454.

A THIRD MEMOIR ON QUARTIC SURFACES.

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THE present Memoir is a continuation of my former researches on Nodal Quartic Surfaces, [445, 451]. The leading idea is, that for a quartic surface with k-nodes, given the nature of the circumscribed, $(k-1)$ nodal, sextic cone belonging to any one node of the surface {for instance, $k=10$, that it is a cone (3, 3) composed of two cubic cones}, we thereby determine the equation of the quartic surface, and consequently the nature of the remaining $(k-1)$ nodes thereof. By means of this general theory I complete, in an essential point, the theory of the *Symmetroid;* viz., I show that a 10-nodal quartic surface having a single node (3, 3) is a Symmetroid; whence, as appears by my second Memoir, [451], each of the remaining nine nodes is also a node (3, 3); and we have the theory of the remarkable system of ten points in space such that, joining any one of them with the remaining nine, the nine lines thus obtained are the intersections of two cubic cones. A large part of the Memoir is devoted to the consideration of the surfaces with 16, 15, 14, and 13 nodes: this is substantially a reproduction of the results obtained by Kummer in the Memoir "Ueber die algebraischen Strahlensysteme, &c.," already referred to; but the results in question are brought into connexion with the theory of the present Memoir, and they are, by a change of the constants, exhibited in a form of much greater symmetry and elegance. I attach importance also to the square diagrams by means of which I have exhibited, in a compendious form, the relation between the several nodes and circumscribed sextic cones.

The paragraphs are numbered consecutively with those of the first and second Memoirs.

Preliminary Considerations and Classification.

125. I call to mind that if a quartic surface has a node (conical point), then there is for this node a tangent quadricone and a circumscribed sextic cone; viz., if the surface has $(k-1)$ other nodes, or in all k nodes, then the sextic cone has $(k-1)$

nodal lines (passing through the other nodes respectively), and we have thus for the different forms of the sextic the table No. 11; viz., this is

CIRCUMSCRIBED SEXTIC CONE.

Nodes of Surface.	Nodal Lines of Cones.								
1	0	6							
2	1	6_1							
3	2	6_2							
4	3	6_3							
5	4	6_4							
6	5	6_5	5 , 1						
7	6	6_6	5_1, 1						
8	7	6_7	5_2, 1						
9	8	6_8	5_3, 1	4 , 2					
10	9	6_9	5_4, 1	4_1, 2	4 , 1, 1	3 , 3			
11	10	6_{10}	5_5, 1	4_2, 2	4_1, 1, 1	3_1, 3			
12	11	...	5_6, 1	4_3, 2	4_2, 1, 1	3_1, 3_1	3 , 2, 1		
13	12	...	...	...	4_3, 1, 1	...	3_1, 2, 1	3 , 1, 1, 1	2 , 2, 2
14	13	...	...	...	...	...	...	3_1, 1, 1, 1	2 , 2, 1, 1
15	14	...	...	...	...	...	...	...	2_1, 1, 1, 1, 1
16	15	...	...	...	...	...	...	...	1 , 1, 1, 1, 1, 1

viz., 6 denotes a proper sextic cone without nodal lines; 6_1 a proper sextic cone with one nodal line; 5, 1 a proper quintic cone and a plane, &c.

We may distinguish the nodes according to the sextic cones; thus, a node 6 means a node for which the circumscribed cone is a proper sextic cone, (1, 1, 1, 1, 1, 1) a node where the circumscribed cone breaks up into six planes, &c.

126. A 16-nodal surface has 16 nodes (1, 1, 1, 1, 1, 1), and a 15-nodal surface has 15 nodes (2, 1, 1, 1, 1); but, for a 14-nodal surface, the question arises how many nodes are (3_1, 1, 1, 1), and how many (2, 2, 1, 1). It was remarked, No. 13, that the only possible cases were 14, 0; 8, 6; or 2, 12; and that we might, in like manner, limit the number of possible cases for other values of k; but that the inquiry was not then further pursued. I resume this inquiry, but without obtaining as yet a complete answer.

127. It is to be observed that a line joining any two nodes is not, in general, a line on the surface, but that it may be so; the surfaces for which this is so (viz., any surface which contains upon it a line through two nodes) form, however,

a class by themselves, which at present I *altogether exclude from consideration.* This being so, it will appear in the sequel that there is but one kind of surface having a node (2, 2, 1, 1), and but one kind of surface having a node $(3_1, 1, 1, 1)$. Now there is a surface, Kummer's 14-nodal, the nodes of which are $8(3_1, 1, 1, 1)+6(2, 2, 1, 1)$; wherefore the two kinds are identical, and are each of them Kummer's 14-nodal surface. Similarly, for the 13-nodal surfaces, there is but one kind having a node $(4_3, 1, 1)$, but one kind having a node (3, 1, 1, 1), and moreover but one kind having a node $(3_1, 2, 1)$; and we have Kummer's 13-nodal surface with the nodes $3(4_3, 1, 1) + 1(3, 1, 1, 1) + 9(3_1, 2, 1)$; hence the three kinds are identical with each other and with Kummer's. Moreover, there is but one kind having a node (2, 2, 2); hence all the other nodes must be (2, 2, 2), and we have a surface 13 (2, 2, 2) not given by Kummer. And in like manner for the 12-nodal surfaces, we have the two kinds given by Kummer, and a third kind $12(4_2, 1, 1)$ not given by him; the arrangement thus far being

No. of Nodes.		Character of Surface.
16		16 (1, 1, 1, 1, 1, 1),
15		15 (2, 1, 1, 1, 1),
14		$8(3_1, 1, 1, 1)+6(2, 2, 1, 1)$,
13	(α)	$3(4_3, 1, 1)+1(3, 1, 1, 1)+9(3_1, 2, 1)$,
„	(β)	13 (2, 2, 2),
12	(α)	$12(4_3, 2)$,
„	(β)	$2(5_6, 1)+6(3_1, 3_1)+4(3, 2, 1)$,
„	(γ)	$12(4_2, 1, 1)$.

128. But in the next following case we have Kummer's surface, viz.

$$11\ (\alpha)\quad 1(6_{10})+10(3_1, 3),$$

and I do not know whether one, two, or three kinds of surface having nodes $(4_1, 1, 1)$, $(4_2, 2)$, and $(5_5, 1)$. And in the next case we have (as will appear) the Symmetroid, viz.,

$$10\ (\alpha)\quad 10(3, 3),$$

and I do not know how many kinds of surfaces having a node or nodes 6_9, $(5_4, 1)$, $(4_1, 2)$, (4, 1, 1).

It will be observed that the present division has nothing to do with the octadic and dianodal division in the former Memoir.

129. I consider a conic $A=0$, and any six tangents thereof, $t_1=0$, $t_2=0$, $t_3=0$, $t_4=0$, $t_5=0$, $t_6=0$; we have an identical equation which might be written $AC-B^2=t_1t_2t_3t_4t_5t_6$, but it will be convenient, introducing a constant factor K, to write it

$$AC-B^2=Kt_1t_2t_3t_4t_5t_6,$$

B being a cubic function and C a quartic function of the coordinates.

Consider now the series of factors, such as

$$\begin{aligned} &t_1, \\ &lA \;\; + m\,t_1t_2, \\ &sA \;\; + m\,t_1t_2t_3, \\ &UA + m\,t_1t_2t_3t_4, \\ &VA + m\,t_1t_2t_3t_4t_5, \\ &WA + m\,t_1t_2t_3t_4t_5t_6, \end{aligned}$$

where m is a constant, l a constant, s a linear function, U, V, W functions of the degrees 2, 3, 4 respectively; and compose with one or more such factors an expression involving the term $Kt_1t_2t_3t_4t_5t_6$; for instance, such an expression is

$$\frac{K}{mm'}(sA + mt_1t_2t_3)(l'A + m't_4t_5)\,t_6;$$

this is, of the form $A\Omega + Kt_1t_2t_3t_4t_5t_6$, viz., $A\Omega + (AC - B^2)$, or $A(\Omega + C) - B^2$, say $A\Gamma - B^2$; or what is the same thing, introducing a new coordinate w, we have a quadric function

$$Aw^2 + 2Bw + \Gamma,$$

the discriminant of which, $A\Gamma - B^2$, is equal to the expression in question.

130. In the sequel (x, y, z, w) are considered as the coordinates of a point in space; $A = 0$ is thus a quadric cone, $t_1 = 0$, $t_2 = 0 \ldots t_6 = 0$, any six tangent planes thereof; and hence $Aw^2 + 2Bw + \Gamma = 0$ a quartic surface, having the point $(x = 0, y = 0, z = 0)$ for a node, whereof the circumscribed cone $A\Gamma - B^2 = 0$ breaks up in the assumed manner.

Thus, in order that the circumscribed cone may be as above

$$(sA + mt_1t_2t_3)(l'A + m't_4t_5)\,t_6,$$

we have only to assume

$$\Gamma = C + \frac{K}{mm'}(sl'A + sm't_4t_5 + l'mt_1t_2t_3)\,t_6,$$

and so in other cases. Observe that $sA + mt_1t_2t_3 = 0$ is a cubic cone, which, so long as s, m are arbitrary, has no nodal line; but establishing a single relation (say s remains arbitrary, but a proper value is assigned to m) it will be a cubic cone having a nodal line. And so $UA + mt_1t_2t_3t_4 = 0$ is a quartic cone without any nodal line, but by particularising the constants it may be made to have one, two, or three nodal lines. Such nodal determinations are obviously required in order that the formula may extend to all the before-mentioned forms of the circumscribed cone. The foregoing analysis is the foundation of the whole theory: I have given it, as above, apart from the theory, in order that the nature of it may be the better perceived; but I have now to bring it into connexion with the theory.

On the Sextic Curves, $A_2B_4 - C_3^2 = 0$.

131. I revert to the consideration of plane curves. The equation of a sextic curve $(*\between x, y, z)^6 = 0$ cannot be in general expressed in the form $AC - B^2 = 0$, where

the degrees of A, B, C are 2, 3, 4 respectively; in fact, the existence of such a form implies that there is a conic $A=0$ touching the sextic 6 times; and since a conic can only be made to satisfy 5 conditions, there is not in general any such conic.

132. Such conic, when it exists, is said to be inscribed in the sextic, and the sextic to be circumscribed about the conic, or to be an "amphigram;" and then, $A=0$ being the equation of the conic, that of the sextic is expressible in the form in question $AC-B^2=0$. It is clear that $B=0$ is a cubic curve passing through the 6 points of contact of the conic with the sextic, and that any such curve may be taken for the curve B; in fact if a particular cubic through the 6 points is $B'=0$, and the equation of the sextic is $AC'-B'^2=0$, then taking p an arbitrary linear function of the coordinates, the equation of the general cubic is $B=B'+pA=0$; and then writing

$$A=A,$$
$$B=B'+pA,$$
$$C=C'+2B'p+Ap^2,$$

we have $AC-B^2=AC'-B'^2$; so that the original form $AC'-B'^2=0$ becomes $AC-B^2=0$. But the cubic $B=0$ being assumed at pleasure, the quartic $C=0$ is a determinate curve.

133. It is to be observed that a sextic curve may be an amphigram in more than one way: certainly in two, three, or four, and possibly in a greater number of ways. For the equation of the curve contains 27 constants, and hence determining the sextic so as to touch 4 given conics each of them 6 times, there are still 3 constants; and the curve will be an amphigram in regard to each of the 4 conics; say it is a quadruple amphigram. But in the sequel we are only concerned with a sextic curve considered as an amphigram in regard to a given conic $A=0$ (no attention being paid to the other inscribed conics, if any); and then, by what precedes, taking $B=0$ any cubic whatever through the 6 points of contact, we have a determinate quartic curve $C=0$, and the equation of the sextic curve assumes the form $AC-B^2=0$.

134. The curves $A=0$, $B=0$, $C=0$ contain respectively 5, 9, 14 constants; whence considering the function B as containing an arbitrary constant factor, for the curve $AC-B^2=0$, the number of constants is *primâ facie* $5+9+14+1=29$; but on account of the arbitrary linear function p, the real number is $29-3=26$: this is right, for a sextic curve contains 27 constants; and the curve being an amphigram, there is one relation between the constants, $27-1=26$.

135. Suppose now that the sextic curve $AC-B^2=0$ breaks up into two or more separate curves, say into the two curves $P=0$, $Q=0$ of the orders f, g respectively; $f+g=6$. We have

$$AC-B^2=PQ=0;$$

and the conic $A=0$ touching the sextic six times, must, it is clear, touch the curves $P=0$, $Q=0$, f and g times respectively. And so when the sextic breaks up into any number of curves, each component curve $P=0$ of the order f must touch the sextic g times.

