462.

A NINTH MEMOIR ON QUANTICS.

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It was shown not long ago by Professor Gordan that the number of the irreducible covariants of a binary quantic of any order is finite (see his memoir "Beweis dass jede Covariante und Invariante einer binären Form eine ganze Function mit numerischen Coefficienten einer endlichen Anzahl solcher Formen ist," *Crelle*, t. LXIX. (1869), Memoir dated 8 June 1868), and in particular that for a binary quintic the number of irreducible covariants (including the quintic and the invariants) is $=23$, and that for a binary sextic the number is $=26$. From the theory given in my "Second Memoir on Quantics," *Phil. Trans.*, 1856, [141], I derived the conclusion, which, as it now appears, was erroneous, that for a binary quintic the number of irreducible covariants was infinite. The theory requires, in fact, a modification, by reason that certain linear relations, which I had assumed to be independent, are really not independent, but, on the contrary, linearly connected together: the interconnexion in question does not occur in regard to the quadric, cubic, or quartic; and for these cases respectively the theory is true as it stands; for the quintic the interconnexion first presents itself in regard to the degree 8 in the coefficients and order 14 in the variables, viz. the theory gives correctly the number of covariants of any degree not exceeding 7, and also those of the degree 8 and order less than 14; but for the order 14 the theory as it stands gives a non-existent irreducible covariant $(a, ..)^8 (x, y)^{14}$, viz. we have, according to the theory, $5=(10-6)+1$, that is, of the form in question there are 10 composite covariants connected by 6 syzygies, and therefore equivalent to $10-6$, $=4$ asyzygetic covariants; but the number of asyzygetic covariants being $=5$, there is left, according to the theory, 1 irreducible covariant of the form in question. The fact is that the 6 syzygies being interconnected and equivalent to 5 independent syzygies only, the composite covariants are equivalent to $10-5$, $=5$, the full number of the asyzygetic covariants. And similarly the theory as it stands gives a non-existent irreducible covariant $(a,..)^8(x, y)^{20}$. The theory being thus in error, by reason that it omits to take account of the interconnexion of the syzygies, there is no difficulty in conceiving that the effect is the introduction of an infinite series of non-existent irreducible covariants, which, when the error is corrected, will disappear, and there will be left only a finite series of irreducible covariants.

Although I am not able to make this correction in a general manner so as to show from the theory that the number of the irreducible covariants is finite, and so to present the theory in a complete form, it nevertheless appears that the theory can be made to accord with the facts; and I reproduce the theory, as well to show that this is so as to exhibit certain new formulæ which appear to me to place the theory in its true light. I remark that although I have in my Second Memoir considered the question of finding the number of irreducible covariants of a given degree θ in the coefficients but of any order whatever in the variables, the better course is to separate these according to their order in the variables, and so consider the question of finding the number of the irreducible covariants of a given degree θ in the coefficients, and of a given order μ in the variables. (This is, of course, what has to be done for the enumeration of the irreducible covariants of a given quantic; and what is done completely for the quadric, the cubic, and the quartic, and for the quintic up to the degree 6 in my Eighth Memoir, *Phil. Trans.* 1867, [405].) The new formulæ exhibit this separation; thus (Second Memoir, No. 49), writing a instead of x, we have for the quadric the expression $\dfrac{1}{(1-a)(1-a^2)}$, showing that we have irreducible covariants of the degrees 1 and 2 respectively, viz. the quadric itself and the discriminant: the new expression is $\dfrac{1}{(1-ax^2)(1-a^2)}$, showing that the covariants in question are of the actual forms $(a,..\between x, y)^2$ and $(a,..)^2$ respectively. Similarly for the cubic, instead of the expression No. 55, $\dfrac{1-a^6}{(1-a)(1-a^2)(1-a^3)(1-a^4)}$, we have $\dfrac{1-a^6x^6}{(1-ax^3)(1-a^2x^2)(1-a^3x^3)(1-a^4)}$, exhibiting the irreducible covariants of the forms $(a,..\between x, y)^3$, $(a,..)^2(x, y)^2$, $(a..)^3(x, y)^3$, and $(a,..)^4$, connected by a syzygy of the form $(a,..)^6(x, y)^6$; and the like for quantics of a higher order.

