459.

ON THE DOUBLE-SIXERS OF A CUBIC SURFACE.

[From the *Quarterly Journal of Pure and Applied Mathematics*, vol. x. (1870), pp. 58—71.]

THE 27 lines on a cubic surface include, and that in 36 different ways, *a double-sixer;* viz. a system of two sets of six lines 1, 2, 3, 4, 5, 6; 1′, 2′, 3′, 4′, 5′, 6′, such that every line of the one set intersects all the non-corresponding lines of the other set, thus

	1	2	3	4	5	6
1′		.	.	.	.	.
2′	.		.	.	.	.
3′	.	.		.	.	.
4′	.	.	.		.	.
5′	.	.	.	.		.
6′	.	.	.	.	.	

there being in all 30 intersections.

Any line say 4, of the one set, intersects five lines 1′, 2′, 3′, 5′, 6′ of the other set; and these six lines being given the double-sixer may be constructed; viz. (besides the line 4) we have a line 1 meeting the lines 2′, 3′, 5′, 6′; a line 2 meeting the lines 3′, 5′, 6′, 1′; a line 3 meeting the lines 5′, 6′, 1′, 2′; a line 5 meeting the lines 6′, 1′, 2′, 3′; and a line 6′ meeting the lines 1′, 2′, 3′, 5′; and then the lines 1, 2, 3, 5, 6 are all of them met by a single line 4′, which completes the system.

We may, if we please, consider the lines 4, 2 as given, and then 1′, 3′, 5′, 6′ will be any four lines each of them meeting the two given lines 4, 2; 2′ will be any line meeting 4; and we have to determine a line 4′ meeting 2, such that there may exist the lines 1, 3, 5, 6, completing the system as above. Or what is the same thing, we have a skew quadrilateral 1′, 2, 3′, 4; 5′ and 6′ meet 2 and 4; 2′ meets

4, and 4′ meets 2: 5 and 6 meet 1′ and 3′; 1 meets 3′ and 3 meets 1′; and the two sets 2′, 4′, 5′, 6′ and 1, 3, 5, 6 meet thus

	1	3	5	6
2′	.	.	.	.
4′	.	.	.	.
5′	.	.		.
6′	.	.	.	

Hence, starting with the skew quadrilateral 1′23′4, and taking $x=0$, $y=0$, $z=0$, $w=0$ for the equations of the four planes 41′, 1′2, 23′, 3′4 respectively; or what is the same thing $x=0$, $y=0$ for the equations of the line 1′; $y=0$, $z=0$ for those of the line 2; $z=0$, $w=0$ for those of the line 3′; and $w=0$, $x=0$ for those of the line 4; the several lines may be determined, each of them by means of its six coordinates, as follows:

	a	b	c	f	g	h
1′	0	0	0	0	0	1
2	0	0	0	1	0	0
3′	0	0	1	0	0	0
4	1	0	0	0	0	0
2′	A_2	B_2	C_2	0	G_2	H_2
4′	0	B_4	C_4	F_4	G_4	H_4
5′	0	B_5	C_5	0	G_5	H_5
6′	0	B_6	C_6	0	G_6	H_6
1	a_1	b_1	c_1	f_1	g_1	0
3	a_3	b_3	0	f_3	g_3	h_3
5	a_5	b_5	0	f_5	g_5	0
6	a_6	b_6	0	f_6	g_6	0

where

$$B_2G_2 + C_2H_2 = 0,$$
$$B_4G_4 + C_4H_4 = 0,$$
$$B_5G_5 + C_5H_5 = 0,$$
$$B_6G_6 + C_6H_6 = 0,$$
$$a_1f_1 + b_1g_1 = 0,$$
$$a_3f_3 + b_3g_3 = 0,$$
$$a_5f_5 + b_5g_5 = 0,$$
$$a_6f_6 + b_6g_6 = 0.$$

The conditions in regard to the intersections of the lines 2′, 4′, 5′, 6′ and 1, 3, 5, 6, are formed by means of the diagram

1	f_1	g_1	0	a_1	b_1	c_1	A_2	B_2	C_2	0	G_2	H_2	2′,
3	f_3	g_3	h_3	a_3	b_3	0	0	B_4	C_4	F_1	G_4	H_4	4′,
5	f_5	g_5	0	a_5	b_5	0	0	B_5	C_5	0	G_5	H_5	5′,
6	f_6	g_6	0	a_6	b_6	0	0	B_6	C_6	0	G_6	H_6	6′,

viz. we have the equations

first set,

$$\begin{aligned} f_1A_2+g_1B_2 \quad\quad &+b_1G_2+c_1H_2=0,\\ g_1B_4+a_1F_4&+b_1G_4+c_1H_4=0,\\ g_1B_5 \quad\quad &+b_1G_5+c_1H_5=0,\\ g_1B_6 \quad\quad &+b_1G_6+c_1H_6=0; \end{aligned}$$