136. It follows that if the sextic break up into six lines, say $AC - B^2 = t_1t_2t_3t_4t_5t_6 = 0$, then that each of the lines $t_1 = 0$, $t_2 = 0, \ldots t_6 = 0$ is a tangent to the conic. And conversely, starting with the conic $A = 0$ and any six tangents thereof $t_1 = 0$, $t_2 = 0, \ldots t_6 = 0$, we have an identity of the form in question. In fact, taking any two of the tangents, say $t_1 = 0$ and $t_2 = 0$, then, if $p = 0$ be the equation of the line joining their points of intersection, the equation of the conic will be of the form $t_1t_2 + p^2 = 0$, that is, we may write $A = t_1t_2 + p^2$, or what is the same thing, $t_1t_2 = A - p^2$. (Considering A as a given quadric function of the coordinates, this of course implies that the implicit constant factors of t_1, t_2, p are properly determined.) Similarly, $q = 0$ being the line through the points of contact of t_3, t_4, and $r = 0$ that through the points of contact of t_5, t_6, we have $t_3t_4 = A - q^2$ and $t_5t_6 = A - r^2$; whence, to satisfy the equation

$$AC - B^2 = t_1t_2t_3t_4t_5t_6,$$

we have only to assume $B = lA + pqr$, l an arbitrary linear function of the coordinates, and the equation then gives

$$C = A^2 - A(p^2 + q^2 + r^2) + (q^2r^2 + r^2p^2 + p^2q^2) + l^2A + 2lpqr.$$

137. It will be observed that the grouping of the six tangents into pairs is arbitrary. By altering this grouping, we merely alter the linear function l, but do not obtain any new solution. Thus, say that the new form is $B = l'A + p'q'r'$, then, by properly determining the linear function l, we can reduce this to the original form $B = lA + pqr$; viz., we can satisfy identically the equation $(l - l')A + pqr - p'q'r' = 0$; or what is the same thing, $\lambda A + pqr - p'q'r' = 0$, where λ is a linear function of the coordinates. We have, in fact, the conic $A = 0$ and the cubic $pqr = 0$ intersecting in the six points of contact any other cubic through these six points; and consequently the cubic $p'q'r' = 0$ must be expressible in the form $\lambda A + pqr = 0$, and we have thus the identity in question.

138. We have just seen that the value of B is necessarily of the form $B = lA + pqr$, but we are not concerned with its expression in this particular form. What we require in the sequel is a value of B, and thence one of C, satisfying the identity in question, $AC - B^2 = t_1t_2t_3t_4t_5t_6$; or what is the same thing, introducing for convenience a constant factor K, the identity

$$AC - B^2 = Kt_1t_2t_3t_4t_5t_6.$$

139. Instead of C, I write Γ, and consider the sextic amphigram $A\Gamma - B^2 = 0$ touched by the conic $A = 0$ in the points of contact of the conic with the six tangents $t_1 = 0$, $t_2 = 0, \ldots t_6 = 0$. Suppose the sextic curve breaks up into factors; if one of these factors is a line, it is one of the six tangents, say the tangent $t_1 = 0$. If there is a conic factor, this is a conic touching the conic $A = 0$ at its points of contact with two of the tangents, say the equation is $lA + mt_1t_2 = 0$. Similarly, if there is a cubic, quartic, or quintic factor, then the equation hereof is $sA + mt_1t_2t_3 = 0$, $UA + mt_1t_2t_3t_4 = 0$, or $VA + m_1t_2t_3t_4t_5 = 0$. Or going on to the next case of a sextic factor (being of course the whole curve), we may say that this is $WA + mt_1t_2t_3t_4t_5t_6 = 0$. (Observe that since $AC - B^2 = Kt_1t_2t_3t_4t_5t_6$, this means only that the equation of the sextic amphigram is of the assumed form $A\Gamma - B^2 = 0$.)

140. By what precedes we can, for a sextic amphigram which breaks up in any assigned manner, determine the value of Γ. For instance, let the amphigram break up into two cubic curves; say these are $sA + mt_1t_2t_3 = 0$, $s'A + m't_4t_5t_6 = 0$. Assume

$$A\Gamma - B^2 = \frac{K}{mm'}(sA + mt_1t_2t_3)(s'A + m't_4t_5t_6),$$

then this equation is

$$A\Gamma - B^2 = \frac{K}{mm'}\{ss'A^2 + (sm't_4t_5t_6 + s'mt_1t_2t_3)A\} + AC - B^2;$$

that is, we have

$$\Gamma = C + \frac{K}{mm'}(ss'A + sm't_4t_5t_6 + s'mt_1t_2t_3),$$

and so in any other case.

I have already adverted to the question of the "nodal determination" of the formulæ, and it might be properly here considered; viz., the question is as to the determination of the constants in such manner that, for instance, $sA + mt_1t_2t_3 = 0$ may be a nodal cubic, $UA + mt_1t_2t_3t_4 = 0$ a nodal, binodal, or trinodal quartic, &c.; but I defer it for the moment in order first to apply the theory to the quartic surfaces.

Application to Quartic Surfaces.

141. If a quartic surface has a node or nodes, we may take for a node the point $x = 0$, $y = 0$, $z = 0$; the equation of the quartic surface is then of the form

$$Aw^2 + 2Bw + \Gamma = 0,$$

where A, B, Γ are functions of x, y, z of the degrees 2, 3, 4 respectively. $A = 0$ is the tangent quadricone at the node in question; and the circumscribed cone is $A\Gamma - B^2 = 0$. By what precedes, this is an amphigram touching the quadricone along six generating lines thereof; say the tangent planes of the cone $A = 0$ along these six lines respectively are $t_1 = 0$, $t_2 = 0, \ldots t_6 = 0$. We have then an identical equation

$$AC - B^2 = Kt_1t_2t_3t_4t_5t_6,$$

viz., regarding for a moment this equation as an equation for the determination of B, and B' as any particular solution thereof, then its general solution is $B' + tA$, where t is an arbitrary linear function of (x, y, z), and the B in the equation of the surface is properly $= B' + tA$. But by the substituting $w - t$ in place of w, the B of the equation of the surface would then be made $= B'$; and it thus appears that we may, without loss of generality, take the B of this equation to be *any particular value* satisfying the identity in question; and then, B having such particular value, C is a quartic function of (x, y, z) completely determined by the same identity. And we then, by what precedes, at once determine Γ so that the circumscribed cone $A\Gamma - B^2 = 0$ may be a cone breaking up in any assigned manner; for instance, if it be a cone

(3, 3), then, as just mentioned, the two cubic cones are $sA + mt_1t_2t_3 = 0$, $s'A + m't_4t_5t_6 = 0$; and Γ has the value

$$\Gamma = C + \frac{K}{mm'}(ss'A + sm't_4t_5t_6 + s'mt_1t_2t_3),$$

above obtained.

On the Nodal Determination.

142. I am not able to discuss with much completeness the question of nodal determination. We have to consider a cubic curve $sA + mt_1t_2t_3 = 0$, a quartic curve $UA + mt_1t_2t_3t_4 = 0$, &c., as the case may be, and to determine the constants so that this shall have a node or nodes. Consider for a moment the form $PA + t_1Q = 0$, where Q denotes the product $mt_2t_3 \ldots$ of all or any of the tangents $t_2, \ldots t_5$; the orders of PA, t_1Q are of course equal, that is, the order of P is less by unity than that of Q. I say that, by establishing a single relation between the constants, this may be made to have a node *at the point of contact* $A = 0$, $t_1 = 0$. In fact, writing $\Delta = \lambda\delta_x + \mu\delta_y + \nu\delta_z$, where λ, μ, ν are arbitrary, there will be a node at any point if for that point $\Delta(PA + t_1Q) = 0$. But for the point $A = 0$, $t_1 = 0$ this becomes $P\Delta A + Q\Delta t_1 = 0$; moreover, if $t = 0$ be any other tangent of the conic $A = 0$, and if $p = 0$ be the line joining the points of contact of the tangents t, t_1, then we may write $A = tt_1 - p^2$, and thence (since at the point in question, $A = 0$, $t_1 = 0$, we have also $p = 0$) we find $\Delta A = t\Delta t_1$, and the foregoing equation thus becomes $(tP + Q)\Delta t_1 = 0$; viz., this equation is satisfied irrespectively of the values of λ, μ, ν, if only at the point in question (that is, for the values of the coordinates which belong to the point $t_1 = 0$, $A = 0$) we have $tP + Q = 0$, which is a single relation between the constants.

143. In particular the cubic curve $sA + mt_1t_2t_3 = 0$ may be made to have a node at the point of contact of any one of the three tangents; the quartic curve $UA + mt_1t_2t_3t_4 = 0$, a node, or two or three nodes, at the point or points of contact of any one, two, or three of the four tangents; and so in other cases. These are not the only solutions, and they are in fact solutions which (as afterwards explained) I propose to reject, attending in each case only to the remaining or proper solutions of the problem.

144. To obtain in a different manner the foregoing result, consider again the cubic curve $sA + mt_1t_2t_3 = 0$; regarding this as a given curve, the conic $A = 0$ is a conic determined (not of course completely) as a conic having therewith 3 points of 2-pointic intersection; viz., if the cubic has a node, then the cone $A = 0$ is either a conic passing through the node and besides touching the curve twice, or else it is a conic touching the curve 3 times; the former is of course the above mentioned case where there is a node at one of the points of contact on the conic $A = 0$; the latter is regarded as the proper solution. So in the case of a quartic curve $UA + mt_1t_2t_3t_4 = 0$, regarding this as a given curve, the conic $A = 0$ is a conic having therewith four points of 2-pointic intersection; viz., if the quartic curve has one, two, or three nodes, then the conic is either a conic passing through one, two, or three nodes, and besides touching the quartic thrice, twice, or once; or else it is a conic touching the quartic

four times. The former is the above mentioned case where there is a node or nodes at a point or points of contact with the conic $A=0$; the latter is regarded as the proper solution.

145. To fix the ideas, and at the same time obtain a result which will be afterwards useful, I work out the formulæ for the cubic curve $sA+mt_1t_2t_3=0$, taking this equation under the form

$$\left(\frac{x}{\lambda}+\frac{y}{\mu}+\frac{z}{\nu}\right)(x^2+y^2+z^2-2yz-2zx-2xy)+m\,xyz=0.$$

This may have a node in two different ways; viz.,

1°. At the point of contact of one of the tangents $x=0$, $y=0$, $z=0$ with the conic $A=0$; say at the point of contact of $x=0$, that is, the point $x=0$, $y-z=0$. The value of m is $=\dfrac{4}{\mu}+\dfrac{4}{\nu}$; hence $\dfrac{1}{\nu}=-\dfrac{1}{\mu}+\tfrac{1}{4}m$; and, substituting, the equation of the curve becomes

$$\left(\frac{x}{\lambda}+\frac{y-z}{\lambda}\right)\{x^2-2x(y+z)+(y-z)^2\}+\tfrac{1}{4}mz(x+y-z)^2=0,$$

which has obviously a node at the point in question.

2°. The node may be at a point not on the conic $A=0$, viz. the value of m is $=\dfrac{(\lambda+\mu+\nu)^2}{\lambda\mu\nu}$, the equation is

$$\left(\frac{x}{\lambda}+\frac{y}{\mu}+\frac{z}{\nu}\right)(x^2+y^2+z^2-2yz-2zx-2xy)+\frac{(\lambda+\mu+\nu)^2}{\lambda\mu\nu}xyz=0.$$

In fact, writing for shortness

$$\begin{aligned}-\lambda+\mu+\nu&=L,\\ \lambda-\mu+\nu&=M,\\ \lambda+\mu-\nu&=N\\ \lambda+\mu+\nu&=P,\end{aligned}$$

the node is at the point $x:y:z=L\lambda:M\mu:N\nu$; which is at once verified, if we remark that, writing for convenience $x, y, z=L\lambda, M\mu, N\nu$, then we have

$$\begin{aligned}-x+y+z&=MN,\\ x-y+z&=NL,\\ x+y-z&=LM,\end{aligned}$$

$$\frac{x}{\lambda}+\frac{y}{\mu}+\frac{z}{\nu}=P,\quad x^2+y^2+z^2-2yz-2zx-2xy=-LMNP\,(=A).$$

For, of the three equations for the coordinates of a node, the first is

$$\frac{1}{\lambda}A+\left(\frac{x}{\lambda}+\frac{y}{\mu}+\frac{z}{\nu}\right)2(x-y-z)+\frac{(\lambda+\mu+\nu)^2}{\lambda\mu\nu}yz=0,$$

that is, for the values in question,

$$-\frac{1}{\lambda} LMNP + P(-2MN) + \frac{P^2}{\lambda\mu\nu} MN\mu\nu = 0,$$

that is

$$-LMNP - 2\lambda MNP + P^2 MN = 0,$$

or finally, $-L - 2\lambda + P = 0$, which is satisfied; and similarly the other two equations are satisfied.