In the present Ninth Memoir I give the last-mentioned formulæ; I carry on the theory of the quintic, extending the Table No. 82 of the Eighth Memoir up to the degree 8, calculating all the syzygies, and thus establishing the interconnexions in virtue of which it appears that there are really no irreducible covariants of the forms $(a,..)^8(x, y)^{14}$, and $(a,..^8\between x, y)^{20}$. I reproduce in part Gordan's theory so far as it applies to the quintic, and I give the expressions of such of the 23 covariants as are not given in my former memoirs; these last were calculated for me by Mr W. Barrett Davis, by the aid of a grant from the Donation Fund at the disposal of the Royal Society. [The expressions referred to are in fact printed, 143.] The paragraphs of the present memoir are numbered consecutively with those of the former memoirs on Quantics.

Article Nos. 328 to 332. *Reproduction of my original Theory as to the Number of the Irreducible Covariants.*

328. I reproduce to some extent the considerations by which, in my Second Memoir on Quantics, I endeavoured to obtain the number of the irreducible covariants of a given binary quantic $(a, b, \dots \between x, y)^n$.

Considering in the first instance the covariants as functions of the coefficients $(a, b, c, \dots)$, without regarding the variables (x, y), and attending only to the following properties—1°, a covariant is a rational and integral homogeneous function of the coefficients; 2°, if $P, Q, R, \dots$ are covariants, any rational and integral function $F(P, Q, R, \dots)$, homogeneous in regard to the coefficients, is also a covariant,—we say that the covariants $X, Y, \dots$ of the same degree in regard to the coefficients, and not connected by any identical equation $\alpha X + \beta Y \dots = 0$ (where $\alpha, \beta, \dots$ are quantities independent of the coefficients $(a, b, c, \dots)$), are *asyzygetic* covariants, and that a covariant not expressible as a rational and integral function of covariants of lower degrees is an *irreducible* covariant; and it is assumed that we know the number of the asyzygetic covariants of the degrees 1, 2, 3, ...; say, these are $A_1, A_2, A_3, \dots$, or, what is the same thing, that the number of the asyzygetic covariants of the degree θ, or form $(a, b, \dots)^\theta$, is equal to the coefficient of a^θ in a given function

$$\phi(a) = 1 + A_1 a + A_2 a^2 \dots + A_\theta a^\theta + \dots,$$

where I have purposely written a, as a representative of the coefficients $(a, b, c, \dots)$, in place of the x of my Second Memoir.

329. The theory was, that determining $\alpha_1, \alpha_2, \dots$ by the conditions

$$A_1 = \alpha_1,$$
$$A_2 = \tfrac{1}{2}\alpha_1(\alpha_1 + 1) + \alpha_2,$$
$$A_3 = \tfrac{1}{6}\alpha_1(\alpha_1 + 1)(\alpha_1 + 2) + \alpha_1\alpha_2 + \alpha_3,$$
$$\vdots$$

that is, throwing

$$1 + A_1 a + A_2 a^2 + A_3 a^3 + \dots$$

into the form

$$(1-a)^{-\alpha_1}(1-a^2)^{-\alpha_2}(1-a^3)^{-\alpha_3}\dots,$$

the index α_r would express the number of irreducible covariants of the degree r *less* the number of the (irreducible) linear relations, or syzygies, between the composite or non-irreducible covariants of the same degree. Thus $A_1 = \alpha_1$, there would be α_1 covariants of the degree 1[1]; these give rise to $\frac{1}{2}\alpha_1(\alpha_1 + 1)$ composite covariants of the degree 2; or, assuming that these are connected by k_2 syzygies, the number of asyzygetic composite covariants of the degree 2 would be $\frac{1}{2}\alpha_1(\alpha_1 + 1) - k_2$; and thence there would be $A_2 - \frac{1}{2}\alpha_1(\alpha_1 + 1) + k_2$, that is, $\alpha_2 + k_2$ irreducible covariants of the same degree; so that (irreducible invariants *less* syzygies) $(\alpha_2 + k_2) - k_2$ is $= \alpha_2$.

[1] For the case of covariants, α_1 is of course $= 1$; but in the investigation the term covariant properly stands for any function satisfying the conditions 1° and 2°.