second set,

$$\begin{aligned} f_3A_2+g_3B_2+h_3C_2 \quad\quad &+b_3G_2=0,\\ g_3B_4+h_3C_4+a_3F_3&+b_3G_4=0,\\ g_3B_5+h_3C_5 \quad\quad &+b_3G_5=0,\\ g_3B_6+h_3C_6 \quad\quad &+b_3G_6=0; \end{aligned}$$

third set,

$$\begin{aligned} f_5A_2+g_5B_2 \quad\quad &+b_5G_2=0,\\ g_5B_4 \quad +a_5F_4&+b_5G_4=0,\\ g_5B_6 \quad\quad &+b_5G_6=0; \end{aligned}$$

fourth set,

$$\begin{aligned} f_6A_2+g_6B_2 \quad\quad &+b_6G_2=0,\\ g_6B_4 \quad +a_6F_4&+b_6G_4=0,\\ g_6B_5 \quad\quad &+b_6G_5=0; \end{aligned}$$

and it is to be shown, that taking as given the coordinates of 2′, 5′, 6′, that is $(A_2, B_2, C_2, G_2, H_2)$, (B_5, C_5, G_5, H_5) and (B_6, C_6, G_6, H_6), we can find the coordinates of the remaining lines 4′, 1, 3, 5, 6.

The first set of equations gives

$$g_1,\ b_1,\ c_1 = \left\| \begin{matrix} B_5, & G_5, & H_5 \\ B_6, & G_6, & H_6 \end{matrix} \right\|,$$

viz. g_1, b_1, c_1 are proportional, but as only the ratios are material, they may be taken equal, to the determinants $G_5H_6-G_6H_5$, $H_5B_6-H_6B_5$, $B_5G_6-B_6G_5$. And then retaining g_1, b_1, c_1 to signify these values respectively, the first equation gives f_1A_2, and the second equation gives a_1F_4; multiplying together these values, and writing $a_1f_1=-b_1g_1$, we find

$$(B_2B_4,\ G_2G_4,\ H_2H_4,\ G_2H_4+G_4H_2,\ H_2B_4+H_4B_2,\ B_2G_4+B_4G_2+A_2F_4 \between g_1,\ b_1,\ c_1)^2=0.$$

Proceeding in a similar manner with the second set of equations, we have first

$$g_3,\ h_3,\ b_3 = \left\| \begin{matrix} B_5, & C_5, & G_5 \\ B_6, & C_6, & G_6 \end{matrix} \right\|,$$

$$\text{(observe } h_3 = G_5 B_6 - G_6 B_5, = -c_1),$$

and then

$$(B_2B_4,\ C_2C_4,\ G_2G_4,\ C_2G_4 + C_4G_2,\ G_2B_4 + G_4B_2 + A_2F_4,\ B_2C_4 + B_4C_2 \between g_3,\ h_3,\ b_3)^2 = 0.$$

The third set gives more simply

$$g_5{}^2 B_2 B_4 + g_5 b_5 (B_2 G_4 + B_4 G_2 + A_2 F_4) + b_5{}^2 G_2 G_4 = 0,$$

or since

$$g_5 : b_5 = G_6 : -B_6,$$

this is

$$G_6{}^2 B_2 B_4 - G_6 B_6 (B_2 G_4 + B_4 G_2 + A_2 F_4) + B_6{}^2 G_2 G_4 = 0,$$

and similarly, the fourth set gives

$$g_6{}^2 B_2 B_4 + g_6 b_6 (B_2 G_4 + B_4 G_2 + A_2 F_4) + b_6{}^2 G_2 G_4 = 0,$$

or since $g_6 : b_6 = G_5 : -B_5$, this is

$$G_5{}^2 B_2 B_4 - G_5 B_5 (B_2 G_4 + B_4 G_2 + A_2 F_4) + B_5{}^2 G_2 G_4 = 0:$$

and these last two results lead to the values of the ratios of B_2B_4, $B_2G_4+B_4G_2+A_2F_4$, G_2G_4; viz. these are proportional to expressions containing the common factor $B_5G_6 - B_6G_5$, and omitting this common factor, and taking them equal instead of merely proportional to the resulting expressions (which is allowable, since the absolute values are not material), we have

$$B_2B_4,\ B_2G_4 + B_4G_2 + A_2F_4,\ G_2G_4 = B_5B_6,\ B_5G_6 + B_6G_5,\ G_5G_6.$$

Returning to the result obtained from the first set of equations, this now becomes

$$(B_5B_6,\ G_5G_6,\ H_2H_4,\ G_2H_4 + G_4H_2,\ H_2B_4 + H_4B_2,\ B_5G_6 + B_6G_5 \between g_1,\ b_1,\ c_1)^2 = 0:$$

but the terms containing g_1, b_1 are $(B_5g_1 + G_5b_1)(B_6g_1 + G_6b_1)$, viz. this is $= -H_5c_1 . -H_6c_1$, that is $H_5H_6c_1{}^2$; the whole equation is thus divisible by c_1, and omitting this factor, it becomes

$$g_1 (H_2 B_4 + H_4 B_2) + b_1 (G_2 H_4 + G_4 H_2) + c_1 (H_2 H_4 + H_5 H_6) = 0.$$

