2 \\ yw, & xw-2yz, & -y^2, & xy \\ 2xy-y^2, & x^2-2xy, & 0, & 0 \\ (z-w)\,y, & (z-w)\,x, & xy, & -xy \end{vmatrix} = 0;$$

that is, $3x^2y^2(x-y)^2(xw-yz)=0$; viz., $x=0$, $y=0$, $x-y=0$ are the planes through the nodal directrix and the three fixed generators respectively (each plane therefore occurring twice); and $xw-yz=0$ is the quadric scroll generated by the lines which meet each of the three generators $(x=0,\ z=0)$, $(y=0,\ w=0)$, $(x-y=0,\ z-w=0)$; this scroll passing also through the nodal directrix $x=0$, $y=0$.

The Cubo-cubic Transformation between Two Spaces.

101. The principal system in the first space is a sextic curve with 7 apparent double points; but this curve may be either a single curve, or it may break up into inferior curves. I have not examined all the cases which may arise; but the two extreme cases are—(A) The sextic curve breaks up into six lines, viz., two non-intersecting lines, and four other lines each meeting each of the two lines (this implies that no two of the four lines meet each other): here the two lines give 1 apparent double point, and the four lines give 6 apparent double points; total number is $=7$, as it should be. (B) The curve is a proper sextic curve, with 7 apparent double points: this gives, as will be shown, the general lineo-linear transformation. The two cases are each of them symmetrical.

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(A) *The Principal System consists of Six Lines.*

102. Taking in the first space, for the equations of the two lines, $(x=0,\ y=0)$ and $(z=0,\ w=0)$, and for the equations of the four lines, $(x=0,\ z=0)$, $(y=0,\ w=0)$, $(x-y=0,\ z-w=0)$, $(x-py=0,\ z-qw=0)$, then, if the equations of transformation are taken to be

$$\begin{aligned} x'-py' : x'-y' : z'-qw' : z'-w' = {} & (x-py)(xw-yz) \\ & : (x-y)(qxw-pyz) \\ & : (z-qw)(xw-yz) \\ & : (z-w)(qxw-pyz); \end{aligned}$$

these lead conversely (see *post*, No. 104) to a like system,

$$\begin{aligned} x-py : x-y : z-qw : z-w = {} & (x'-py')M' \\ & : (x'-y')N' \\ & : (z'-qw')M' \\ & : (z'-w')N', \end{aligned}$$

where for shortness

$$\begin{aligned} M' &= p(q-1)^2 x'w' - q(p-1)^2 y'z' + (pq-1)(p-q)y'w', \\ N' &= (q-1)^2 x'w' - (p-1)^2 y'z' + (pq-1)(p-q)y'w'; \end{aligned}$$

or, as these are better written,

$$\begin{aligned} M' &= -q(p-1)y'\{(p-1)z'-(pq-1)w'\} + p(q-1)w'\{(q-1)x'-(pq-1)y'\}, \\ N' &= -(p-1)y'\{(p-1)z'-(pq-1)w'\} + (q-1)w'\{(q-1)x'-(pq-1)y'\}. \end{aligned}$$

Hence the principal system in the second plane is composed of the two non-intersecting lines $(x'=0,\ y'=0)$, $(z'=0,\ w'=0)$ and the four lines $\{(p-1)z'-(pq-1)w'=0,\ (q-1)x'-(pq-1)y'=0\}$, $(y'=0,\ w'=0)$, $(x'-y'=0,\ z'-w'=0)$, $(x'-py'=0,\ z'-qw'=0)$, each meeting each of the two lines.

103. The Jacobian of the first space is

$$\begin{vmatrix} 2xw-yz-pyw, & -xz-pxw+2pyz, & -y(x-py), & x(x-py) \\ 2qxw-pyz-qyw, & -pxz-qxw+2pyz, & -py(x-y), & qx(x-y) \\ w(z-qw), & -z(z-qw), & xw-2yz+qyw, & xz-2qxw+qyz \\ qw(z-w), & -pz(z-w), & qxw-2pyz+pyw, & qxz-2qxw+pyz \end{vmatrix} = 0,$$

viz., this is $xyzw(x-y)(x-py)(z-w)(z-qw)=0$, the equation of the planes each passing through one of the four lines and one of the two lines.