330. The k_2 syzygies are here irreducible syzygies; for, calling $P, Q, R, \ldots$ the covariants of the degree 1, there is no identical relations between the terms P^2, Q^2, $PQ, \ldots$: imagine for a moment that we could have l_2 such identical relations (viz. this might very well be the case if instead of the $\frac{1}{2}\alpha_1(\alpha_1+1)$ functions P^2, Q^2, $PQ, \ldots$, we were dealing with the same number of other quadric functions of these quantities), that is, relations not establishing any relation between P^2, Q^2, $PQ, \ldots$, and besides these k_2 non-identical relations as above; then the number of irreducible invariants would be $\alpha_2 + k_2 + l_2$, and the number of irreducible syzygies being as before k_2, the difference would be not α_2 but $\alpha_2 + l_2$. The l_2 identical relations are here relations between composite covariants, and the effect (if any such relation could subsist) would, it appears, be to increase α_2; between syzygies such identical relations do actually exist, and the effect is contrariwise to diminish the α; we may, for instance, for the degree s have irreducible covariants *less* irreducible syzygies $= \alpha_s - l_s$.

331. Assume for a moment that, for a given value of s, α_s is positive; but for the term l_s it would of course follow that there was for the degree in question a certain number of irreducible covariants; and it was in this manner that I was led to infer that the number of the covariants of a quintic was infinite—viz. the transformed expression for the number of asyzygetic covariants is

$$= \text{coeff. } a^\theta \text{ in } (1-a^4)^{-1}(1-a^8)^{-3}(1-a^{12})^{-6}(1-a^{14})^{-4}\ldots,$$

a product which does not terminate, and as to which it is also assumed that the series of negative indices does not terminate.

332. The principle is the same, but the discussion as to the number of the irreducible covariants becomes more precise, if we attend to the covariants as involving not only the coefficients $(a, b, \ldots)$ but also the variables (x, y); we have then to consider the covariants of the form $(a, b, \ldots)^\theta (x, y)^\mu$, or, say, of the form $a^\theta x^\mu$ (degree θ and order μ), and the number of the asyzygetic covariants of this form is given as the coefficient of $a^\theta x^\mu$ in a given function of (a, x), (I write a instead of the z of my Second Memoir in the formulæ which contain x and z): by taking account of the composite covariants and syzygies, we successively determine, from the given number of asyzygetic covariants for each value of θ and μ, the number of the irreducible covariants for the same values of θ and μ. This is, in fact, done for the quintic in my Eighth Memoir up to the covariants and syzygies of the degree 6. But before resuming the discussion for the quintic, I will consider the preceding cases of the quadric, the cubic, and the quartic.

Article Nos. 333 to 336. *New formulæ for the number of Asyzygetic Covariants.*

333. For the quadric $(a, b, c \between x, y)^2$, the number of asyzygetic covariants $a^\theta x^\mu$

$$= \text{coeff. } a^\theta x^{\theta - \frac{1}{2}\mu} \text{ in } \frac{1-x}{(1-a)(1-ax)(1-ax^2)},$$

(see Second Memoir, No. 35, observing that q is there $= \theta - \frac{1}{2}\mu$, and that the subtraction of successive coefficients is effected by means of the factor $1-x$ in the numerator. See also Eighth Memoir, No. 251, where a like form is used for the quintic). Writing ax^2 for a, and $\frac{1}{x^2}$ for x, this is

$$= \text{coeff. } a^\theta x^\mu \text{ in } \frac{1-\frac{1}{x^2}}{(1-ax^2)(1-a)\left(1-\frac{a}{x^2}\right)}.$$

The development is

$$\begin{array}{l|l} 1 & -\frac{1}{x^2} \\ +ax^2 & +a\left(\frac{1}{x^2}\right) \\ +a^2(x^4+1) & +a^2\left(\frac{1}{x^4}+1\right) \\ +a^3(x^6+x^2) & +a^3\left(\frac{1}{x^6}+\frac{1}{x^2}\right) \\ +a^4(x^8+x^4+1) & +a^4\left(\frac{1}{x^8}+\frac{1}{x^4}+1\right) \\ \vdots & \vdots \end{array}$$

which is

$$= A(x) - \frac{1}{x^2} A\left(\frac{1}{x}\right),$$

where

$$A(x) = \frac{1}{(1-ax^2)(1-a^2)};$$

and, since $\frac{1}{x^2} A\left(\frac{1}{x}\right)$ contains only negative powers, the required number is

$$= \text{coeff. } a^\theta x^\mu \text{ in } \frac{1}{(1-ax^2)(1-a^2)},$$

indicating that the covariants are powers and products of (ax^2 and a^2), the quadric itself, and the discriminant. Compare Second Memoir, No. 49, according to which, writing therein a for x, the number of asyzygetic covariants is