Proceeding in like manner with the result obtained from the second set of equations, this becomes

$$(B_5B_6,\ C_2C_4,\ G_5G_6,\ C_2G_4 + C_4G_2,\ B_5G_6 + B_6G_5,\ B_2G_4 + B_4C_2 \between g_3,\ h_3,\ b_3)^2 = 0,$$

where the terms containing g_3, b_3 are $(Bg_3+G_5b_3)(Bg_3+G_3b)$, viz. this is $-h_3C_5 . -h_3C_6 = h_3{}^2C_5C_6$; the whole equation divides by h_3, and it then becomes

$$g_3 (B_2C_4 + B_4C_2) + h_3 (C_2C_4 + C_5C_6) + b_3 (C_2G_4 + C_4G_2) = 0.$$

Considering B_4, G_4, F_4 as given by the equations

$$B_2B_4 = B_5B_6,\ G_2G_4 = G_5G_6,\ B_2G_4 + G_2B_4 + A_2F_4 = B_5G_6 + B_6G_5,$$

the equations last obtained determine the values of H_4 and C_4, viz. these equations may be written

$$(g_1B_2 + b_1G_2 + c_1H_2)\,H_4 + H_2\,(g_1B_4 + b_1G_4) + c_1H_5H_6 = 0,$$
$$(g_3B_2 + h_3C_2 + b_3G_2)\,C_4 + C_2\,(g_3B_4 + b_3G_4) + h_3C_5\,C_6 = 0\,;$$

but in order that the values $(B_4,\ C_4,\ F_4,\ G_4,\ H_4)$ given by these five equations may belong to a line $(0,\ B_4,\ C_4,\ F_4,\ G_4,\ H_4)$, they must satisfy the equation

$$B_4G_4 + C_4H_4 = 0,$$

viz. in order to the existence of the line 4, this equation must be satisfied identically by the foregoing values; and I proceed to show that it is in fact thus satisfied. Multiplying the values of C_4, H_4, and writing $C_4H_4 = -B_4G_4$, the identity to be verified is

$$(g_1B_2 + b_1G_2 + c_1H_2)(g_3B_2 + h_3C_2 + b_3G_2)\,B_4G_4$$
$$+ [H_2\,(g_1B_4 + b_1G_4) + c_1H_5H_6]\,[C_2\,(g_3B_4 + b_3G_4) + h_3C_5C_6] = 0.$$

The first line includes the terms

$$\{g_1g_3B_2^2 + b_1b_3G_2^2 + (b_1g_3 + b_3g_1)\,B_2G_4 + c_1h_3C_2H_2\}\,B_4G_4,$$

which, writing $C_2H_2 = -B_2G_2$ and $B_2B_4 = B_5B_6$, $G_2G_4 = G_5G_6$, are

$$= g_1g_3B_2G_4B_5B_6 + b_1b_3G_2B_4G_5G_6 + (b_1g_3 + g_1b_3 - c_1h_3)\,B_5B_6G_5G_6.$$

The second line includes the terms

$$C_2H_2\,(g_1B_4 + b_1G_4)\,(g_3B_4 + b_3G_4) + c_1h_3C_5H_5C_6H_6,$$

which, reducing in like manner, are

$$= -g_1g_3G_2B_4B_5B_6 - b_1b_3B_2G_4G_5G_6 - (b_1g_3 + g_1b_3 - c_1h_3)\,B_5B_6G_5G_6,$$

and these are together

$$= (g_1g_3B_5B_6 - b_1b_3G_5G_6)\,(B_2G_4 - B_4G_2).$$

The remaining terms from the first line are at once reduced to

$$(g_1h_3C_2G_4 + c_1g_3H_2G_4)\,B_5B_6 + (b_1h_3C_2B_4 + c_1b_3H_2B_4)\,G_5G_6,$$

and those from the second line are

$$C_5C_6\,H_2\,(g_1h_3B_4 + b_1h_3G_4) + H_5H_6C_2\,(c_1g_3B_4 + c_1b_3G_4).$$

Hence, attending to the relation $c_1 = -h_3$, and collecting and arranging, the equation to be verified is

$$(g_1g_3B_5B_6 - b_1b_3G_5G_6)\,(B_2G_4 - B_4G_2)$$
$$+ h_3C_2\,G_4\,(g_1B_5B_6 - b_3H_5H_6)$$
$$+ h_3H_2B_4\,(g_1C_5C_6 - b_3G_5G_6)$$
$$+ h_3C_2\,B_4\,(b_1\,G_5G_6 - g_3H_5H_6)$$
$$+ h_3H_2G_4\,(b_1\,C_5C_6 - g_3B_5\,B_6) = 0.$$

