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NOTE ON THE CARTESIAN WITH TWO IMAGINARY AXIAL FOCI.

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LET A, A', B, B' be a pair of points and antipoints; viz.,

(A, A') the two imaginary points, coordinates $(\pm \beta i, 0)$,

(B, B') the two real points, coordinates $(0, \pm \beta)$;

and write $\rho, \rho', \sigma, \sigma'$ for the distances of a point (x, y) from the four points respectively; say

$$\rho = \sqrt{(x+\beta i)^2 + y^2}, \quad \sigma = \sqrt{x^2 + (y+\beta)^2},$$

$$\rho' = \sqrt{(x-\beta i)^2 + y^2}, \quad \sigma' = \sqrt{x^2 + (y-\beta)^2}.$$

We have

$$\rho^2 + \rho'^2 = 2x^2 + 2y^2 - 2\beta^2 = \sigma^2 + \sigma'^2 - 4\beta^2,$$

$$\rho\rho' = \sqrt{(x+\beta i + yi)(x+\beta i - yi)(x-\beta i + yi)(x - \beta i - yi)} = \sigma\sigma';$$

and thence

$$(\rho + \rho')^2 = (\sigma + \sigma')^2 - 4\beta^2,$$

$$(\rho - \rho')^2 = (\sigma - \sigma')^2 - 4\beta^2;$$

or say

$$\rho + \rho' = \sqrt{(\sigma+\sigma')^2 - 4\beta^2},$$

$$i(\rho - \rho') = \sqrt{4\beta^2 - (\sigma - \sigma')^2}.$$

The equation of a Cartesian having the two imaginary axial foci A, A' is

$$(p+qi)\rho + (p-qi)\rho' + 2k^2 = 0;$$

viz., this is

$$p(\rho+\rho')+qi(\rho-\rho')+2k^2=0;$$

or, what is the same thing, it is

$$p\sqrt{(\sigma+\sigma')^2+4\beta^2}+q\sqrt{4\beta^2-(\sigma-\sigma')^2}+2k^2=0,$$

which is the equation expressed in terms of the distances σ, σ' from the non-axial real foci B, B'. Of course, the radicals are to be taken with the signs $\pm$. This equation gives, however, the Cartesian in combination with an equal curve situate symmetrically therewith in regard to the axis of y.

The distances σ, σ' may conveniently be expressed in terms of a single variable parameter θ; in fact, we may write

$$\pm p\sqrt{(\sigma+\sigma')^2-4\beta^2}=-k^2-k\theta,$$
$$\pm q\sqrt{4\beta^2-(\sigma-\sigma')^2}=-k^2+k\theta;$$

that is

$$(\sigma+\sigma')^2-4\beta^2=\frac{k^2}{p^2}(k+\theta)^2,$$

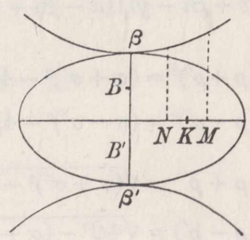
$$4\beta^2-(\sigma-\sigma')^2=\frac{k^2}{q^2}(k-\theta)^2;$$

and therefore

$$\sigma+\sigma'=\sqrt{4\beta^2+\frac{k^2}{p^2}(k+\theta)^2},$$

$$\sigma-\sigma'=\pm\sqrt{4\beta^2-\frac{k^2}{q^2}(k-\theta)^2}:$$

so that, assigning to θ any given value, we have σ, σ', and thence the position of the point on the curve. We may draw the hyperbola $y^2=4\beta^2+\frac{k^2}{p^2}x^2$, and the ellipse $y^2=4\beta^2-\frac{k^2}{q^2}x^2$; and then measuring off in these two curves respectively the ordinates which belong to the abscissæ $k+\theta$ for the hyperbola, $k-\theta$ for the ellipse, we have



the values $\sigma+\sigma'$ and $\sigma-\sigma'$, which determine the point on the curve. Considering k, p, q, β as disposable quantities, the conics may be any ellipse and hyperbola whatever, having a pair of vertices in common; and the complete construction is,—From the fixed point K in the axis of x, measure off in opposite directions the equal distances KM, KN, and take

$$\sigma + \sigma' \text{ the ordinate at } M \text{ in the hyperbola,}$$
$$\pm(\sigma - \sigma') \quad ,, \quad ,, \quad N \quad ,, \quad \text{ellipse;}$$

where σ, σ' denote the distances of the required point from the fixed points B and B' respectively, the distance of each of these from the origin being $= \frac{1}{2}$ the common semi-axis. We may imagine N travelling from one extremity of the x-axis of the ellipse to the other, the value of $\sigma + \sigma'$ will be real and greater than BB', that of $\sigma - \sigma'$ real and less than BB', and the point (σ, σ') will be real. The construction gives, it will be observed, the two symmetrically situated curves.

The x-semi-axis of the ellipse is $\frac{q}{k} 2\beta$, and the form of the curve depends chiefly on the value of the ratio $k : \frac{q}{k} 2\beta$; or, what is the same thing, $k^2 : 2\beta q$. We see, for instance, that, in order that the curve may meet the axis of y in two real points between the foci, the value $\theta = -k$ must give a real value of $\sigma - \sigma'$; viz., that we must have $4\beta^2 > \frac{4k^4}{q^2}$; that is, $\beta^2 q^2 > k^4$, or $k^2 < \beta q$. If k has this value, viz., $k = \frac{1}{2} \frac{q}{k} 2\beta =$ $\frac{1}{2}$ semi-axis, the curve *touches* the axis of y at the origin; if $k < \frac{1}{2}$ semi-axis, the curve cuts the axis of y in two real points between the foci; if $k > \frac{1}{2}$ semi-axis, the curve does not cut the axis of y between the foci.