$$= \text{coeff. } a^\theta \text{ in } \frac{1}{(1-a)(1-a^2)}.$$

334. For the cubic $(a, b, c, d)(x, y)^3$ the number of asyzygetic covariants $a^\theta x^\mu$ is

$$= \text{coeff. } a^\theta x^{\theta - \frac{1}{2}\mu} \text{ in } \frac{1-x}{(1-a)(1-ax)(1-ax^2)(1-ax^3)};$$

or transforming as before, this is

$$= \text{coeff. } a^\theta x^\mu \text{ in } \frac{1-\frac{1}{x^2}}{(1-ax^3)(1-ax)(1-ax^{-1})(1-ax^{-3})}:$$

the function is here

$$= A(x) - \frac{1}{x^2} A\left(\frac{1}{x}\right),$$

where

$$A(x) = \frac{1 - a^6x^6}{(1 - ax^3)(1 - a^2x^2)(1 - a^3x^3)(1 - a^4)}$$

(that this is so may be easily verified); and since the second term contains only negative powers, the required number is $=$ coeff. $a^\theta x^\mu$ in $A(x)$. The formula, in fact, indicates that the covariants are made up of $(ax^3, a^2x^2, a^3x^3, a^4)$, the cubic itself, the Hessian, the cubicovariant, and the discriminant, these being connected by a syzygy (a^6x^6) of the degree 6 and order 6. Compare Second Memoir, No. 50, according to which the number of covariants of degree θ is

$$= \text{coeff. } a^\theta \text{ in } \frac{1 - a^6}{(1 - a)(1 - a^2)(1 - a^3)(1 - a^4)}.$$

335. For the quartic $(a, b, c, d, e \between x, y)^4$ the number of asyzygetic covariants $a^\theta x^\mu$ is

$$= \text{coeff. } a^\theta x^{\theta - \frac{1}{2}\mu} \text{ in } \frac{1 - x}{(1 - a)(1 - ax)(1 - ax^2)(1 - ax^3)(1 - ax^4)};$$

or transforming as before, this is

$$= \text{coeff. } a^\theta x^\mu \text{ in } \frac{1 - x^{-2}}{(1 - ax^4)(1 - ax^2)(1 - a)(1 - ax^{-2})(1 - ax^{-4})}:$$

the function is here

$$= A(x) - \frac{1}{x^2} A\left(\frac{1}{x}\right),$$

where

$$A(x) = \frac{1 - a^6x^{12}}{(1 - ax^4)(1 - a^2x^4)(1 - a^2)(1 - a^3)(1 - a^3x^6)};$$

and the second term containing only negative powers, the required number is $=$ coeff. $a^\theta x^\mu$ in $A(x)$. The formula indicates that the covariants are made up of $(ax^4, a^2x^4, a^2, a^3, a^3x^6)$, the quartic itself, the Hessian, the quadrinvariant, the cubinvariant, and the cubicovariant, these being connected by a syzygy (a^6x^{12}) of the degree 6 and order 12. Compare Second Memoir, No. 51, according to which the number of covariants of degree θ is

$$= \text{coeff. } a^\theta \text{ in } \frac{1 - a^6}{(1 - a)(1 - a^2)^2(1 - a^3)^2}.$$

336. For the quintic $(a, b, c, d, e, f \between x, y)^5$ the number of asyzygetic covariants $a^\theta x^\mu$ is

$$= \text{coeff. } a^\theta x^{\theta - \frac{1}{2}\mu} \text{ in } \frac{1 - x}{(1 - a)(1 - ax)(1 - ax^2)(1 - ax^3)(1 - ax^4)(1 - ax^5)};$$

or transforming as before, this is

$$= \text{coeff. } a^\theta a^\mu \text{ in } \frac{1-x^{-2}}{(1-ax^5)(1-ax^3)(1-ax)(1-ax^{-1})(1-ax^{-3})(1-ax^{-5})}.$$

The developed expression is

$$\begin{array}{lc|l} 1 & -\dfrac{1}{x^2} & 1 \\ +ax^5 & & +ax^{-5} \\ +a^2(x^{10}+x^6+x^2) & & +a^2(x^{-10}+x^{-6}+x^{-2}) \\ \vdots & & \vdots \end{array}$$

but here there is not any *finite* function $A(x)$ such that this development is

$$= A(x) \qquad -\frac{1}{x^2} A\left(\frac{1}{x}\right).$$

The numerical coefficients are of course the same as those in the development of the untransformed function; viz. they are the numbers given in the third column of Table No. 82 (Eighth Memoir), and also (carried further) in the third column of the following Table, No. 87. And we can, from the discussion of these coefficients, deduce the form of $A(x)$, viz. this is