But we have

$$g_1 B_5 B_6 - b_3 H_5 H_6 = B_5 B_6 (G_5 H_6 - G_6 H_5) - H_5 H_6 (B_5 C_6 - B_6 C_5)$$
$$= B_6 H_6 (B_5 G_5 + C_5 H_5) - B_5 H_5 (B_6 G_6 + C_6 H_6) = 0,$$

and similarly

$$g_1 C_5 C_6 - b_3 G_5 G_6 = 0,$$
$$b_1 G_5 G_6 - g_3 H_5 H_6 = 0,$$
$$b_1 C_5 C_6 - g_3 B_5 B_6 = 0.$$

Moreover, writing $g_1 B_5 B_6 = b_3 H_5 H_6$, we have

$$g_1 g_3 B_5 B_6 - b_1 b_3 G_5 G_6$$
$$= b_3 (g_3 H_5 H_6 - b_1 G_5 G_4) = 0,$$

and the five terms of the equation in question thus separately vanish; and the equation is consequently verified.

We may collect the results as follows:

Data are lines 1′, 2, 3′, 4, 2′, 5′, 6′;

and then, for the remaining lines, 1, 3, 5, 6, 4′ the coordinates are as follows:

For 4′,

$$B_2 B_4 = B_5 B_6, \quad G_2 G_4 = G_5 G_6, \quad A_2 F_4 = B_5 G_6 + B_6 G_5 - B_2 G_4 - B_4 G_2, \quad A_4 = 0,$$

$$\begin{vmatrix} B_2, & G_2, & H_2 \\ B_5, & G_5, & H_5 \\ B_6, & G_6, & H_6 \end{vmatrix} H_4 + \begin{vmatrix} H_2 B_4, & H_2 G_4, & H_5 H_6 \\ B_5, & G_5, & H_5 \\ B_6, & G_6, & H_6 \end{vmatrix} = 0,$$

$$\begin{vmatrix} B_2, & C_2, & G_2 \\ B_5, & C_5, & G_5 \\ B_6, & C_6, & G_6 \end{vmatrix} C_4 + \begin{vmatrix} C_2 B_4, & C_5 C_6, & C_2 G_4 \\ B_5, & C_5, & G_5 \\ B_6, & C_6, & G_6 \end{vmatrix} = 0,$$

$(B_4 G_4 + C_4 H_4 = 0$, identity).

For 1,

$$(g_1, b_1, c_1) = \begin{Vmatrix} B_5, & G_5, & H_5 \\ B_6, & G_6, & H_6 \end{Vmatrix}, \quad A_2 f_2 = -\begin{vmatrix} B_2, & G_2, & H_2 \\ B_5, & G_5, & H_5 \\ B_6, & G_6, & H_6 \end{vmatrix}, \quad F_4 a_1 = -\begin{vmatrix} B_4, & G_4, & H_4 \\ B_5, & G_5, & H_5 \\ B_6, & G_6, & H_6 \end{vmatrix}, \quad h_1 = 0,$$

$(a_1 f_1 + b_1 g_1 = 0$, identity).

For 3,

$$(g_3, h_3, b_3) = \begin{Vmatrix} B_5, & C_5, & G_5 \\ B_6, & C_6, & G_6 \end{Vmatrix}, \quad A_2 f_3 = -\begin{vmatrix} B_2, & C_2, & G_2 \\ B_5, & C_5, & G_5 \\ B_6, & C_6, & G_6 \end{vmatrix}, \quad F_4 a_3 = -\begin{vmatrix} B_4, & C_4, & G_4 \\ B_5, & C_5, & G_5 \\ B_6, & C_6, & G_6 \end{vmatrix}, \quad c_3 = 0,$$

$(a_3 f_3 + b_3 g_3 = 0$, identity).

For 5,

$$g_5 = G_6,\quad b_5 = -B_6,\quad A_2 f_5 = B_6 G_2 - B_2 G_6,\quad F_4 a_5 = B_6 G_4 - B_4 G_6,\quad c_5 = 0,\quad h_5 = 0,$$

$$(a_5 f_5 + b_5 g_5 = 0,\ \text{identity}).$$

For 6,

$$g_6 = G_5,\quad b_6 = -B_5,\quad A_2 f_6 = B_5 G_2 - B_2 G_5,\quad F_4 a_6 = B_5 G_4 - B_4 G_5,\quad c_6 = 0,\quad h_6 = 0;$$

and, for actual calculation, it is convenient to remark that as only the ratios are material, a set of six coordinates may be multiplied or divided by any common number at pleasure.