Quartic Surfaces resumed.

146. Passing now from curves to cones, and to the theory of the quartic surface, suppose that there is a component cone having a nodal line, say the cubic cone $sA + mt_1t_2t_3 = 0$: if the remaining factor is $t_4t_5t_6$, then we have

$$A\Gamma - B^2 = \frac{K}{m}(sA + mt_1t_2t_3)\, t_4t_5t_6.$$

Suppose the nodal line is a line of contact with the cone $A = 0$, say its equations are $t_1 = 0$, $p = 0$ (p a linear function), then $sA + mt_1t_2t_3$ is a quadric function $(*\between t_1, p^2)$, (of course with variable coefficients); hence $A\Gamma - B^2$ is a quadric function; and A being a linear function $(*\between t_1, p)$, it follows that B is a linear function, and thence that Γ is also a linear function; that is, A, B, Γ are each of them a linear function $(*\between t_1, p)$, or the line in question (viz. the line of contact $A = 0$, $t_1 = 0$) is a line on the quartic surface $Aw^2 + 2Bw + \Gamma = 0$. As already mentioned, *I exclude from consideration* the surfaces which have upon them a line through two nodes; that is, I exclude from consideration the case in question where any component cone, or say where the sextic cone, has a nodal line which is a line on the tangent cone $A = 0$.

147. Now, excluding the case just referred to, *I assume as a postulate* that there is but one way in which the cubic cone $sA + mt_1t_2t_3 = 0$ can be made to have a nodal line, or the quartic cone $UA + mt_1t_2t_3t_4 = 0$ one, two, or three nodal lines &c., as the case may be. It is to be understood that this does not mean that the constants are in any of these cases completely determined, but that there is between them a relation or relations constituting a general solution which includes in itself every particular solution whatever. I have no doubt that as regards the cubic cone at least the assumption is correct. This being so, the character of a single node determines the nature of the surface; for instance, if there is a node $(3_1, 3)$, then taking this as the point ($x = 0$, $y = 0$, $z = 0$) the equation of the surface is $Az^2 + 2Bw + \Gamma = 0$, where

$$\Gamma = C + \frac{K}{mm'}(ss'A + l'st_4t_5t_6 + ls't_1t_2t_3),$$

a surface of a determinate nature; so that the character of all the remaining nodes is completely determined.

148. The point to be attended to is, that if for instance there were two essentially distinct ways of giving the cubic cone $sA + mt_1t_2t_3 = 0$ a nodal line (such as there would be if the excluded case were considered admissible), then the foregoing equation

of the surface would or might include two distinct forms of equation applying to different kinds of surface. The conclusion is that there is but one kind of quartic surface having a node $(3_1, 3)$. Admitting this, and similarly that there is but one kind of quartic surface having a node 6_{10}, it follows that if (as the fact is) there is a surface having the nodes $1(6_{10}) + 10(3_1, 3)$ (Kummer's 11-nodal surface), then that the two first-mentioned kinds are in fact each of them this last-mentioned kind of surface; and it was in this manner that I arrived at the enumeration given near the beginning of the present Memoir.

149. The reasoning is, of course, in place of a direct demonstration which would consist in showing that a surface having a node $(3_1, 3)$ has 9 other like nodes, and also a node 6_{10}; and that a surface having a node 6_{10} has 10 other nodes $(3_1, 3)$; and that, starting from either form of equation, we could, by passing to a node of the other kind, obtain the other form of equation.

Enumeration of the Cases.

150. I collect the results as follows: I call to mind that we have always the identical equation $AC - B^2 = Kt_1t_2t_3t_4t_5t_6$, that the equation of the surface is $Aw^2 + 2Bw + \Gamma = 0$, and that the circumscribed cone is $A\Gamma - B^2 = 0$. The equation of a surface having different kinds of nodes will assume different forms according as the origin (or point $x = 0$, $y = 0$, $z = 0$) is taken to be at a node of one or other of these kinds; these forms of the equations are distinguished as "node-forms,"—viz., we speak of the node-form $(3_1, 3)$ when the origin is a node $(3_1, 3)$, and so in other cases.

The 16-nodal surface

$$16\,(1,\ 1,\ 1,\ 1,\ 1,\ 1),$$

node-form

$$(1,\ 1,\ 1,\ 1,\ 1,\ 1),$$

cone is

$$t_1t_2t_3t_4t_5t_6 = 0,$$

and

$$\Gamma = C;$$

viz., equation is

$$Aw^2 + 2Bw + C = 0.$$

The 15-nodal surface

$$15\,(2,\ 1,\ 1,\ 1,\ 1),$$

node-form

$$(2,\ 1,\ 1,\ 1,\ 1)$$

cone is

$$(lA + mt_1t_2)\,t_3t_4t_5t_6 = 0,$$

and

$$\Gamma = C + \frac{Kl}{m}\,t_3t_4t_5t_6 = 0.$$

The 14-nodal surface

$$8(3_1,\ 1,\ 1,\ 1) + 6(2,\ 2,\ 1,\ 1),$$

node-form

$$(3_1,\ 1,\ 1,\ 1),$$

cone is

$$(sA + mt_1t_2t_3)\, t_4t_5t_6 = 0,$$

where

$$sA + mt_1t_2t_3 = 0 \text{ is a nodal cubic } 3_1,$$

and

$$\Gamma = C + \frac{Ks}{m} t_4t_5t_6\,;$$

node-form

$$(2,\ 2,\ 1,\ 1),$$

cone is

$$(lA + mt_1t_2)\,(l'A + m't_3t_4)\, t_5t_6 = 0,$$

and

$$\Gamma = C + \frac{K}{mm'} (ll'A + lm't_3t_4 + l'mt_1t_2)\, t_5t_6.$$

The 13 (α)-nodal surface

$$3(4_3,\ 1,\ 1) + 1(3,\ 1,\ 1,\ 1) + 9(3_1,\ 2,\ 1),$$

node-form

$$(4_3,\ 1,\ 1),$$

cone is

$$(UA + mt_1t_2t_3t_4)\, t_5t_6 = 0,$$

where

$$UA + mt_1t_2t_3t_4 = 0 \text{ is a trinodal quartic } 4_3,$$

and

$$\Gamma = C + \frac{K}{m} Ut_5t_6\,;$$

node-form

$$(3,\ 1,\ 1,\ 1),$$

cone is

$$(sA + mt_1t_2t_3)\, t_4t_5t_6 = 0,$$

and

$$\Gamma = C + \frac{K}{m} st_4t_5t_6\,;$$

node-form

$$(3_1,\ 2,\ 1),$$

cone is

$$(sA + mt_1t_2t_3)\,(l'A + m't_4t_5)\, t_6 = 0,$$

where

$$sA + mt_1t_2t_3 = 0 \text{ is a nodal cubic } 3_1,$$

and

$$\Gamma = C + \frac{K}{mm'} (sl'A + m'st_4t_5 + ml't_1t_2t_3)\, t_6.$$

The 13 (β)-nodal surface

$$13\,(2,\ 2,\ 2),$$

node-form

$$(2,\ 2,\ 2),$$

cone is

$$(lA + mt_1t_2)\,(l'A + m't_3t_4)\,(l''A + m''t_5t_6) = 0,$$

and

$$\Gamma = C + \frac{K}{mm'm''}\{ll'l''\,A^2 + (l'l''mt_1t_2 + l''lm't_3t_4 + ll'm''t_5t_6)\,A$$
$$+ lm'm''t_3t_4t_5t_6 + l'm''mt_5t_6t_1t_2 + l''mm't_1t_2t_3t_4\}.$$

The 12 (α)-nodal surface

$$12\,(4_3,\ 2),$$

node-form

$$(4_3,\ 2),$$

cone is

$$(UA + mt_1t_2t_3t_4)\,(l'A + m't_5t_6) = 0,$$

where

$$UA + mt_1t_2t_3t_4 = 0 \text{ is a trinodal quartic } 4_3,$$

and

$$\Gamma = C + \frac{K}{mm'}(l'UA + m'Ut_5t_6 + l'mt_1t_2t_3t_4).$$

The 12 (β)-nodal surface

$$2\,(5_6,\ 1) + 6\,(3_1,\ 3_1) + 4\,(3,\ 2,\ 1),$$

node-form

$$(5_6,\ 1),$$

cone is

$$(VA + mt_1t_2t_3t_4t_5)\,t_6 = 0,$$

where

$$VA + mt_1t_2t_3t_4t_5 = 0 \text{ is a 6-nodal quintic } 5_6,$$

and

$$\Gamma = C + \frac{K}{m}\,Vt_6\,;$$

node-form

$$(3_1,\ 3_1),$$

cone is

$$(sA + mt_1t_2t_3)\,(s'A + m't_4t_5t_6) = 0,$$

where

$$sA + mt_1t_2t_3 = 0 \text{ and } s'A + m't_4t_5t_6 = 0 \text{ are each of them a nodal cubic } 3_1,$$

and

$$\Gamma = C + \frac{K}{mm'}(ss'A + m'st_4t_5t_6 + ms't_1t_2t_3)\,;$$

node-form

$$(3,\ 2,\ 1),$$

cone is

$$(sA + mt_1t_2t_3)\,(l'A + m't_4t_5)\,t_6 = 0,$$

and

$$\Gamma = C + \frac{K}{mm'}(l'sA + m'st_4t_5 + l'mt_1t_2t_3)\,t_6.$$

The 12 (γ)-nodal surface,

$$12\,(4_2,\ 1,\ 1),$$

node-form

$$(4_2,\ 1,\ 1),$$

cone is

$$(UA + mt_1t_2t_3t_4)\,t_5t_6 = 0,$$

where

$$UA + mt_1t_2t_3t_4 = 0 \text{ is a binodal quartic } 4_2,$$

and

$$\Gamma = C + \frac{K}{m}\,Ut_5t_6.$$

The 11 (α)-nodal surface,

$$1\,(6_{10}) + 10\,(3_1,\ 3)$$

node-form

$$(6_{10}),$$

cone is

$$WA + mt_1t_2t_3t_4t_5t_6 = 0,$$

where this is a 10-nodal sextic 6_{10},

and

$$\Gamma = C + \frac{K}{m}\,W;$$

node-form

$$(3_1,\ 3),$$

cone is

$$(sA + mt_1t_2t_3)(s'A + m't_4t_5t_6) = 0,$$

where

$$sA + mt_1t_2t_3 = 0 \text{ is a nodal cubic } 3_1,$$

and

$$\Gamma = C + \frac{K}{mm'}\,(ss'A + m'st_4t_5t_6 + ms't_1t_2t_3).$$

Other 11-nodal surfaces,

node-form

$$(5_5,\ 1),$$

cone is

$$(VA + mt_1t_2t_3t_4t_5)\,t_6 = 0,$$

where

$$VA + mt_1t_2t_3t_4t_5 = 0 \text{ is a 5-nodal quintic } 5_5,$$

and

$$C = \Gamma + \frac{K}{m}\,Vt_6 = 0;$$

node-form

$$(4_2,\ 2),$$

cone is

$$(UA + mt_1t_2t_3t_4)(l'A + m't_5t_6) = 0,$$

where

$$UA + mt_1t_2t_3t_4 = 0 \text{ is a binodal quartic } 4_2,$$

and

$$\Gamma = C + \frac{K}{mm'}\,(l'UA + m'Ut_5t_6 + l'mt_1t_2t_3t_4);$$

node-form

$$(4, 1, 1),$$

cone is

$$(UA + mt_1t_2t_3t_4)\,t_5t_6 = 0,$$

where

$$UA + mt_1t_2t_3t_4 \text{ is a nodal quartic } 4_1,$$

and

$$\Gamma = C + \frac{K}{m}\,Ut_5t_6\,;$$

but whether these node-forms belong to the same or to different surfaces is not ascertained.

The enumeration is not extended to the 10-nodal surfaces, but I consider one case of these surfaces.

The 10 (α)-*nodal surface* 10 (3, 3).

