# 461.

## ON THE GEOMETRICAL INTERPRETATION OF THE COVARIANTS OF A BINARY CUBIC.

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CONSIDER the binary cubic $U = (a, b, c, d \between x, y)^3$, and its covariants, viz. the discriminant (invariant)

$$\nabla = a^2d^2 - 6abcd + 4ac^3 + 4b^3d - 3b^2c^2,$$

the Hessian

$$H = (ac - b^2)\, x^2 + (ad - bc)\, xy + (bd - c^2)\, y^2,$$

and the cubicovariant

$$\begin{aligned} \Phi = &\ (\quad a^2d - 3abc + 2b^3\ )\, x^3 \\ &- (-3abd + 6ac^2 - 3b^2c)\, x^2y \\ &+ (-3acd + 6bd^2 - 3bc^2)\, xy^2 \\ &- (\quad ad^2 - 3bcd + 2c^3\ )\, y^3, \end{aligned}$$

connected by the identical equation

$$\Phi^2 - \nabla U^2 = -4H^3.$$

Then if we regard $(a, b, c, d)$ as the coordinates of a point in space, but $(x, y)$ as variable parameters, the equation

$$\nabla = 0$$

represents a quartic torse, having for its cuspidal curve the skew cubic $ac - b^2 = 0$, $ad - bc = 0$, $bd - c^2 = 0$; the equation

$$U = 0$$

is that of the tangent plane to the torse along the line $ax^2+2bxy+cy^2=0$, $bx^2+2cxy+dy^2=0$: this line meets the cuspidal curve in the point whose coordinates are $a:b:c:d=y^3:-xy^2:x^2y:-y^3$. The equation

$$H=0$$

is that of a quadric cone having the last mentioned point for its vertex, and passing through the cuspidal curve: and the equation

$$\Phi=0$$

is that of the cubic surface which is the first polar of the same point in regard to the torse.

The equation $\Phi^2-\nabla U^2=-4H^3$, writing therein $U=0$, gives $\Phi^2=-4H^3$, a result which implies that $U=0$, $H=0$ is a certain curve repeated twice, and that $U=0$, $\Phi=0$ is the same curve repeated three times. The curve in question is at once seen to be the line of contact $\delta_x U=0$, $\delta_y U=0$; it thus appears that the tangent plane $U=0$ meets the cubic surface $\Phi=0$ in this line taken three times. This can only be the case if the equation $\Phi=0$ be expressible in the form $MU+(\delta_x U)^3=0$, or, what is the same thing,

$$MU+(\alpha\delta_x U+\beta\delta_y U)^3=0,$$

$\alpha$ and $\beta$ constants, $M$ a quadric function of $(a, b, c, d)$; that is, $\Phi$ must be equal to a function of the form

$$MU+(\alpha\delta_x U+\beta\delta_y U)^2.$$

Seeking for this expression of $\Phi$, and writing the symbols out at length, I find that the required identical equation is

$$-(\beta x-\alpha y)^3\left\{\begin{array}{l}\phantom{-}(\phantom{-3abd+}a^2d-3abc+2b^3)\,x^3\\ -(-3abd+6ac^2-3b^2c)\,x^2y\\ +(-3acd+6bd^2-3bc^2)\,xy^2\\ -(-\phantom{3acd+}ad^2-3bcd+2c^3)\,y^3\end{array}\right\}+2\{\alpha(ax^2+2bxy+cy^2)+\beta(bx^2+2cxy+dy^2)\}^3=$$

$$(a, b, c, d\between x, y)^3\,.\left(\begin{array}{llll}2a^2, & 6ab, & 6b^2, & -ad+3bc\\ 6ab, & 12ac+6b^2, & 3ad+15bc, & 6c^2\\ 6b^2, & 3ad+15bc, & 12bd+6c^2, & 6cd\\ -ad+3bc, & 6c^2, & 6cd, & 2d^2\end{array}\right)^{\dagger}(x, y)^3(\alpha, \beta)^3,$$

(where the † indicates that the binomial coefficients are *not* to be inserted, viz. the function on the right hand is $\{2a^2x^3+6abx^2y+6b^2xy^3+(-ad+3bc)\,y^3\}\,\alpha^3+\&c.$). As a verification remark that for $x=\alpha$, $y=\beta$, the equation becomes simply $2U^3=U\,.\,2U^2$.