But these results may be further reduced. Writing

$$(g_3,\ h_3,\ b_3) = \left\| \begin{matrix} B_5, & C_5, & G_5 \\ B_6, & C_6, & G_6 \end{matrix} \right\|,$$

we have

$$-(g_3 B_2 + h_3 C_2 + b_3 G_2) C_4 = C_2 \left(g_3 \frac{B_5 B_6}{B_2} + b_3 \frac{G_5 G_6}{C_2} \right) + h_3 C_5 C_6 = \frac{C_2}{B_2 G_2} (g_3 B_5 B_6 G_2 + b_3 G_5 G_6 B_2) + h_3 C_5 C_6.$$

But

$$g_3 B_5 B_6 G_2 + b_3 G_5 G_6 B_2 = \left\{ \begin{matrix} B_5 B_6 G_2 (C_5 G_6 - C_6 G_5) \\ + G_5 G_6 B_2 (B_5 C_6 - C_5 B_6) \end{matrix} \right\} = \left\{ \begin{matrix} B_6 G_6 C_5 (B_5 G_2 - B_2 G_5) \\ + B_5 G_5 C_6 (G_6 B_2 - G_2 B_6) \end{matrix} \right\}$$

$$= C_5 C_6 \left\{ \begin{matrix} -H_6 (B_5 G_2 - B_2 G_5) \\ -H_5 (G_6 B_2 - G_2 B_6) \end{matrix} \right\}$$

$$= C_5 C_6 \{ B_2 (G_5 H_6 - G_6 H_5) + G_2 (B_6 H_5 - B_5 H_6) \}$$

$$= C_5 C_6 (B_2 g_1 + G_2 b_1),$$

since

$$(g_1,\ b_1,\ c_1) = \left\| \begin{matrix} B_5, & G_5, & H_5 \\ B_6, & G_6, & H_6 \end{matrix} \right\|;$$

the equation obtained is thus

$$g_3 B_4 + b_3 G_4 = \frac{C_5 C_6}{B_2 G_2} (g_1 B_2 + b_1 G_2).$$

We then have

$$-(g_3 B_2 + h_3 C_2 + b_3 G_2) C_4 = \frac{C_5 C_6}{B_2 G_2} [C_2 (g_1 B_2 + b_1 G_2) + h_3 B_2 G_2]$$

$$= \frac{-C_5 C_6}{C_2 H_2} [C_2 (g_1 B_2 + b_1 G_2) + c_1 C_2 H_2]$$

$$= -\frac{C_5 C_6}{H_2} (g_1 B_2 + b_1 G_2 + c_1 H_2),$$

that is

$$C_4 = \frac{C_5 C_6}{H_2} \frac{g_1 B_2 + b_1 G_2 + c_1 H_2}{g_3 B_2 + h_3 C_2 + b_3 G_2};$$

and in like manner

$$-(g_1B_2+b_1G_2+c_1H_2)\,H_4=\frac{H_2}{B_2G_2}(g_1G_2B_5B_6+b_1B_2G_5G_6)+c_1H_5H_6,$$

$$g_1G_2B_5B_6+b_1B_2G_5G_6=\left\{\begin{array}{l}G_2B_5B_6\,(G_5H_6-G_5H_5)\\+B_2G_5G_6\,(H_5B_6-H_6G_5)\end{array}\right\}=\left\{\begin{array}{l}B_5G_5H_6\,(G_2B_6-G_6B_2)\\+B_6G_6H_5\,(B_2G_5-B_5G_2)\end{array}\right\}$$

$$=H_5H_6\left\{\begin{array}{l}-C_5\,(G_2B_6-G_6B_2)\\-C_6\,(B_2G_5-B_5G_2)\end{array}\right\}$$

$$=H_5H_6\,\{B_2\,(C_5C_6=C_6G_5)+G_2\,(B_5C_6-B_6C_5)\}$$

$$=H_5H_3\,(B_2g_3+G_2b_3)\,;$$

the equation obtained is thus

$$g_1B_4+b_1G_4=\frac{H_5H_6}{B_2G_2}(B_2g_3+G_2b_3);$$

and then

$$-(g_1B_2+b_1G_2+c_1H_2)\,H_4=\frac{H_5H_6}{B_2G_2}\{H_2\,(B_2g_3+G_2b_3)+c_1B_2G_2\}$$

$$=-\frac{H_5H_6}{C_2H_2}\{H_2\,(B_2g_3+G_2b_3)+h_3C_2H_2\}$$

$$=-\frac{H_5H_6}{C_2}(B_2g_3+C_2h_3+G_2b_3),$$

that is

$$H_4=\frac{H_5H_6}{C_2}\,\frac{g_3B_2+h_3C_2+b_3G_2}{g_1B_2+b_1G_2+c_1H_2},$$

which values of C_4, H_4 satisfy, as they should do, the relation

$$C_4H_4=-B_4G_4.$$

We have also

$$A_2F_4=B_5G_6+B_6G_5-\frac{B_2G_5G_6}{G_2}-\frac{G_2B_5C_6}{B_2}$$

$$=-\frac{1}{B_2G_2}(G_2B_5-B_2G_5)(G_2B_6-B_2G_6)$$

$$=\frac{1}{C_2H_2}(G_2B_5-B_2G_5)(G_2B_6-B_2G_6),$$

which gives F_4.