Similarly, the Jacobian of the second space is

$$\{(q-1)x'-(pq-1)y'\}\{(p-1)z'-(pq-1)w'\}(x'-y')(z'-w')(x'-py')(z'-qw')y'w'=0;$$

viz., this is the equation of the eight planes each passing through one of the four lines and one of the two lines.

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104. To effect the foregoing transformation, writing

$$\begin{aligned} x' : y' : z' : w' = \ & (x-py)(\ xw-\ yz) \\ &: (x-\ y)(qxw-pyz) \\ &: (z-qw)(\ xw-\ yz) \\ &: (z-\ w)(qxw-pyz); \end{aligned}$$

or what will ultimately be the same thing, but it is more convenient for working with,

$$\begin{aligned} x' &= (x-py)(\ xw-\ yz), \\ y' &= (x-\ y)(qxw-pyz), \\ z' &= (z-qw)(\ xw-\ yz), \\ w' &= (z-\ w)(qxw-pyz); \end{aligned}$$

these give

$$\begin{aligned} x-py &= M'x', \\ x-\ y &= N'y', \\ z-qw &= M'z', \\ z-\ w &= N'w', \end{aligned}$$

where M', N' are quantities which have to be determined; and thence

$$\begin{aligned} (1-p)\,x &= M'x'-pN'y', \\ (1-p)\,y &= M'x'-\ N'y', \\ (1-q)\,z &= M'z'-qN'w', \\ (1-q)\,w &= M'z'-\ N'w'; \end{aligned}$$

whence also

$$(1-p)(1-q)(\ xw-\ yz) = N'[\ \{(q-1)\,x'w'-(p-1)\,y'z'\}\,M'+(p-q)\,y'w'N'],$$

$$(1-p)(1-q)(qxw-pyz) = M'[-(p-q)\,x'z'M'+\{(pq-q)\,x'w'-(pq-p)\,y'z'\}\,N'];$$

but we have

$$\frac{xw-yz}{qxw-pyz} = \frac{x'}{y'} \div \frac{x-py}{x-y} = \frac{x'}{y'} \div \frac{M'x'}{N'y'} = \frac{N'}{M'};$$

or, substituting,

$$\begin{aligned} & M'\{\ \ (q-1)\,x'w'-(p-1)\,y'z'\}+N'(p-q)\,y'w' \\ &= M'\{-(p-q)\,x'z'\}+N'\{(pq-q)\,x'w'-(pq-p)\,y'z'\}; \end{aligned}$$

that is

$$M'\{(q-1)\,x'w'-(p-1)\,y'z'+(p-q)\,x'z'\} = N'\{(pq-q)\,x'w'-(pq-p)\,y'z'-(p-q)\,y'w'\};$$

or, what is the same thing,

$$\begin{aligned} M' &= (pq-q)\,x'w'-(pq-p)\,y'z'-(p-q)\,y'w', \\ N' &= (\ q-1)\,x'w'-(\ p-1)\,y'z'+(p-q)\,x'z'; \end{aligned}$$

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viz., M', N' having these values, the original equations

$$\begin{aligned} x' : y' : z' : w' = \ & (x-py)(\ xw - \ yz) \\ : \ & (x - \ y)(pxw - qyz) \\ : \ & (z-qw)(\ xw - \ yz) \\ : \ & (z - \ w)(pxw - qyz), \end{aligned}$$

give

$$x-py : x-y : z-qw : z-w = M'x' : N'y' : M'z' : N'w'.$$

If, in these equations, in place of (x', y', z', w') we write $(x'-py', x'-y', z'-qw', z'-w')$, the new values of M', N' are found to be

$$\begin{aligned} M' &= p(q-1)^2 x'w' - q(p-1)^2 y'z' + (pq-1)(p-q)\, y'w', \\ N' &= \ \ (q-1)^2 x'w' - \ \ (p-1)^2 y'z' + (pq-1)(p-q)\, y'w', \end{aligned}$$

and we have the formulæ of No. 102.

(B) *The Principal System of a Proper Sextic Curve; the Lineo-linear Transformation between Two Spaces.*

105. I start with the lineo-linear transformation, and show that this is in fact a transformation such that the principal system in either space is a sextic curve with seven apparent double points. I do not attempt any formal proof, but assume that the lineo-linear transformation is the most general one which gives rise to such a principal system.