151. I assume only that there is a single node (3, 3): taking the cone to be

$$(sA + mt_1t_2t_3)\,(s'A + m't_4t_5t_6) = 0\,;$$

then for the equation of the surface, in the node-form (3, 3) in question, we have

$$\Gamma = C + \frac{K}{mm'}\,(ss'A + sm't_4t_5t_6 + s'mt_1t_2t_3).$$

But I present this result under a different form, as follows: I write

$$A = p^2 + ft_1t_2 = q^2 + gt_3t_4 = r^2 + ht_5t_6,$$

where f, g, h are constants, and, as before, $p = 0$, $q = 0$, $r = 0$ are the lines joining the points of contact of t_1, t_2; t_3, t_4; and t_5, t_6 respectively: we have

$$sA + mt_1t_2t_3 = sA + mt_3\left(\frac{A - p^2}{f}\right), \text{ and } s'A + m't_4t_5t_6 = s'A + m't_4\left(\frac{A - r^2}{h}\right);$$

or in place of s, s' introducing new linear functions σ, σ', the cubic curves may be taken to be $\sigma A - \frac{m}{f}\,p^2t_3$, $\sigma' A - \frac{m'}{h}\,r^2t_4$, so that we have

$$\begin{aligned}
A\Gamma - B^2 &= \frac{K}{mm'}\left(\sigma A - \frac{m}{f}\,p^2t_3\right)\left(\sigma' A - \frac{m'}{h}\,r^2t_4\right),\\
&= \frac{K}{mm'}\left(\sigma\sigma' A^2 - \sigma A\,\frac{m'}{h}\,r^2t_4 - \sigma' A\,\frac{m}{f}\,p^2t_3 + \frac{mm'}{fh}\,p^2r^2\,\frac{A - q^2}{g}\right),\\
&= \frac{K}{mm'}\left(\sigma\sigma' A^2 - \sigma A\,\frac{m'}{h}\,r^2t_4 - \sigma' A\,\frac{m}{f}\,p^2t_3 + \frac{mm'}{fgh}\,p^2r^2A\right) - \frac{K}{fgh}\,p^2q^2r^2\,;
\end{aligned}$$

whence $B = \left(\frac{K}{fgh}\right)^{\frac{1}{2}}(pqr + tA)$, where t is a linear function of the coordinates; and we then have

$$\Gamma = \frac{K}{mm'}\left(\sigma\sigma' A - \sigma\,\frac{m'}{h}\,r^2t_4 - \sigma'\,\frac{m}{f}\,p^2t_3 + \frac{mm'}{fgh}\,p^2r^2\right) + \frac{K}{fgh}\,(t^2A + 2tpqr),$$

where A may be considered as standing for $q^2+gt_3t_4$. The equation $Aw^2+2Bw+\Gamma=0$ of the surface, substituting throughout for A its value, is therefore

$$(q^2+gt_3t_4)\,w^2+2\left(\frac{K}{fgh}\right)^{\frac{1}{2}}\{pqr+t\,(q^2+gt_3t_4)\}\,w$$

$$+\frac{K}{mm'}\left[\sigma\sigma'(q^2+gt_3t_4)-\sigma\frac{m'}{h}r^2t_4-\sigma'\frac{m}{f}p^2t_3+\frac{mm'}{fgh}p^2r^2\right]+\frac{K}{fgh}\left[t^2(q^2+gt_3t_4)+2tpqr\right]=0,$$

where the cone is

$$\left\{\sigma\,(q^2+gt_3t_4)-\frac{m}{f}p^2t_3\right\}\left\{\sigma'(q^2+gt_3t_4)-\frac{m'}{h}r^2t_4\right\}=0.$$

152. Writing in the equation of the surface $w\left(\frac{K}{fgh}\right)^{\frac{1}{2}}$ instead of w, it becomes

$$(q^2+gt_3t_4)\,w^2+2\left[pqr+t\,(q^2+gt_3t_4)\right]w$$

$$+\frac{fgh}{mm'}\left[\sigma\sigma'(q^2+gt_3t_4)-\sigma\frac{m'}{h}r^2t_4-\sigma'\frac{m}{f}p^2t_3+\frac{mm'}{fgh}p^2r^2\right]+t^2(q^2+gt_3t_4)+2tpqr=0\,;$$

and then writing $\frac{m}{f}\sigma$ and $\frac{m'}{h}\sigma'$ for σ and σ' respectively, this is

$$(q^2+gt_3t_4)\,w^2+2pqrw+2tw\,(q^2+gt_3t_4)$$

$$+g\left[\sigma\sigma'(q^2+gt_3t_4)-\sigma r^2t_4-\sigma'p^2t_3+\frac{1}{g}p^2r^2\right]+t^2(q^2+gt_3t_4)+2tpqr=0.$$

We may consider t_3, t_4 as denoting not the functions originally so represented, but these functions each multiplied by a suitable constant, and thereupon write $g=-1$; viz., $t_3=0$, $t_4=0$, will now denote any two tangents to the conic $A=0$, the implicit factors being so determined that $A=q^2-t_3t_4$. The equation of the surface is

$$(q^2-t_3t_4)\,w^2+2pqrw+2tw\,(q^2-t_3t_4)$$

$$-\sigma\sigma'(q^2-t_3t_4)+\sigma r^2t_4+\sigma'p^2t_3+p^2r^2+t^2(q^2-t_3t_4)+2pqrt=0\,;$$

viz., this is

$$(q^2-t_3t_4)\left[(w+t)^2-\sigma\sigma'\right]+2pqr\,(w+t)+\sigma r^2t_4+\sigma'p^2t_3+p^2r^2=0,$$

the sextic cone being

$$\{\sigma\,(q^2-t_3t_4)-p^2t_3\}\,\{\sigma'(q^2-t_3t_4)-r^2t_4\}=0.$$

153. But the foregoing equation of the surface is

$$\begin{vmatrix} -\sigma', & w+t, & ., & r \\ w+t, & -\sigma, & p, & . \\ ., & p, & t_4, & -q \\ r, & ., & -q, & t_3 \end{vmatrix}=0,$$

as is at once seen by developing the determinant; the functions $w+t$, σ, σ', p, q, r, t_3, t_4 are all of them linear; and the determinant is thus a symmetrical quartic determinant the terms whereof are linear functions of the coordinates; viz. the surface is a symmetroid. That is, a surface having a single node (3, 3) is a symmetroid; but I have shown (*Second Memoir*, No. 116) that a symmetroid has each of its ten nodes (3, 3); wherefore the surface having a single node (3, 3) is the $10(\alpha)$-nodal surface, nodes 10 (3, 3).

154. Start from two cubic cones $U=0$, $V=0$, having each the same vertex ($x=0$, $y=0$, $z=0$); we may in a variety of ways determine the two cones $\alpha U+\beta V=0$, $\gamma U+\delta V=0$, having a common inscribed quadric cone $A=0$ (viz., $\alpha:\beta$ being assumed at pleasure, then $\gamma:\delta$ will be determined; not, I believe, uniquely, but I do not know what the multiplicity is). This being so, the quadric cone $A=0$ is uniquely determined; and then, assuming at pleasure the plane $w=0$, the $10(\alpha)$-nodal surface $Aw^2+2Bw+\Gamma=0$ is uniquely determined: consequently the remaining nine nodes are determinate points on the nine lines $U=0$, $V=0$ respectively. And we have thus a system of ten points in space such that, joining any one of them with the remaining nine, the nine lines so obtained are the intersections of two cubic cones, or say that they are an ennead of lines.

Notation for the Cases afterwards considered.

155. I proceed to further develope the theory of some of the different surfaces. The same node-form of equation will, of course, assume different shapes according to the actual expressions in terms of the coordinates (x, y, z) of the several functions A, &c., which enter into it. I have found it convenient to attribute to A and B certain specific values which are not in every case those of the coefficients of w^2, w in the equation of the surface: this means that we must, in the equation of the surface, substitute new symbols for these coefficients, and write the equation say in the form $A'w^2+2B'w+\Gamma=0$; the change of notation, when it occurs, will be duly explained.

156. It is in general (but not always) convenient to take the equation of the tangent cone to be $x^2+y^2+z^2-2yz-2zx-2xy=0$; for then any plane $\dfrac{x}{\alpha}+\dfrac{y}{\beta}+\dfrac{z}{\gamma}=0$, where $\alpha+\beta+\gamma=0$, will be a tangent plane; so that six tangent planes may be represented by $x=0$, $y=0$, $z=0$, and by three equations of the form just referred to. And in reference to this assumed form of the equation of the tangent cone, and to what follows, I write

$$\alpha+\beta+\gamma=0,$$
$$\alpha'+\beta'+\gamma'=0,$$
$$\alpha''+\beta''+\gamma''=0,$$

$$P=\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma},$$

$$P'=\frac{x}{\alpha'}+\frac{y}{\beta'}+\frac{z}{\gamma'},$$

$$P''=\frac{x}{\alpha''}+\frac{y}{\beta''}+\frac{z}{\gamma''},$$

$$\begin{aligned}
X &= \alpha\ (\gamma'\gamma''\, y - \beta'\beta'' z),\\
Y &= \beta\ (\alpha'\alpha''\, z - \gamma'\gamma'' x),\\
Z &= \gamma\ (\beta'\beta'' x - \alpha'\alpha'' y),\\[4pt]
X' &= \alpha'\ (\gamma''\gamma\ y - \beta''\beta\, z),\\
Y' &= \beta'\ (\alpha''\alpha\ z - \gamma''\gamma\ x),\\
Z' &= \gamma'\ (\beta''\beta\ x - \alpha''\alpha\ y),\\[4pt]
X'' &= \alpha''\ (\gamma\,\gamma' y - \beta\,\beta' z),\\
Y'' &= \beta''\ (\alpha\,\alpha' z - \gamma\,\gamma' x),\\
Z'' &= \gamma''\ (\beta\,\beta' x - \alpha\,\alpha' y),
\end{aligned}$$

$$\begin{aligned}
A &= x^2 + y^2 + z^2 - 2yz - 2zx - 2xy,\\
B &= \alpha\alpha'\alpha''(y^2z - yz^2) + \beta\beta'\beta''(z^2x - zx^2) + \gamma\gamma'\gamma''(x^2y - xy^2) + Mxyz,\\
C &= (\alpha\alpha'\alpha'' yz + \beta\beta'\beta'' zx + \gamma\gamma'\gamma'' xy)^2;
\end{aligned}$$

where

$$\begin{aligned}
M &= (\beta - \gamma)\,\alpha'\alpha'' + (\gamma - \alpha)\,\beta'\beta'' + (\alpha - \beta)\,\gamma'\gamma'',\\
&= (\beta' - \gamma')\,\alpha''\alpha + (\gamma' - \alpha')\,\beta''\beta + (\alpha' - \beta')\,\gamma''\gamma,\\
&= (\beta'' - \gamma'')\,\alpha\alpha' + (\gamma'' - \alpha'')\,\beta\beta' + (\alpha'' - \beta'')\,\gamma\gamma',\\
&= -\tfrac{1}{3}\{(\beta-\gamma)(\beta'-\gamma')(\beta''-\gamma'') + (\gamma-\alpha)(\gamma'-\alpha')(\gamma''-\alpha'') + (\alpha-\beta)(\alpha'-\beta')(\alpha''-\beta'')\};
\end{aligned}$$

also

$$K = 4\alpha\alpha'\alpha''\beta\beta'\beta''\gamma\gamma'\gamma'':$$

and we have identically

$$AC - B^2 = Kxyz\left(\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma}\right)\left(\frac{x}{\alpha'} + \frac{y}{\beta'} + \frac{z}{\gamma'}\right)\left(\frac{x}{\alpha''} + \frac{y}{\beta''} + \frac{z}{\gamma''}\right).$$

The 16-*nodal Surface* 16 (1, 1, 1, 1, 1, 1).