Moreover

$$-F_4a_1=g_1B_4+b_1G_4+c_1H_4=-\frac{H_5H_6}{C_2H_2}(g_3B_2+b_2G_3)-\frac{h_3H_5H_6}{C_2}\,\frac{g_3B_2+b_3G_2+h_3C_2}{g_1B_2+b_1G_2+c_1H_2}$$

$$=\frac{-H_5H_6}{C_2H_2(g_1B_2+b_1G_2+c_1H_2)}\left\{\begin{array}{l}(g_3B_2+b_3G_2)\,(g_1B_2+b_1G_2+c_1H_2)\\+(g_3B_2+b_3G_2+h_3C_2)\,h_3H_2\end{array}\right\}$$

$$=\frac{-H_5H_6}{C_2H_2(g_1B_2+b_1G_2+c_1H_2)}\{(g_3B_2+b_3G_2)\,(g_1B_2+b_1G_2)+c_1h_3B_2G_2\},$$

which combined with the foregoing value of F_4, gives a_1.

Again,

$$-F_4a_3 = g_3B_4 + h_3C_4 + b_3G_4 = -\frac{C_5C_6}{C_2H_2}(g_1B_2 + b_1G_2) = \frac{c_1C_5C_6}{H_2}\,\frac{g_1B_2 + b_1G_2 + c_1H_2}{g_3B_2 + h_3C_2 + b_3G_2}$$

$$= -\frac{C_5C_6}{C_2H_2(g_3B_2 + h_3C_2 + b_3G_2)}\left\{\begin{array}{l}(g_1B_2 + b_1G_2)(g_3B_2 + h_3C_2 + b_3G_2)\\ +(g_1B_2 + b_1G_2 + c_1H_2)c_1C_2\end{array}\right\}$$

$$= \frac{-C_5C_6}{C_2H_2(g_3B_2 + h_3C_2 + b_3G_2)}\{(g_1B_2 + b_1G_2)(g_1B_2 + b_3G_2) + c_1h_3B_2C_2\},$$

which combined with the foregoing value of F_4, gives a_3.

Write

$$\omega = B_5G_6 + B_6G_5 + C_5H_6 + C_3H_5,$$

we have

$$\begin{aligned} b_1g_1 &= (H_5B_6 - H_6B_5)(G_5H_6 - G_6H_5) \\ &= -H_5^2B_6G_6 - H_6^2B_5G_5 + H_5H_6(B_5G_6 + B_6G_5) \\ &= H_5H_6(C_6H_5 + C_5H_6 + B_5G_6 + B_6G_5), \end{aligned}$$

that is

$$b_1g_1 = H_5H_3\omega,$$

and similarly

$$b_3g_3 = C_5C_6\,\omega,$$

$$b_1b_3 = B_5B_6\,\omega,$$

$$g_1g_3 = G_5G_6\,\omega,$$

$$b_1g_3 + b_3g_1 + c_1h_3 = -(B_5G_6 + B_3G_5)\,\omega,$$

$$\begin{aligned}(b_1G_2 + g_1B_2)(b_3G_2 + g_3B_2) + c_1h_2B_2G_2 &= \{G_2^2B_5B_6 + B_2^2G_5G_6 - B_2G_2(B_5G_6 + B_6G_5)\}\,\omega \\ &= (G_2B_5 - B_2G_5)(G_2B_6 - B_2G_6)\,\omega,\end{aligned}$$

which last value is to be substituted for the left-hand function in the formulæ for a_1 and a_3 respectively.

Whence, finally recollecting that

$$B_2G_2 + C_2H_2 = 0,\quad B_5G_5 + C_5H_5 = 0,\quad B_6G_6 + C_6H_6 = 0,$$

and

$$\omega = C_6H_5 + C_5H_6 + B_6G_5 + B_5G_6,$$

we have

$$\text{For 1}\left\{\begin{array}{ll}(g_1,\ b_1,\ c_1) = \left\Vert\begin{array}{ccc}B_5, & G_5, & H_5\\ B_6, & G_6, & H_6\end{array}\right\Vert, & f_1 = -\dfrac{1}{A_2}(g_1B_2 + b_1G_2 + c_1H_2),\\[2ex] b_1g_1 = H_5H_6\omega. & a_1 = \dfrac{A_2H_5H_6\omega}{g_1B_2 + h_1G_2 + c_1H_2},\quad h_1 = 0.\end{array}\right.$$

For 3
$$\left\{\begin{array}{ll} (g_3, h_3, b_3) = \left\| \begin{array}{ccc} B_5, & C_5, & G_6 \\ B_6, & C_6, & G_6 \end{array} \right\|, & f_3 = -\dfrac{1}{A_2}(g_3B_2 + h_3C_2 + b_3G_2), \\ b_3g_3 = C_5C_6\omega. & a_3 = \dfrac{A_2C_5C_6\omega}{g_3B_2 + h_3C_2 + b_3G_2}, \quad h_3 = 0, \end{array}\right.$$

where observe that $c_1 + h_3 = 0$.