We have between (x, y, z, w), (x', y', z', w') three lineo-linear equations; writing these first under the form

$$\begin{aligned} (P_1, Q_1, R_1, S_1 \between x', y', z', w') &= 0, \\ (P_2, Q_2, R_2, S_2 \between x', y', z', w') &= 0, \\ (P_3, Q_3, R_3, S_3 \between x', y', z', w') &= 0, \end{aligned}$$

we have $x' : y' : z' : w' = X : Y : Z : W$, where X, Y, Z, W are the determinants (each with its proper sign) formed out of the matrix

$$\left| \begin{matrix} P_1, & Q_1, & R_1, & S_1 \\ P_2, & Q_2, & R_2, & S_2 \\ P_3, & Q_3, & R_3, & S_3 \end{matrix} \right| .$$

106. Each of the surfaces $X=0$, $Y=0$, $Z=0$, $W=0$, or generally any surface $aX+bY+cZ+dW=0$, is thus a cubic surface passing through the curve

$$\left\| \begin{matrix} P_1, & Q_1, & R_1, & S_1 \\ P_2, & Q_2, & R_2, & S_2 \\ P_3, & Q_3, & R_3, & S_3 \end{matrix} \right\| = 0,$$

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which is at once seen to be of the order 6. In fact any two of these surfaces, for instance

$$\begin{vmatrix} P_1, & Q_1, & R_1 \\ P_2, & Q_2, & R_2 \\ P_3, & Q_3, & R_3 \end{vmatrix} = 0 \text{ and } \begin{vmatrix} P_1, & Q_1, & S_1 \\ P_2, & Q_2, & S_2 \\ P_3, & Q_3, & S_3 \end{vmatrix} = 0,$$

have in common a curve

$$\left\|\begin{matrix} P_1, & P_2, & P_3 \\ Q_1, & Q_2, & Q_3 \end{matrix}\right\| = 0,$$

which is of the order 3; they consequently besides intersect in a curve of the order 6, which is the before mentioned curve of intersection of all the surfaces. And it further appears that the number of the apparent double points is $=7$; in fact the formula in the case of two surfaces of the orders μ, ν, the complete intersection of which consists of a curve of the order m with h apparent double points, and of a curve of the order m' with h' apparent double points, the numbers m, m', h, h' are connected by the equation $2(h-h') = (m-m')(\mu-1)(\nu-1)$. (Salmon's *Solid Geometry*, 2nd Ed., p. 273 [Ed. 4, p. 311]). Hence, in the case of the two cubic surfaces intersecting as above (since for the cubic curve we have $m'=3$, $h'=1$, and for the sextic $m=6$), the formula becomes $2(h-1)=12$, that is $h=1+6=7$; or the number of apparent double points is $=7$.

107. It thus appears that the principal system in the first plane is a curve of the order 6, with seven apparent double points: it is to be added that there are not in general any actual double points or stationary points, so that the class of the curve is $6.5-2.7, =16$, and its deficiency is $\tfrac{1}{2}5.4-7, =3$. For convenience I will refer to this as the curve Σ.

The transformation is obviously a symmetrical one; hence the principal system in the second space is in like manner a curve of the order 6, with seven apparent double points; say it is the curve Σ'.

108. Consider in the first space any point P on the curve Σ; for this point the three equations

$$(P_1, Q_1, R_1, S_1 \between x', y', z', w') = 0,$$
$$(P_2, Q_2, R_2, S_2 \between \quad ,, \quad) = 0,$$
$$(P_3, Q_3, R_3, S_3 \between \quad ,, \quad) = 0,$$

are not independent, but are equivalent to two linear equations in (x', y', z', w'); that is, to the point P on the curve Σ there corresponds in the second space, not a determinate point P', but any point whatever on a certain line L'; or say to the point P on Σ there corresponds a line L'; and as P describes the curve Σ, L' describes a scroll Π'; that is, to the curve Σ there corresponds a scroll Π', the principal counter-system in the second space. Similarly to the curve Σ' there corresponds a scroll Π, the principal counter-system of the first space.