157. Kummer starts from an irrational equation, which is readily converted into the following

$$\sqrt{x(X-w)} + \sqrt{y(Y-w)} + \sqrt{z(Z-w)} = 0,$$

and then, rationalizing, we have

$$Aw^2 + 2Bw + C = 0,$$

where as above

$$AC - B^2 = Kxyz\, PP'P''.$$

This agrees with the foregoing theory; viz., the point ($x = 0$, $y = 0$, $z = 0$) being a node, the rationalized equation must, of course, be in the node-form (1, 1, 1, 1, 1, 1), (being the only node-form); and the symmetry of the formulæ enables us at once to write down the equations of the 16 singular planes, and thence to deduce the coordinates of the 16 nodes; viz.,

the singular planes are	and the nodes are
(1) $x = 0$,	(1) $(0, -\beta, \gamma, \alpha'\alpha''\beta\gamma)$,
(2) $y = 0$,	(2) $(\alpha, 0, -\gamma, \beta'\beta''\gamma\alpha)$,
(3) $z = 0$,	(3) $(-\alpha, \beta, 0, \gamma'\gamma''\alpha\beta)$,
(4) $w = 0$,	(4) $(\alpha'\alpha'', \beta'\beta'', \gamma'\gamma'', 0)$,
(5) $X - w = 0$,	(5) $(1, 0, 0, 0)$,
(6) $Y - w = 0$,	(6) $(0, 1, 0, 0)$,
(7) $Z - w = 0$,	(7) $(0, 0, 1, 0)$,
(8) $P = 0$,	(8) $(0, 0, 0, 1)$,
(9) $X' - w = 0$,	(9) $(0, -\beta', \gamma', \alpha''\alpha\beta'\gamma')$,
(10) $Y' - w = 0$,	(10) $(\alpha', 0, -\gamma', \beta''\beta\gamma'\alpha')$,
(11) $Z' - w = 0$,	(11) $(-\alpha', \beta', 0, \gamma''\gamma\alpha'\beta')$,
(12) $P' = 0$,	(12) $(\alpha''\alpha, \beta''\beta, \gamma''\gamma, 0)$,
(13) $X'' - w = 0$,	(13) $(0, -\beta'', \gamma'', \alpha\alpha'\beta''\gamma'')$,
(14) $Y'' - w = 0$,	(14) $(\alpha'', 0, -\gamma'', \beta\beta'\gamma''\alpha'')$,
(15) $Z'' - w = 0$,	(15) $(-\alpha'', \beta'', 0, \gamma\gamma'\alpha''\beta'')$,
(16) $P'' = 0$,	(16) $(\alpha\alpha', \beta\beta', \gamma\gamma', 0)$,

where the nodes and planes are numbered as by Kummer; and by means of his (differently arranged) diagram of the relation between the several nodes and planes, I was enabled to form the following square diagram, which exhibits this relation in, I think, the most convenient form. To explain this, observe that in the upper and left-hand margins, the numbers refer to the nodes; in the body of the table, and in the right-hand margin to the planes, the table shows that for the node 1, the circumscribed cone is made up of the planes 1, 6, 7, 8, 9, 13; and that the remaining 15 nodes are situate on the nodal lines of this cone, the node 2 on the intersection of the planes 7, 8; the node 3 on the intersection of the planes 6, 8, and so on; and the like as regards the other lines of the table.

NODES	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	CONES
1	*	7, 8	6, 8	6, 7	9, 13	1, 6	1, 7	1, 8	1, 9	6, 13	7, 13	8, 13	1, 13	6, 9	7, 9	8, 9	1, 9, 13, 8, 7, 6
2	7, 8	*	5, 8	5, 7	2, 5	10, 14	2, 7	2, 8	5, 14	2, 10	7, 14	8, 14	5, 10	2, 14	7, 10	8, 10	2, 10, 14, 7, 8, 5
3	6, 8	5, 8	*	5, 6	3, 5	3, 6	11, 15	3, 8	5, 15	6, 15	3, 11	8, 15	5, 11	6, 11	3, 15	8, 11	3, 11, 15, 6, 5, 8
4	6, 7	5, 7	5, 6	*	4, 5	4, 6	4, 7	12, 16	5, 16	6, 16	7, 16	4, 12	5, 12	6, 12	7, 12	4, 16	4, 12, 16, 5, 6, 7
5	9, 13	2, 5	3, 5	4, 5	*	3, 4	2, 4	2, 3	5, 9	2, 13	3, 13	4, 13	5, 13	2, 9	3, 9	4, 9	5, 13, 9, 4, 3, 2
6	1, 6	10, 14	3, 6	4, 6	3, 4	*	1, 4	1, 3	1, 14	6, 10	3, 14	4, 14	1, 10	6, 14	3, 10	4, 10	6, 14, 10, 3, 4, 1
7	1, 7	2, 7	11, 15	4, 7	2, 4	1, 4	*	1, 2	1, 15	2, 15	7, 11	4, 15	1, 11	2, 11	7, 15	4, 11	7, 15, 11, 2, 1, 4
8	1, 8	2, 8	3, 8	12, 16	2, 3	1, 3	1, 2	*	1, 16	2, 16	3, 16	8, 12	1, 12	2, 12	3, 12	8, 16	8, 16, 12, 1, 2, 3
9	1, 9	5, 14	5, 15	5, 16	5, 9	1, 14	1, 15	1, 16	*	15, 16	14, 16	14, 15	1, 5	9, 14	9, 15	9, 16	9, 1, 5, 16, 15, 4
10	6, 13	2, 10	6, 15	6, 16	2, 13	6, 10	2, 15	2, 16	15, 16	*	13, 16	15, 13	10, 13	2, 6	10, 15	10, 16	10, 2, 6, 15, 16, 13
11	7, 13	7, 14	3, 11	7, 16	3, 13	3, 14	7, 11	3, 16	14, 16	13, 16	*	13, 14	11, 13	11, 14	3, 7	11, 16	11, 3, 7, 14, 13, 16
12	8, 13	8, 14	8, 15	4, 12	4, 13	4, 14	4, 15	8, 12	14, 15	13, 15	13, 14	*	12, 13	12, 14	12, 15	4, 8	12, 4, 8, 13, 14, 15
13	1, 13	5, 10	5, 11	5, 12	5, 13	1, 10	1, 11	1, 12	1, 5	10, 13	11, 13	12, 13	*	11, 12	10, 12	10, 11	13, 5, 1, 12, 11, 10
14	6, 9	2, 14	6, 11	6, 12	2, 9	6, 14	2, 11	2, 12	9, 14	2, 6	11, 14	12, 14	11, 12	*	9, 12	9, 11	14, 6, 2, 11, 12, 9
15	7, 9	7, 10	3, 15	7, 12	3, 9	3, 10	7, 15	3, 12	9, 15	10, 15	3, 7	12, 15	10, 12	9, 12	*	9, 10	15, 7, 3, 10, 9, 12
16	8, 9	8, 10	8, 11	4, 16	4, 9	4, 10	4, 11	8, 16	9, 16	10, 16	11, 16	4, 8	10, 11	9, 11	9, 10	*	16, 8, 4, 9, 10, 11

158. The before mentioned irrational equation may be written

$$\sqrt{1\,.\,5}+\sqrt{2\,.\,6}+\sqrt{3\,.\,7}=0,$$

and by symmetry we see that also

$$\sqrt{1\,.\,9}+\sqrt{2\,.\,10}+\sqrt{3\,.\,11}=0,$$

$$\sqrt{1\,.\,13}+\sqrt{2\,.\,14}+\sqrt{3\,.\,15}=0;$$

viz., these are three equations each containing the planes 1, 2, 3, which are three of the planes belonging to the node 1; the other three planes in any such equation (for instance, the planes 5, 6, 7, in the first equation) being three planes belonging to another node. Instead of the planes 1, 2, 3, we may have any other three planes belonging to the node 1; and instead of the node 1, any other node; but each equation belongs to two nodes: the number of equations is thus

$$\frac{6\,.\,5\,.\,4}{1\,.\,2\,.\,3}\times 16\times 3\div 2, =480.$$

159. To obtain the planes belonging to any such equation, combine any two of the outside right-hand lines of the diagram, these contain in common two numbers the places of which are interchanged; striking these out, we have four columns, and taking out of these any three columns, we have the corresponding sets of planes. For instance, lines 1 and 2 contain 78 and 87 respectively; striking these out, the lines are

1, 9, 13, 6;

2, 10, 14, 5;

whence we have the sets (1, 9, 13) and (2, 10, 14); viz., there is an irrational equation of the form

$$\sqrt{1\,.\,2}+\sqrt{9\,.\,10}+\sqrt{13\,.\,14}=0,$$

but it is probably necessary to introduce constant factors along with the products 1 . 2, 9 . 10, and 13 . 14 respectively. There are $\frac{1}{2}$ 16 . 15, = 120 pairs of lines, and each line gives 4 equations; in all 120 × 4, = 480 equations, as above.

160. I stop to remark that Kummer gives for his 13-nodal surface an equation containing three arbitrary constants, say λ, μ, ν, such that, putting one of these = 0, we have the 14-nodal surface; putting two of them each = 0, the 15-nodal surface; putting all three of them = 0, the 16-nodal surface. The equations for the 16-nodal surface and that for the 14-nodal surface, made use of by Kummer, are, in fact, those deduced as above from the equation of the 13 (α)-nodal surface; and the like form might have been used for the 15-nodal surface. But the form actually used by Kummer, as presently appearing, is an equivalent form not thus deducible from the equation of the 13 (α)-nodal surface.

The 15-*nodal Surface* 15 (2, 1, 1, 1, 1).

161. Kummer's equation is readily converted into the following:

$$Aw^2 + 2Bw + C + \frac{Kl}{m}\, xyz\, P = 0,$$

the circumscribed cone being thus

$$(lA + mP'P'')\, xyz\, P = 0,$$

and the equation being in the node-form (2, 1, 1, 1, 1).

The formulæ for the 15 nodes and the 10 singular planes depend upon a quadric equation, for the symmetrical expression of which I write

$$\alpha'\alpha'' - \beta''\gamma' = \beta'\beta'' - \gamma''\alpha' = \gamma'\gamma'' - \alpha''\beta' = \omega,$$
$$\alpha'\alpha'' - \beta'\gamma'' = \beta'\beta'' - \gamma'\alpha'' = \gamma'\gamma'' - \alpha'\beta'' = \varpi;$$

so that

$$\omega - \varpi = \beta'\gamma'' - \beta''\gamma' = \gamma'\alpha'' - \gamma''\alpha' = \alpha'\beta'' - \alpha''\beta',$$
$$\omega + \varpi = \alpha'\alpha'' + \beta'\beta'' + \gamma'\gamma'':$$

the equation in question then is

$$(\rho - \omega)(\rho - \varpi) + \frac{Kl}{4\alpha\beta\gamma m} = 0;$$

so that, calling the roots of it ρ_1, ρ_2, we have

$$\rho_1 + \rho_2 = \omega + \varpi, \quad \rho_1\rho_2 = \omega\varpi + \frac{Kl}{4\alpha\beta\gamma m};$$

or we may write

$$\rho_1 = \tfrac{1}{2}(\omega + \varpi + \sqrt{\Omega}), \quad = \tfrac{1}{2}(\alpha'\alpha'' + \beta'\beta'' + \gamma'\gamma'' + \sqrt{\Omega}),$$
$$\rho_2 = \tfrac{1}{2}(\omega + \varpi - \sqrt{\Omega}), \quad = \tfrac{1}{2}(\alpha'\alpha'' + \beta'\beta'' + \gamma'\gamma'' - \sqrt{\Omega}),$$

if for shortness

$$\Omega = (\omega - \varpi)^2 - \frac{Kl}{\alpha\beta\gamma m}.$$

162. I write also for shortness

$$\mathrm{a} = (\beta - \gamma)\,\alpha'\alpha'' + \alpha\,(\beta'\beta'' - \gamma'\gamma''),$$
$$\mathrm{b} = (\gamma - \alpha)\,\beta'\beta'' + \beta\,(\gamma'\gamma'' - \alpha'\alpha''),$$
$$\mathrm{c} = (\alpha - \beta)\,\gamma'\gamma'' + \gamma\,(\alpha'\alpha'' - \beta'\beta'');$$

and I say that the singular planes are

(1) (1) $x = 0$,

(2) (2) $y = 0$,

(3) (3) $z = 0$,

(4) (9) $-\frac{1}{\beta\gamma}(X'-w)+(\rho_1-\omega)\left(\frac{y}{\beta}+\frac{z}{\gamma}\right)=0,$

(5) (10) $-\frac{1}{\gamma\alpha}(Y'-w)+(\rho_1-\omega)\left(\frac{z}{\gamma}+\frac{x}{\alpha}\right)=0,$

(6) (11) $-\frac{1}{\alpha\beta}(Z'-w)+(\rho_1-\omega)\left(\frac{x}{\alpha}+\frac{y}{\beta}\right)=0,$

(7) (13) $-\frac{1}{\beta\gamma}(X''-w)+(\rho_2-\varpi)\left(\frac{y}{\beta}+\frac{z}{\gamma}\right)=0,$

(8) (14) $-\frac{1}{\gamma\alpha}(Y''-w)+(\rho_2-\varpi)\left(\frac{z}{\gamma}+\frac{x}{\alpha}\right)=0,$

(9) (15) $-\frac{1}{\alpha\beta}(Z''-w)+(\rho_2-\varpi)\left(\frac{x}{\alpha}+\frac{y}{\beta}\right)=0,$