For 5
$$\left\{\begin{array}{ll} g_5 = G_6, \quad b_5 = -B_6, & f_5 = \dfrac{1}{A_2}(G_2B_6 - B_2G_6), \\ c_5 = 0\ , \quad h_5 = \quad 0, & a_5 = \dfrac{A_2B_6G_6}{G_2B_6 - B_2G_6}. \end{array}\right.$$

For 6
$$\left\{\begin{array}{ll} g_6 = G_5, \quad b_6 = -B_5, & f_6 = \dfrac{1}{A_2}(G_2B_5 - B_2G_5), \\ c_6 = 0\ , \quad h_6 = \quad 0, & a_6 = \dfrac{A_2B_5G_5}{G_2B_5 - B_2G_5}. \end{array}\right.$$

For 4′
$$\left\{\begin{array}{ll} A_4 = 0, & B_4 = \dfrac{B_5B_6}{B_2}, \\ F_4 = \dfrac{(G_2B_5 - G_5B_2)(G_2B_6 - G_6B_2)}{A_2C_2H_2}, & G_4 = \dfrac{G_5G_6}{G_2}, \\ C_4 = \dfrac{C_5C_6}{H_2}\ \dfrac{g_1B_2 + b_1G_2 + c_1H_2}{g_3B_2 + h_3C_2 + b_3G_2}, & \\ H_4 = \dfrac{H_5H_6}{C_2}\ \dfrac{g_3B_2 + h_3C_2 + b_3G_2}{g_1B_2 + b_1G_2 + c_1H_2}. & \end{array}\right.$$

I have thought it worth while to effect the numerical calculations for enabling the construction of a drawing or model. For this purpose taking X, Y, Z as ordinary rectangular coordinates, I write

$$\begin{aligned} x &= \quad X + Y + Z - 10, \\ y &= \qquad\qquad\ \ Z, \\ z &= -X + Y + Z - 10, \\ w &= \qquad\quad Y, \end{aligned}$$

that is, I take 1′ and 2 to be lines in the plane of XY, defined by the equations $X + Y = 10$ and $X - Y = -10$ respectively, and 3′ and 4 to be lines in the plane of XZ defined by the equations $X - Z = -10$, $X + Z = 10$ respectively. And I take 5′ to be the line joining the points (2, 0, 8) and (−9, 1, 0); 6′ the line joining the points (3, 0, 7) and (−8, 2, 0); 2′ the line joining the points (9, 0, 1) and (−3, 7, 4). We calculate then for each of the lines the *xyzw* coordinates of two points thereof; and thence the six coordinates of the line, viz.:

	x y z w	x y z w	A B C F G H	reduced A B C F G H
5′	0, 8, − 4, 0	− 18, 0, 0, 1	0, 72, 144, 0, 8, − 4	0, 18, 36, 0, 2, − 1
6′	0, 7, − 6, 0	− 16, 0, 0, 2	0, 96, 112, 0, 14, − 12	0, 48, 56, 0, 7, − 6
2′	0, 1, − 18, 0	− 2, 4, 4, 7	76, 36, 2, 0, 7, −126	76, 36, 2, 0, 7, −126

and effecting the calculations for the remaining lines, we have

	A	B	C	F	G	H
4′	0	24	− 944	$-\frac{9}{38}$	2	$\frac{3}{59}$
1	$\frac{380}{59}$	60	30	$\frac{385}{19}$	− 5	0
3	127680	− 720	0	$\frac{15}{19}$	140	− 30
5	304	− 48	0	$\frac{21}{19}$	7	0
6	$\frac{152}{3}$	− 18	0	$\frac{27}{38}$	0	0
	a	*b*	*c*	*f*	*g*	*h*

or reducing to integers, the values are

	A	B	C	F	G	H
5′	0	18	36	0	2	− 1
6′	0	48	56	0	7	− 6
2′	76	36	2	0	7	− 126
4′	0	53808	− 2116448	− 531	4484	114
1	1444	13452	6726	10443	− 1121	0
3	485184	− 2736	0	3	532	− 114
5	5776	− 912	0	21	133	0
6	5776	− 2052	0	81	228	0
	a	*b*	*c*	*f*	*g*	*h*