109. The scroll Π is the Jacobian of the first space; and as such it is of the order 8, having the curve Σ for a triple line—and it thus appears that the Jacobian

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of the first space is a scroll (a theorem the analytical verification of which seems by no means easy). But without assuming the identity of the scroll Π with this Jacobian, or taking the order of the scroll to be known, I proceed to show that the scroll Π is the scroll generated by the lines each of which meets the curve Σ three times; it will thereby appear that the order is $=8$, and that the curve is a triple line on the scroll.

Consider a point P' on Σ', and the corresponding line L of the first space: take Θ' a plane in the second space; corresponding to it the cubic surface Θ in the first space. By imposing a single relation on the coefficients (a, b, c, d) in the equation $ax' + by' + cz' + dw' = 0$ of the plane Θ', we make it pass through the point P'; therefore by imposing this same *single* relation on the coefficients (a, b, c, d) of the cubic surface Θ, we make it pass through the line L; Θ is a cubic surface through Σ; and it is easy to see that the effect will be as above only if the line L cuts the curve Σ three times; this being so, the general cubic surface Θ meets L in three points (viz., the three intersections of L with Σ), and if Θ be made to pass through a fourth point on the line L, it will pass through the line L; it thus appears that the line L meets Σ three times, and consequently that the scroll Π is generated by the lines which meet Σ three times.

110. The theory of a scroll so generated is considered in my "Memoir on Skew Surfaces, otherwise Scrolls"(1). Writing $m=6$, $h=7$ and therefore $M\,[= -\tfrac{1}{2}m(m-1)+h], = -8$, the order of the scroll is $\left(\tfrac{1}{3}[m]^3 + (m-2)M = 40 - 32\right) = 8$; but calculating the values of
$NG\,(m^3) = \tfrac{1}{2}[m]^4 + 6m + M(3[m]^2 - 12m + 33) + M^2 . 3$,
$NR\,(m^3) = \tfrac{1}{18}[m]^6 + \tfrac{3}{8}[m]^5 - \tfrac{1}{2}[m]^3 - 3m + M(\tfrac{1}{3}[m]^4 - \tfrac{1}{6}[m]^3 - \tfrac{5}{2}m^2 + 8m - 20) + M^2(\tfrac{1}{2}[m]^2 - 2m)$;
these are found to be respectively $=0$; viz., there are no nodal generators, and no nodal residue; the sextic curve Σ is a triple curve on the surface, and there is not any other multiple line.

111. It may be remarked that any plane Θ' meets the sextic curve Σ' in six points; hence the corresponding cubic surface Θ contains six lines, generatrices of Π, and, therefore, each meeting the curve Σ three times; say six lines L. Through one of these lines L, draw to the cubic surface Θ a triple tangent plane meeting it in the line L and in two other lines, say M, N; this plane must meet Σ in three new points which must lie on the lines M, N; viz., one of these lines must pass through two of the points, and the other line through the third point.

Addition—September, 1870.

[Some corrections have been made in accordance with the concluding paragraph of a paper "Note on the Rational Transformation and on Special Systems of Points," 450.]

The formulæ of No. 84 are included in the following more general formulæ; viz., if the principal system consist of α_1 points, each a simple point, α_2 points each a

1 *Phil. Trans.* vol. CLIII. 1863, pp. 453—483, [339]. See the Table $S\,(m^3)$ &c., p. 457; in the value of $NR\,(m^3)$ instead of term $+3m$ read $-3m$. [This correction should have been made in the present Reprint.]

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quadri-conical point, α_3 points each a cubi-conical point, &c., and of a simple curve order m_1 with h_1 apparent double points, a double curve order m_2 with h_2 apparent double points, and so on; and if moreover, the curves m_1, m_2 intersect in $k_{1,2}$ points, the curves m_1, m_3 in $k_{1,3}$ points, &c.; then writing in general $\rho = \frac{1}{2}m(m-1) - h$; that is, $\rho_1 = \frac{1}{2}m_1(m_1-1) - h_1$, $\rho_2 = \frac{1}{2}m_2(m_2-1) - h_2$, &c., I find that the general condition of equivalence is