(10) (8) $P=0$;

and that the nodes are

(1) (1) $(\;0\;,\; -\beta,\; \gamma\;,\; \alpha'\alpha''\beta\gamma\;),$

(2) (2) $(\;\alpha\;,\; 0\;,\; -\gamma,\; \beta'\beta''\gamma\alpha),$

(3) (3) $(-\alpha,\; \beta\;,\; 0\;,\; \gamma\gamma''\alpha\beta\;),$

(4) (9) $\{\; 0\;,\; \rho_1-\gamma'\gamma'',\; -(\rho_2-\beta'\beta''),\; \alpha\,(\rho_1-\gamma'\gamma'')(\rho_2-\beta'\beta'')\},$

(5) (10) $\{-(\rho_2-\gamma'\gamma''),\; 0\;,\; \rho_1-\alpha'\alpha''\;,\; \beta\,(\rho_1-\alpha'\alpha'')(\rho_2-\gamma'\gamma'')\},$

(6) (11) $\{\;\rho_1-\beta'\beta''\;,\; -(\rho_2-\alpha'\alpha''),\; 0\;,\; \gamma\,(\rho_1-\beta'\beta'')(\rho_2-\alpha'\alpha'')\},$

(7) (13) $\{\; 0\;,\; \rho_2-\gamma'\gamma''\;,\; -(\rho_1-\beta'\beta''),\; \alpha\,(\rho_2-\gamma'\gamma'')(\rho_1-\beta'\beta'')\},$

(8) (14) $\{-(\rho_1-\gamma'\gamma''),\; 0\;,\; \rho_2-\alpha'\alpha''\;,\; \beta\,(\rho_2-\alpha'\alpha'')(\rho_1-\gamma'\gamma'')\},$

(9) (15) $\{\;\rho_2-\beta'\beta''\;,\; -(\rho_1-\alpha'\alpha''),\; 0\;,\; \gamma\,(\rho_2-\beta'\beta'')(\rho_1-\alpha'\alpha'')\},$

(10) (8) $(0, 0, 0, 1),$

(11) (5) $(1, 0, 0, 0),$

(12) (6) $(0, 1, 0, 0),$

(13) (7) $(0, 0, 1, 0),$

(14) (16) $\left(\tfrac{1}{2}\alpha\,(\mathrm{a}-\alpha\sqrt{\Omega}),\; \tfrac{1}{2}\beta\,(\mathrm{b}-\beta\sqrt{\Omega}),\; \tfrac{1}{2}\gamma\,(\mathrm{c}-\gamma\sqrt{\Omega}),\; \frac{kl}{4m}\right),$

(15) (12) $\left(\tfrac{1}{2}\alpha\,(\mathrm{a}+\alpha\sqrt{\Omega}),\; \tfrac{1}{2}\beta\,(\mathrm{b}+\beta\sqrt{\Omega}),\; \tfrac{1}{2}\gamma\,(\mathrm{c}+\gamma\sqrt{\Omega}),\; \frac{kl}{4m}\right).$

163. The small reference numbers are those used by Kummer. It is, I think, better to retain the reference numbers belonging to the case of the 16-nodal surface; viz., there are here given, large, 1, 2, 3, 8, 9, 10, 11, 13, 14, 15 for the planes, and 1, 2, 3, 5, 6, ... to 16 for the nodes. Belonging to each node (that is, with the node as vertex) there is a quadric cone passing through 8 other nodes; and each node lies (exclusively of the cone whose vertex it is) in 8 such cones. We have thus the following square diagram:

NODES	1	2	3	5	6	7	8	9	10	11	12	13	14	15	16	CONES
1	*	$C8$	$C8$	9, 13	$C1$	$C1$	1, 8	1, 9	$C13$	$C13$	8, 13	1, 13	$C9$	$C9$	8, 9	1, 8, 9, 13 (6, 7)
2	$C8$	*	$C8$	$C2$	10, 14	$C2$	2, 8	$C14$	2, 10	$C14$	8, 14	$C10$	2, 14	$C10$	8, 10	2, 8, 13, 14 (5, 7)
3	$C8$	$C8$	*	$C3$	$C3$	11, 15	3, 8	$C15$	$C15$	3, 11	8, 15	$C11$	$C11$	3, 15	8, 11	3, 8, 11, 15 (5, 6)
5	9, 13	$C2$	$C3$	*	$C3$	$C2$	2, 3	$C9$	2, 13	3, 13	$C13$	$C13$	2, 9	3, 9	$C9$	2, 3, 9, 13 (4, 5)
6	$C1$	10, 14	$C3$	$C3$	*	$C1$	1, 3	1, 14	$C10$	3, 14	$C14$	1, 10	$C14$	3, 10	$C10$	1, 3, 10, 14 (4, 6)
7	$C1$	$C2$	11, 15	$C2$	$C1$	*	1, 2	1, 15	2, 15	$C11$	$C15$	1, 11	2, 11	$C15$	$C11$	1, 2, 11, 15 (4, 7)
8	1, 8	2, 8	3, 8	2, 3	1, 3	1, 2	*	$C1$	$C2$	$C3$	$C8$	$C1$	$C2$	$C3$	$C8$	1, 2, 3, 8 (12, 16)
9	1, 9	$C14$	$C15$	$C9$	1, 14	1, 15	$C1$	*	$C15$	$C14$	14, 15	$C1$	9, 14	9, 15	$C9$	1, 9, 14, 15 (5, 16)
10	$C13$	2, 10	$C15$	2, 13	$C10$	2, 15	$C2$	$C15$	*	$C13$	13, 15	10, 13	$C2$	10, 15	$C10$	2, 10, 13, 15 (6, 16)
11	$C13$	$C14$	3, 11	3, 13	3, 14	$C11$	$C3$	$C14$	$C13$	*	13, 14	11, 13	11, 14	$C3$	$C11$	3, 11, 13, 14 (7, 16)
12	8, 13	8, 14	8, 15	$C13$	$C14$	$C15$	$C8$	14, 15	13, 15	13, 14	*	$C13$	$C14$	$C15$	$C8$	8, 13, 14, 15 (4, 12)
13	1, 13	$C10$	$C11$	$C13$	1, 10	1, 11	$C1$	$C1$	10, 13	11, 13	$C13$	*	$C11$	$C10$	10, 11	1, 10, 11, 13 (5, 12)
14	$C9$	2, 14	$C11$	2, 9	$C14$	2, 11	$C2$	9, 14	$C2$	11, 14	$C14$	$C11$	*	$C9$	9, 11	2, 9, 11, 14 (6, 12)
15	$C9$	$C10$	3, 15	3, 9	3, 10	$C15$	$C3$	9, 15	10, 15	$C3$	$C15$	$C10$	$C9$	*	9, 10	3, 9, 10, 15 (7, 12)
16	8, 9	8, 10	8, 11	$C9$	$C10$	$C11$	$C8$	$C9$	$C10$	$C11$	$C8$	10, 11	9, 11	9, 10	*	8, 9, 10, 11 (4, 16)

164. The arrangement is the same as in the 16-nodal square diagram; only, in the right-hand margin, a bracket (6, 7) denotes that instead of the planes (6, 7) we have a quadric cone; which cones are, in the body of the table, denoted by C. Thus for the node 1 the sextic cone is made up of the planes 1, 8, 9, 13 and of a quadric cone (6, 7), $=C$: the remaining 14 nodes lie on the nodal lines of the sextic cone, viz., the node 2 on an intersection of the cone C with the plane 8, the node 3 on an intersection of the same cone and plane, the node 5 on the intersection of the planes 9, 13, and so on.

The Equation of the 15-*nodal Surface, as deduced from that of the* 13 (α)-*nodal.*

165. If, in the equation hereafter given for the 13-nodal surface, we write $\nu=0$, $\mu=0$, or (what is the same thing) in that of the 14-nodal surface we write $\mu=0$, the form is

$$\begin{aligned} &w^2(A+4\lambda yz)\\ &+2w\;(B-2\lambda yz\,X)\\ &+\qquad C=0. \end{aligned}$$

The circumscribed sextic cone is thus (2, 1, 1, 1, 1),

$$(\lambda\alpha^2yz+\beta\gamma x^2+\gamma\alpha xy+\alpha\beta xz)\,yz\,(P'-\rho_1P'')\,(P'-\rho_2P'')=0,$$

where ρ_1, ρ_2 now denote the roots of the equation

$$\lambda(\rho\alpha-\alpha'')^2-\frac{1}{\beta'\beta''\gamma'\gamma''}(\omega-\varpi)^2\,\rho\alpha'\alpha''=0.$$

The singular planes are

$$\begin{array}{ll}
(4) & w=0,\\
(12) & P'-\rho_1P''=0,\\
(16) & P'-\rho_2P''=0,\\
(3) & z=0,\\
(2) & y=0,\\
(15) & \dfrac{\gamma''\alpha'\beta'}{\gamma'\alpha''\beta''}\rho_1(Z'-w)-(Z''-w)=0,\\
(11) & \dfrac{\gamma''\alpha'\beta'}{\gamma'\alpha''\beta''}\rho_2(Z'-w)-(Z''-w)=0,\\
(14) & \dfrac{\beta''\gamma'\alpha'}{\beta'\gamma''\alpha''}\rho_1(Y'-w)-(Y''-w)=0,\\
(10) & \dfrac{\beta''\gamma'\alpha'}{\beta'\gamma''\alpha''}\rho_2(Y'-w)-(Y''-w)=0,\\
(5) & X-w=0,
\end{array}$$

and we have then the same square table as before: the coordinates of the 15 nodes may be obtained without difficulty.

166. The form is really equivalent to that first considered in regard to the 15-nodal surface. To show that this is so, we have only to arrange according to powers of x; viz., the equation thus becomes

$$\begin{aligned}&x^2\{(\gamma\gamma'\gamma'')^2y^2+(\beta\beta'\beta'')^2z^2+w^2-2\beta\beta'\beta''zw+2\gamma\gamma'\gamma''wy+2\beta\beta'\beta''\gamma\gamma'\gamma''yz\}\\&+2x\{\gamma\gamma'\gamma''\alpha\alpha'\alpha''y^2z+\alpha\alpha'\alpha''\beta\beta'\beta''yz^2+\beta\beta'\beta''z^2w-zw^2-w^2y-\gamma\gamma'\gamma''wy^2+Myzw\}\\&+\quad(\alpha\alpha'\alpha''yz+wy-wz)^2-4\lambda yzw(X-w)=0,\end{aligned}$$

where, if for a moment A denotes the coefficient of x^2, we have $y=0$, $z=0$, $w=0$, $X-w=0$, four tangent planes of the quadric cone $A=0$.

14-*nodal Surface* $8(3_1, 1, 1, 1)+6(2, 2, 1, 1)$, *Node-form* $(3_1, 1, 1, 1)$.

167. In the equation hereafter given for the 13-nodal surface, writing $\nu=0$, the sextic cone becomes

$$4z(\lambda\alpha^2y^2z+\mu\beta^2zx^2+\beta\gamma x^2y+\gamma\alpha xy^2+\alpha\beta xyz)(P'-\rho_1P'')(P'-\rho_2P'')=0;$$

viz., this is of the form in question $(3_1, 1, 1, 1)$; and the equation of the surface is

$$\begin{aligned}&w^2A+4(\lambda y+\mu x)z-4\lambda\mu z^2\\&+2w\{B-2(\lambda yX+\mu xY)z\}\\&+\quad C=0.\end{aligned}$$