The line (a, b, c, f, g, h) is given as the intersection of any two of the four planes

$$\left(\begin{array}{cccc} \cdot & h, & -g, & a \\ -h, & \cdot & f, & b \\ g, & -f, & \cdot & c \\ -a, & -b, & -c, & \cdot \end{array}\right|\!\left(x, y, z, w\right)=0,$$

or substituting for x, y, z, w the values $X+Y+Z-10,\ Z,\ -X+Y+Z-10,\ Y$, these become

$$\left(\begin{array}{cccc} g\ , & a-g\ , & h-g\ , & g \\ -f-h, & b+f-h, & f-h\ , & h-g \\ g\ , & c+g\ , & g-f\ , & -g \\ c-a, & -c-a\ , & -c-a-b, & c+a \end{array}\right|\!\left(X, Y, Z, 10\right)=0,$$

or, what is the same thing,

$$\left(\begin{array}{cccc} \cdot\ , & 2g-a+c\ , & 2g-f-h\ , & -2g \\ -2g+a-c, & \cdot\ , & a+b+c+f-h, & -a-c \\ -2g+f+h, & -a-b-c-f+h, & \cdot\ , & -f+h \\ 2g\ , & a+c\ , & -f+h\ , & \cdot \end{array}\right|\!\left(X, Y, Z, 10\right)=0.$$

And substituting, we have the equations of the several lines, viz.:

(1′) $X+Y=10,\ Z=0,$

(2) $-X+Y=10,\ Z=0,$

(3′) $-X+Z=10,\ Y=0,$

(4) $X+Z=10,\ Y=0,$

(5′) $$\left(\begin{array}{cccc} \cdot\ , & 40, & 5, & -4 \\ -40, & \cdot\ , & 55, & -36 \\ -5, & -55, & \cdot\ , & 1 \\ 4, & 36, & -1, & \cdot \end{array}\right|\!\left(X, Y, Z, 10\right)=0,$$

(6′) $$\left(\begin{array}{cccc} \cdot\ , & -35, & 10, & -7 \\ -35, & \cdot\ , & 55, & -28 \\ -10, & -55, & \cdot\ , & 3 \\ 7, & 28, & -3, & \cdot \end{array}\right|\!\left(X, Y, Z, 10\right)=0,$$

(2′)
$$\left(\begin{array}{cccc} . , & -30, & 70, & -7 \\ 30, & . , & 120, & -39 \\ -70, & -120, & . , & 63 \\ 7, & 39, & -63, & . \end{array}\right)(X, Y, Z, 10) = 0,$$

(4′)
$$\left(\begin{array}{cccc} . , & -2107480, & 9385, & -8968 \\ 2107480, & . , & -2063285, & 2116448 \\ -9385, & 2063285, & . , & -645 \\ 8968, & -2116448, & 645, & . \end{array}\right)(X, Y, Z, 10) = 0,$$

(1)
$$\left(\begin{array}{cccc} . , & 3040, & -12685, & 2242 \\ -3040, & . , & 32065, & -8170 \\ 12685, & -32065, & . , & 10443 \\ -2242, & 8170, & -10443, & . \end{array}\right)(X, Y, Z, 10) = 0,$$

(3)
$$\left(\begin{array}{cccc} . , & -484120, & 1175, & -1064 \\ 484120, & . , & 482565, & -485184 \\ -1175, & -482565, & . , & 117 \\ 1064, & 485184, & -117, & . \end{array}\right)(X, Y, Z, 10) = 0,$$

(5)
$$\left(\begin{array}{cccc} . , & 5510, & 245, & -266 \\ -5510, & . , & 4885, & -5776 \\ -245, & -4885, & . , & 21 \\ 266, & 5776, & -21, & . \end{array}\right)(X, Y, Z, 10) = 0,$$

(6)
$$\left(\begin{array}{cccc} . , & -5320, & 375, & -456 \\ 5320, & . , & 3805, & -5776 \\ -3751, & -3805, & . , & 81 \\ 456, & 5776, & -81, & . \end{array}\right)(X, Y, Z, 10) = 0.$$

The several lines intersect as they should do, the coordinates of the points of intersection being as follows:

	1′	2′	3′	4′	5′	6′
1	*	621 1239 ÷ 305 836	−354 0 ÷ 43 76	42417 −177 ÷ 5098 8968	−493 59 ÷ 78 152	−1188 354 ÷ 253 532
2	0 10 0	*	−10 0 0	−472 −2 ÷ 47 0	−9 1 0	−8 2 0
3	912 −2 ÷ 91 0	9693 −21 ÷ 1073 1064	*	−2484 66 ÷ 24727 251464	807 −1 ÷ 398 3192	3672 −6 ÷ 1213 8512
4	10 0 0	9 0 1	0 0 10	*	2 0 8	3 0 7
5	304 −14 ÷ 29 0	459 −21 ÷ 47 38	6 0 ÷ 7 76	4752 42 ÷ 6521 71744	*	1080 −42 ÷ 283 2128
6	76 −6 ÷ 7 0	2421 −189 ÷ 233 152	54 0 ÷ 25 304	1479 9 ÷ 727 8968	65 −3 ÷ 16 154	*

viz. the coordinates of 12′ (intersection of lines 1 and 2′) are $(\frac{621}{305}, \frac{1239}{305}, \frac{836}{305})$, and so in other cases; where there is no divisor the coordinates are integers. I find however, on laying down the figure, that the lines 3 and 4, 3′ and 4′ come so close together, that the figure cannot be obtained with any accuracy.