$$\left.\begin{array}{l} \alpha_1 + (\ 3n - \ 2)\, m_1 - 2\rho_1 \\ \quad + 8\alpha_2 + (12n - 16)\, m_2 - 16\rho_2 \\ \qquad \vdots \\ \quad + r^3\alpha_r + (3r^2n - 2r^3)\, m_r - 2r^3\rho_r \\ \quad - 5k_{1,2} - 8k_{1,3} \ldots - (3r-1)\, k_{1,r} \\ \qquad - 28k_{2,3} \\ \qquad\qquad \ldots - s^2(3r-s)\, k_{s,r}\, (s < r) \end{array}\right\} = n^3 - 1;$$

and that the general condition of postulation is

$$\left.\begin{array}{l} \alpha_1 + (\ n+1)\, m_1 - \ \rho_1 \\ \quad + 4\alpha_2 + (3n+1)\, m_2 - 5\rho_2 \\ \qquad \vdots \\ \quad + \frac{1}{6}r(r+1)(r+2)\,\alpha_r \\ \quad + [\frac{1}{2}r(r+1)\,n - \frac{1}{6}r(r+1)(2r-5)]\, m_r \\ \qquad - \frac{1}{24}[(r-1)(r-2)(r-3)(r-4) \\ \qquad\qquad + 4r(r+1)(2r+1)]\,\rho_r \\ - 2k_{1,2} - 3k_{1,3} \ldots - rk_{1,r} \\ \quad - 8k_{2,3} \\ \qquad \vdots \\ \quad - \frac{1}{2}s(s+1)\{r+1-\frac{1}{3}(s+2)\}\, k_{s,r}\, (s<r) \end{array}\right\} = \frac{1}{6}(n+1)(n+2)(n+3) - 4:$$

in which formulæ it is however assumed that the curves have not any actual multiple points. This implies that if any one of the curves, say m_r, break up into two or more curves, the component curves do not intersect each other; for, of course, any such point of intersection would be an actual double point on the curve m_r. I believe, however, that the formulæ will extend to this case by admitting for s the value $s = r$; viz., if we suppose the curve m_r to be the aggregate of the two curves m_r', m_r'' intersecting in K_r points, then that the corresponding terms in the equivalence-equation are

$$(3r^2n - 2r^3)(m_r' + m_r'') - 2r^3(\rho_r' + \rho_r'') - 2r^3K_r,$$

and that those in the postulation-equation are

$$\begin{array}{l} [\frac{1}{2}r(r+1)\,n - \frac{1}{6}r(r+1)(2r-5)]\,(m_r' + m_r'') \\ \qquad - \frac{1}{24}[(r-1)(r-2)(r-3)(r-4) + 4r(r+1)(2r+1)]\,(\rho_r' + \rho_r'') \\ \quad - \frac{1}{6}r(r+1)(2r+1)\,K_r. \end{array}$$

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Let the r-tuple curve consist of three right lines meeting in a point: this is an actual triple point, and the formulæ do not apply. But calculating the postulation-terms by the formula, we have $m_r = 3$, $\rho_r = \tfrac{1}{2}3 \,.\, 2 - 0, = 3$; and the terms are

$$[\tfrac{1}{2}r(r+1)n - \tfrac{1}{6}r(r+1)(2r-5)]\,3 - \tfrac{1}{8}\,[(r-1)(r-2)(r-3)(r-4) + 4r(r+1)(2r+1)],$$

which are

$$= \tfrac{1}{2}r(r+1)(3n-4r+4) - \tfrac{1}{8}(r-1)(r-2)(r-3)(r-4),$$

or say

$$= \tfrac{1}{2}r(r+1)(3n-4r+4) + \tfrac{1}{8}(-r^4 + 10r^3 - 35r^2 + 50r - 24).$$

I have found by an independent investigation that this value requires the correction

$$+\tfrac{1}{8}\,[r^4 - 8r^3 + 30r^2 - 56r + 24 + \tfrac{1}{2}\{1-(-)^r 1\}],$$

and that the true value of the postulation is

$$= \tfrac{1}{2}r(r+1)(3n-4r+6) + \tfrac{1}{8}[\quad 2r^3 - 5r^2 - 6r \qquad + \tfrac{1}{2}\{1-(-)^r 1\}],$$

viz., that this is the number of the conditions to be satisfied that a surface of the order n may have for an r-tuple curve three given right lines meeting in a point.

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