The singular planes are

$$w=0, \qquad (4) \quad (4)$$
$$P'-\rho_1P''=0, \qquad (12) \quad (2)$$
$$P'-\rho_2P''=0, \qquad (16) \quad (3)$$
$$z=0, \qquad (3) \quad (1)$$
$$\frac{\gamma''\alpha'\beta'}{\gamma'\alpha''\beta''}\rho_1(Z'-w)-(Z''-w)=0, \qquad (15) \quad (6)$$
$$\frac{\gamma''\alpha'\beta'}{\gamma'\alpha''\beta''}\rho_2(Z'-w)-(Z''-w)=0, \qquad (11) \quad (5)$$

and the nodes are

$$(0, 0, 0, 1), \qquad (8) \quad (1)$$
$$(1, 0, 0, 0), \qquad (5) \quad (13)$$
$$(0, 1, 0, 0), \qquad (6) \quad (14)$$
$$(0, 0, 1, 0), \qquad (7) \quad (5)$$
$$\left(-\frac{\beta''-\beta'\rho_1}{\beta'\beta''}, \frac{\alpha''-\alpha'\rho_1}{\alpha'\alpha''}, 0, \frac{\gamma\gamma'\gamma''(\beta''-\beta'\rho_1)(\alpha''-\alpha'\rho_1)}{\alpha''\beta''\gamma'-\alpha'\beta'\gamma''\rho_1}\right), \qquad (11) \quad (4)$$
$$\left(-\frac{\beta''-\beta'\rho_2}{\beta'\beta''}, \frac{\alpha''-\alpha'\rho_2}{\alpha'\alpha''}, 0, \frac{\gamma\gamma'\gamma''(\beta''-\beta'\rho_2)(\alpha''-\alpha'\rho_2)}{\alpha''\beta''\gamma'-\alpha'\beta'\gamma''\rho_2}\right), \qquad (15) \quad (3)$$

$$\left(\frac{\alpha\alpha'\alpha''}{\alpha''-\alpha'\rho_1},\ \frac{\beta\beta'\beta''}{\beta''-\beta'\rho_1},\ \frac{\gamma\gamma'\gamma''}{\gamma''-\gamma'\rho_1},\ 0\right),\qquad (16)\quad {\scriptstyle (7)}$$

$$\left(\frac{\alpha\alpha'\alpha''}{\alpha''-\alpha'\rho_2},\ \frac{\beta\beta'\beta''}{\beta''-\beta'\rho_2},\ \frac{\gamma\gamma'\gamma''}{\gamma''-\gamma'\rho_2},\ 0\right),\qquad (12)\quad {\scriptstyle (8)}$$

$$(\ \alpha'\alpha'',\ \beta'\beta'',\ \gamma'\gamma'',\ 0\ ),\qquad (4)\quad {\scriptstyle (6)}$$

$$(\ -\alpha,\ \beta,\ 0,\ \gamma'\gamma''\alpha\beta),\qquad (3)\quad {\scriptstyle (2)}$$

two nodes on line

$$\frac{\gamma''\alpha'\beta'}{\gamma'\alpha''\beta''}\rho_2(Z'-w)-(Z''-w)=0,\quad P'-\rho_1P''=0,\qquad (13,\ 14)\quad {\scriptstyle (9,\ 10)}$$

and two nodes on line

$$\frac{\gamma''\alpha'\beta'}{\gamma'\alpha''\beta''}\rho_1(Z'-w)-(Z''-w)=0,\quad P'-\rho_2P''=0,\qquad (9,\ 10)\quad {\scriptstyle (11,\ 12)}$$

where the large numbers are those for the 16-nodal surface, the small numbers are Kummer's.

168. In these formulæ, ρ_1, ρ_2 denote the roots of the equation

$$\begin{aligned}&(\lambda\beta''^2+\mu\alpha''^2)(\rho\alpha'\beta')^2\\ &+(\lambda\beta'^2+\mu\alpha'^2)(\alpha''\beta'')^2\\ &-2\left[\lambda\beta'\beta''+\mu\alpha'\alpha''+\frac{1}{2\gamma'\gamma''}(\omega-\varpi)^2\right]\rho\alpha'\beta'\alpha''\beta''=0.\end{aligned}$$

169. The relation of the nodes and sextic cones is given by means of the square diagram on the opposite page:

NODES	3	4	5	6	7	8	9	10	11	12	13	14	15	16	CONES
3	*	×	$C\,3$	$C\,3$	11, 15	$C\,3$	$C\,15$	$C\,15$	3, 11	$C\,15$	$C\,11$	$C\,11$	3, 15	$C\,11$	3, 11, 15, (5, 6, 8)
4	×	*	$C\,4$	$C\,4$	$C\,4$	12, 16	$C\,16$	$C\,16$	$C\,16$	4, 12	$C\,12$	$C\,12$	$C\,12$	4, 16	4, 12, 16, (5, 6, 7)
5	$D\,3$	$D\,4$	*	3, 4	$D\,4$	$D\,3$			$D'\,3$	$D'\,4$			$D'\,3$	$D'\,4$	3, 4, (2, 5), (9, 13)
6	$D\,3$	$D\,4$	3, 4	*	$D\,4$	$D\,3$			$D'\,3$	$D'\,4$			$D'\,3$	$D'\,4$	3, 4, (1, 6), (10, 14)
7	11, 15	$C\,4$	$C\,4$	$C\,4$	*	×	$C\,15$	$C\,15$	$C\,11$	4, 15	$C\,11$	$C\,11$	$C\,15$	4, 11	4, 11, 15(1, 2, 7)
8	$C\,3$	12, 16	$C\,3$	$C\,3$	×	*	$C\,16$	$C\,16$	3, 16	$C\,12$	$C\,12$	$C\,12$	3, 12	$C\,16$	3, 12, 16(1, 2, 8)
9	$D\,15$	$D\,16$			$D'\,15$	$D'\,16$	*	15, 16	$D\,16$	$D\,15$			$D'\,15$	$D'\,16$	5, 16(5, 14), (1, 9)
10	$D\,15$	$D\,16$			$D'\,15$	$D'\,16$	15, 16	*	$D\,16$	$D\,15$			$D'\,15$	$D'\,16$	15, 16(6, 13), (2, 10)
11	3, 11	$C\,16$	$C\,3$	$C\,3$	$C\,11$	3, 16	$C\,16$	$C\,16$	*	×	$C\,11$	$C\,11$	$C\,3$	11, 16	3, 11, 16(7, 13, 14)
12	$C\,15$	4, 12	$C\,4$	$C\,4$	4, 15	$C\,12$	$C\,15$	$C\,15$	×	*	$C\,12$	$C\,12$	12, 15	$C\,4$	4, 12, 15(8, 13, 14)
13	$D\,11$	$D\,12$			$D'\,11$	$D'\,12$			$D'\,11$	$D'\,12$	*	11, 12	$D\,12$	$D\,11$	11, 12(5, 10), (1, 13)
14	$D\,11$	$D\,12$			$D'\,11$	$D'\,12$			$D'\,11$	$D'\,12$	11, 12	*	$D\,12$	$D\,11$	11, 12(6, 9), (2, 14)
15	3, 15	$C\,12$	$C\,3$	$C\,3$	$C\,15$	3, 12	$C\,15$	$C\,15$	$C\,3$	12, 15	$C\,12$	$C\,12$	*	×	3, 12, 15(7, 9, 10)
16	$C\,11$	4, 16	$C\,4$	$C\,4$	4, 11	$C\,16$	$C\,16$	$C\,16$	11, 16	$C\,4$	$C\,11$	$C\,11$	×	*	4, 11, 16(8, 9, 10)

where the arrangement is the same as before; only in the right-hand margin, a bracket (5, 6, 8) denotes that, instead of the planes 5, 6, 8, we have a nodal cubic cone (5, 6, 8), which is in the body of the table referred to as C, and the nodal line thereof by $\times$; and the brackets (2, 5), (9, 13) denote that, instead of the planes 2, 5, we have a quadric cone (2, 5), and instead of the planes 9, 13, a quadric cone (9, 13); which cones are in the body of the table referred to as D, D' respectively. And it is to be understood that each vacant square of the table should contain the symbol (D, D'), this being omitted only for the purpose of better exhibiting the form of the table.

170. The equation of the surface may be presented in the irrational form

$$\begin{aligned}&\sqrt{(P'-\rho_1P'')\,[\gamma''\alpha'\beta'\rho_2\,(Z'-w)-\gamma'\alpha''\beta''\,(Z''-w)]}\\ +&\sqrt{(P'-\rho_2P'')\,[\gamma''\alpha'\beta'\rho_1\,(Z'-w)-\gamma'\alpha''\beta''\,(Z''-w)]}\\ +&\ (\rho_1-\rho_2)\sqrt{\alpha'\beta'\alpha''\beta''\left(\frac{\lambda}{\alpha''^2}+\frac{\mu}{\beta''^2}\right)zw}=0.\end{aligned}$$

In fact the norm of the left-hand side is

$$\begin{aligned}=(\rho_1-\rho_2)^2\,(\omega-\varpi)^2\,\{&w^2\,[A+4z\,(\lambda y+\mu x)-4\lambda\mu z^2]\\ &+2w\,[B-2z\,(\lambda yX+\mu xY)]\\ &+C\qquad\qquad\qquad\qquad\}.\end{aligned}$$

To partially verify this, observe that, writing the equation under the form $\sqrt{R}+\sqrt{S}+\sqrt{T}=0$, on writing therein $w=0$, we ought to have

$$(R-S)^2=(\rho_1-\rho_2)^2\,(\omega-\varpi)^2\,(\alpha\alpha'\alpha''yz+\beta\beta'\beta''zx+\gamma\gamma'\gamma''xy)^2.$$

But writing $w=0$, we have

$$\begin{aligned}R-S=&\ \ (P'-\rho_1P'')\,(\gamma''\alpha'\beta'\rho_2Z'-\gamma'\alpha''\beta''Z'')\\ &-(P'-\rho_2P'')\,(\gamma''\alpha'\beta'\rho_1Z'-\gamma'\alpha''\beta''Z''),\\ =&\ \ (\rho_2-\rho_1)\,(\gamma''\alpha'\beta'P'Z'-\gamma'\alpha''\beta''P''Z''),\\ =&\ \ (\rho_2-\rho_1)\,\{\gamma''\,(\beta'\gamma'x+\gamma'\alpha'y+\alpha'\beta'z)\,(\beta\beta''x-\alpha\alpha''y)\\ &\qquad\qquad-\gamma'\,(\beta''\gamma''x+\gamma''\alpha''y+\alpha''\beta''z)\,(\beta\beta'x-\alpha\alpha'y)\},\end{aligned}$$

which is easily found to be

$$=-(\rho_2-\rho_1)\,(\omega-\varpi)\,(\alpha\alpha'\alpha''yz+\beta\beta'\beta''zx+\gamma\gamma'\gamma''xy)\,;$$

and thus $(R-S)^2$ has the value in question.

171. In further verification, observe that, writing $x,\ y,\ z=\alpha'\alpha'',\ \beta'\beta'',\ \gamma'\gamma''$, and therefore $P'=0$, $P''=0$, we ought to have

$$(\rho_1-\rho_2)^2\,(\alpha'\beta'\alpha''\beta'')^2\left(\frac{\lambda}{\alpha''^2}+\frac{\mu}{\beta''^2}\right)z^2w^2=(\rho_1-\rho_2)^2\,(\omega-\varpi)^2\,w^2\,[A+4z\,(\lambda y+\mu x)-4\lambda\mu z^2],$$

observing that, for the values in question, B, X, Y, C all vanish.

This is

$$(\rho_1-\rho_2)^2(\alpha'\beta'\alpha''\beta'')^2\left(\frac{\lambda}{\alpha''^2}+\frac{\mu}{\beta''^2}\right)^2(\gamma'\gamma'')^2=(\omega-\varpi)^2\{(\omega-\varpi)^2+4\gamma'\gamma''(\lambda\beta'\beta''+\mu\alpha'\alpha''-\lambda\mu\gamma'\gamma'')\},$$

which is in fact the value of $(\rho_1-\rho_2)^2$ obtained from the equation in ρ.

The 13 (α)-*nodal Surface* $3(4_3, 1, 1)+1(3, 1, 1, 1)+9(3_1, 2, 1)$.

172. The equation, node-form $(4_3, 1, 1)$, is

$$\begin{aligned} &w^2\{A+4(\lambda yz+\mu zx+\nu xy)-4(\mu\nu x^2+\nu\lambda y^2+\lambda\mu z^2)\}\\ &+2w\{B-2(\lambda yzX+\mu zxY+\nu xyZ)\}\\ &+\quad C=0\,; \end{aligned}$$

viz., for the circumscribed cone we have

$$\begin{aligned} &\{A+4(\lambda yz+\mu zx+\nu xy)-4(\mu\nu x^2+\nu\lambda y^2+\lambda\mu z^2)\}C\\ &-\{B-2(\lambda yzX+\mu zxY+\nu xyZ)\}^2\\ &=4\{\lambda\alpha^2y^2z^2+\mu\beta^2z^2x^2+\nu\gamma^2x^2y^2+\beta\gamma x^2yz+\gamma\alpha y^2zx+\alpha\beta z^2xy\}\\ &\qquad\times\left\{-\lambda\frac{X^2}{\alpha^2}-\mu\frac{Y^2}{\beta^2}-\nu\frac{Z^2}{\gamma^2}+\alpha'\alpha''\beta'\beta''\gamma'\gamma''P'P''\right\}, \end{aligned}$$

where, on the right-hand side, the first factor, equated to zero, represents a trinodal quartic cone, the nodal lines whereof are $(y=0,\ z=0)$, $(z=0,\ x=0)$, $(x=0,\ y=0)$.

173. As regards the second factor, it is to be observed that, writing as above

$$\omega-\varpi=\beta'\gamma''-\beta''\gamma',\quad =\gamma'\alpha''-\gamma''\alpha',\quad =\alpha'\beta''-\alpha''\beta',$$

we have identically

$$\alpha'P'-\alpha''P''=\frac{\omega-\varpi}{\beta'\beta''\gamma'\gamma''}\,\frac{X}{\alpha},$$

$$\beta'P'-\beta''P''=\frac{\omega-\varpi}{\gamma'\gamma''\alpha'\alpha''}\,\frac{Y}{\beta},$$

$$\gamma'P'-\gamma''P''=\frac{\omega-\varpi}{\alpha'\alpha''\beta'\beta''}\,\frac{Z}{\gamma}\,;$$

so that the second factor is

$$\begin{aligned} =\frac{1}{(\omega-\varpi)^2}\{&-\lambda(\beta'\beta''\gamma'\gamma'')^2(\alpha'P'-\alpha''P'')^2-\mu(\gamma'\gamma''\alpha'\alpha'')^2(\beta'P'-\beta''P'')^2\\ &-\nu(\alpha'\alpha''\beta'\beta'')^2(\gamma'P'-\gamma''P'')^2+(\omega-\varpi)^2\alpha'\alpha''\beta'\beta''\gamma'\gamma''P'P''\}\,; \end{aligned}$$

or, what is the same thing,

$$\frac{1}{(\omega-\varpi)^2}\{-\ [\lambda(\beta''\gamma'')^2+\mu(\gamma''\alpha'')^2+\nu(\alpha''\beta'')^2](\alpha'\beta'\gamma'P')^2$$

$$-\ [\lambda(\beta'\gamma')^2+\mu(\gamma'\alpha')^2+\nu(\alpha'\beta')^2](\alpha''\beta''\gamma''P'')^2$$

$$+2[\lambda(\beta'\gamma'\beta''\gamma'')+\mu(\gamma'\alpha'\gamma''\alpha'')+\nu(\alpha'\beta'\alpha''\beta'')+\tfrac{1}{2}(\omega-\varpi)^2]\times\alpha'\beta'\gamma'P'\alpha''\beta''\gamma''P''\},$$

so that, equating to zero, we have a pair of planes, each passing through the line $P'=0$, $P''=0$, and which it is clear must be the tangent planes from this line to the quadric cone $A'=0$. I will presently return to this, but I consider first the foregoing identical equation in regard to the circumscribed cone.

174. In verification hereof, observe first that, if $x=0$, the equation becomes

$$\{(y-z)^2+4\lambda yz-4\lambda(\nu y^2+\mu z^2)\}(\alpha\alpha'\alpha'')^2y^2z^2-\{\alpha\alpha'\alpha''(y^2z-yz^2)-2\lambda yz\alpha(\gamma'\gamma''y-\beta'\beta''z)\}^2$$

$$=4\lambda y^2z^2\{-\lambda\alpha^2(\gamma'\gamma''y-\beta'\beta''z)^2-(\alpha\alpha'\alpha'')^2(\nu y^2+\mu z^2)-\alpha^2\alpha'\alpha''(\gamma'y+\beta'z)(\gamma''y+\beta''z)\},$$

viz., omitting terms which destroy each other, this is

$$[(y-z)^2+4\lambda yz](\alpha\alpha'\alpha'')^2y^2z^2-[\alpha\alpha'\alpha''(y^2z-yz^2)-2\lambda yz\alpha(\gamma'\gamma''y-\beta'\beta''z)]^2$$

$$=4\lambda y^2z^2[-\lambda\alpha^2(\gamma'\gamma''y-\beta'\beta''z)^2-\alpha^2\alpha'\alpha''(\gamma'y+\beta'z)(\gamma''y+\beta''z)].$$

Or again, this is

$$4\lambda yz(\alpha\alpha'\alpha'')^2y^2z^2+4\lambda yz\,\alpha^2\alpha'\alpha''(y^2z-yz^2)(\gamma'\gamma''y-\beta'\beta''z)$$

$$=-4\lambda y^2z^2\alpha^2\alpha'\alpha''(\gamma'y+\beta'z)(\gamma''y+\beta''z);$$

viz., this is

$$\alpha'\alpha''yz+(y-z)(\gamma'\gamma''y-\beta'\beta''z)=(\gamma'y+\beta'z)(\gamma''y+\beta''z),$$

which is at once seen to be true.

175. Again, compare the terms which contain x^4yz. On the right-hand side, we have

$$4\beta\gamma\,x^2yz\times\text{term in }x^2\text{ of }\left(-\lambda\frac{X^2}{\alpha^2}-\mu\frac{Y^2}{\beta^2}-\nu\frac{Z^2}{\gamma^2}+\alpha'\alpha''\beta'\beta''\gamma'\gamma''P'P''\right);$$

viz., the coefficient is

$$=4\beta\gamma\{-\mu(\gamma'\gamma'')^2-\nu(\beta'\beta'')^2+\beta'\beta''\gamma'\gamma''\}.$$

On the left-hand side, that is in $A'C-B'^2$, the only terms which give rise to the terms in question are

$$\text{in }A',\ (1-4\mu\nu)x^2;\ \text{ in }C,\ 2\beta\beta'\beta''\gamma\gamma'\gamma''x^2yz;$$

and in B'

$$(\gamma\gamma'\gamma''-2\nu\gamma\beta'\beta'')x^2y,\quad -(\beta\beta'\beta''-2\mu\beta\gamma'\gamma'')x^2z;$$

whence the coefficient is

$$2\beta\beta'\beta''\gamma\gamma'\gamma''(1-4\mu\nu)+2(\gamma\gamma'\gamma''-2\nu\gamma\beta'\beta'')(\beta\beta'\beta''-2\mu\beta\gamma'\gamma''),$$

which is in fact

$$= 4\beta\gamma \{-\mu(\gamma'\gamma'')^2 - \nu(\beta'\beta'')^2 + \beta'\beta''\gamma'\gamma''\},$$

which is right; and the verification may be completed without difficulty.

176. The singular planes are

$$\begin{array}{lr} w = 0, & (4) \\ P' - \rho_1 P'' = 0, & (12) \\ P' - \rho_2 P'' = 0, & (16); \end{array}$$

and the nodes are

$$\begin{array}{lrr}
(0,\ 0,\ 0,\ 1), & (8) & {\scriptstyle (1)} \\
(1,\ 0,\ 0,\ 0), & (5) & {\scriptstyle (5)} \\
(0,\ 1,\ 0,\ 0), & (6) & {\scriptstyle (6)} \\
(0,\ 0,\ 1,\ 0), & (7) & {\scriptstyle (7)} \\
\left(\dfrac{\alpha\alpha'\alpha''}{\alpha''-\alpha'\rho_1},\ \dfrac{\beta\beta'\beta''}{\beta''-\beta'\rho_1},\ \dfrac{\gamma\gamma'\gamma''}{\gamma''-\gamma'\rho_1},\ 0\right), & (16) & {\scriptstyle (2)} \\
\left(\dfrac{\alpha\alpha'\alpha''}{\alpha''-\alpha'\rho_2},\ \dfrac{\beta\beta'\beta''}{\beta''-\beta'\rho_2},\ \dfrac{\gamma\gamma'\gamma''}{\gamma''-\gamma'\rho_2},\ 0\right), & (12) & {\scriptstyle (3)} \\
(\ \alpha'\alpha'',\ \beta'\beta'',\ \gamma'\gamma'',\ 0), & (4) & {\scriptstyle (4)} \\
\text{three nodes in } P' - \rho_1 P'' = 0 & (13,\ 14,\ 15) & {\scriptstyle (8,\ 9,\ 10)} \\
\text{,, \quad ,, } \quad P' - \rho_2 P'' = 0 & (9,\ 10,\ 11) & {\scriptstyle (11,\ 12,\ 13)};
\end{array}$$

where the small numbers are those used by Kummer, the large ones are those referring to the 16-nodal surface, and are here adopted. In the foregoing formulæ ρ_1, ρ_2 are the roots of

$$\begin{aligned}
&[\lambda(\beta''\gamma'')^2 + \mu(\gamma''\alpha'')^2 + \nu(\alpha''\beta'')^2](\rho\alpha'\beta'\gamma')^2 \\
&+ [\lambda(\beta'\gamma')^2 + \mu(\gamma'\alpha')^2 + \nu(\alpha'\beta')^2](\alpha''\beta''\gamma'')^2 \\
&- 2[\lambda\beta'\gamma'\beta''\gamma'' + \mu\gamma'\alpha'\gamma''\alpha'' + \nu\alpha'\beta'\alpha''\beta'' + \tfrac{1}{2}(\omega - \varpi)^2]\rho\alpha'\beta'\gamma'\alpha''\beta''\gamma'' = 0.
\end{aligned}$$

177. We have the square diagram in the following page:

NODES	4	5	6	7	8	9	10	11	12	13	14	15	16	CONES
4	*	$C4$	$C4$	$C4$	12, 16	$C16$	$C16$	$C16$	4, 12	$C12$	$C12$	$C12$	4, 16	4, 12, 16, (5, 6, 7)
5	$C4$	*	$C4$	$C4$	×				$C'4$				$C'4$	4, (2, 3; 5) nodal, (9, 13)
6	$C4$	$C4$	*	$C4$	×				$C'4$				$C'4$	4, (1, 3; 6) nodal, (10, 14)
7	$C4$	$C4$	$C4$	*	×				$C'4$				$C'4$	4, (1, 2; 7) nodal, (11, 15)
8	12, 16	×	×	×	*	$C16$	$C16$	$C16$	$C16$	$C12$	$C12$	$C12$	$C12$	12, 16 (8; 1, 2, 3) trinodal
9	$C16$				$C'16$	*	$C16$	$C16$	×				$C'16$	16, (5; 14, 15) nodal, (1, 9)
10	$C16$				$C'16$	$C16$	*	$C16$	×				$C'16$	16, (6; 13, 15) nodal, (2, 10)
11	$C16$				$C'16$	$C16$	$C16$	*	×				$C'16$	16, (7; 13, 14) nodal, (3, 11)
12	4, 12	$C4$	$C4$	$C4$	$C12$	×	×	×	*	$C12$	$C12$	$C12$	$C4$	4, 12 (8; 13, 14, 15) trinodal
13	$C12$				$C'12$				$C'12$	*	$C12$	$C12$	×	12, (5; 10, 11) nodal, (1, 13)
14	$C12$				$C'12$				$C'12$	$C12$	*	$C12$	×	12, (6; 9, 11) nodal, (2, 14)
15	$C12$				$C'12$				$C'12$	$C12$	$C12$	*	×	12, (7; 9, 10) nodal, (3, 15)
16	4, 16	$C4$	$C4$	$C4$	$C16$	$C16$	$C16$	$C16$	$C4$	×	×	×	*	4, 16, (8; 9, 10, 11) trinodal

where for greater clearness I have omitted the symbol CC', which is to be understood as occupying each of the vacant squares.

The arrangement is the same as before; the right-hand margin shows the sextic cone; viz., for the node 4 this is made up of the singular planes 4, 12, 16 and of a cubic cone represented by (5, 6, 7) (as replacing the planes 5, 6, 7 in the case of the 16-nodal surface). Similarly for the node 5, the sextic cone is made up of the singular plane 4, the nodal cubic cone (2, 3; 5), and the quadric cone (9, 13) (the numbers in these last symbols indicating the planes in the case of the 16-nodal surface, which are here replaced by cones). So for the node 8, the sextic cone is made up of the singular planes 12, 16 and of the trinodal quartic cone (8; 1, 2, 3). As regards a nodal cubic cone, for example (2, 3; 5), the semicolon is used to indicate that the nodal line replaces the intersection of the planes 2, 3; the other intersections 2, 5 and 3, 5 having disappeared. And so for a trinodal quartic cone (8; 1, 2, 3), the semicolon is used to indicate that the nodal lines replace the intersection (1, 2), the intersection (1, 3), and the intersection (2, 3) respectively; the other intersections 1, 2; 2, 8; and 3, 8 having disappeared. Finally, in the body of the table, C is used to denote the cubic or the quartic cone (as the case may be); $\times$ to denote a nodal line of either of these cones; and C' the quadric cone; as already mentioned, the vacant places are considered to be CC'.

The reading of the table is then as follows; viz., for the node 4, the remaining twelve nodes lie on the nodal lines of the sextic cone 4, 12, 16, (5, 6, 7), as shown; viz., 5, 6, 7 are each of them on the intersection of the cubic cone with the plane 4; 8 is on the intersection of the planes 12 and 16; and so on.

I reserve for another Memoir the discussion of the 13 (β)-nodal surface, and the surfaces with less than 13 nodes.