$$= \frac{\begin{array}{c|c|c|c|c} 1-a^5x^{11} & 1-a^6x^{18} & 1-a^7x^{15} & (1-a^8x^{12})^3 & \dots \\ & 14 & 13 & (10)^3 & \\ & 12 & 11 & (\ 8)^2 & \\ & 10 & (9)^3 & (\ 6)^2 & \\ & 8 & 7 & & \\ & 6 & & & \end{array}}{\begin{array}{c|c|c|c|c|c|c|c|c|c} 1-ax^5 & 1-a^2x^6 & 1-a^3x^9 & 1-a^4x^6 & 1-a^5x^7 & 1-a^6x^4 & 1-a^7x^3 & 1-a^8x^3 & 1-a^{18} & \dots \\ & 2 & 5 & 4 & 3 & 2 & 1 & 0 & & \\ & & 3 & 0 & 1 & & & 20 & & \\ & & & & & & & 14 & & \end{array}}$$

where, for shortness, I have written $\underset{2}{1-a^2x^6}$ to stand for $(1-a^2x^6)(1-a^2x^2)$, and so in other cases: moreover in the third column of the numerator the $(9)^3$ shows that the factor is $(1-a^7x^9)^3$, and so in other cases: this will be further explained presently. Compare herewith the form, Second Memoir, No. 52, viz. the number of asyzygetic covariants of the degree θ is

$$= \text{coeff. } a^\theta \text{ in } (1-a)^{-1}(1-a^2)^{-2}(1-a^3)^{-3}(1-a^4)^{-3}(1-a^5)^{-2}(1-a^6)^4(1-a^7)^5(1-a^8)^6 \dots$$

each index being, it will be observed, equal to the number of factors in the numerator, less the number of factors in the denominator, in the corresponding column of the new formula.

Article Nos. 337 to 346. *The 23 Fundamental Covariants.*

337. Gordan's result is that the entire number of the irreducible covariants of the binary quintic is $=23$. I represent these by the letters $A, B, C, \ldots, W$, identifying such of them as were given in my former Memoirs on Quantics with the Tables of these Memoirs, and the new ones, O, P, R, S, T, V, with the Tables Nos. 90, 91, 92, 93, 94, 95 of the present Memoir.

Table No. 87. Identification of the 23 irreducible covariants of the binary quintic.

			Table No.
A	$(a, b, c, d, e, f \between x, y)^5$	f	13
$B = \frac{1}{28000}(A, A)^4$	$(\quad)^2(\quad)^2$	$\iota = (ff)^4$	14
$C = \frac{1}{800}(A, A)^2$	$(\quad)^2(\quad)^6$	$\phi = (ff)^2$	15
$D = -\frac{1}{3}(A, B)^2$	$(\quad)^3(\quad)^3$	$j = (f\iota)^2$	16
$E = \frac{1}{5}(A, B)$	$(\quad)^3(\quad)^5$	$(f\iota)$	17
$F = \frac{1}{15}(A, C)$	$(\quad)^3(\quad)^9$	$(f\phi)$	18
$G = -\frac{1}{2}(B, B)^2$	$(\quad)^4(\quad)^0$	$(\iota\,\iota)^2$	19
$H = -\frac{1}{5}(B, C)^2 + \frac{2}{5}B^2$	$(\quad)^4(\quad)^4$	$p = (\phi\,\iota)^2$	20
$I = -\frac{1}{6}(B, C)$	$(\quad)^4(\quad)^6$	$(\phi\,\iota)$	21
$J = -\frac{1}{4}(B, D)^2$	$(\quad)^5(\quad)^1$	$\alpha = (j\,\iota)^2$	22
$K = -(B, D)$	$(\quad)^5(\quad)^3$	$(j\,\iota)$	23
$L = -\frac{1}{20}(A, H) + \frac{1}{2}BE$	$(\quad)^5(\quad)^7$	(fp)	24
$M = -\frac{1}{48}(B, H)^2 - \frac{1}{6}BG$	$(\quad)^6(\quad)^2$	$\tau = (p\iota)^2$	83
$N = \frac{1}{4}(B, H)$	$(\quad)^6(\quad)^4$	$(p\iota)$	84
$O = -(B, J)$	$(\quad)^7(\quad)^1$	$(\iota\alpha)$	*90
$P = -\frac{1}{5}(A, M) - BK$	$(\quad)^7(\quad)^5$	$(f\tau)$	*91
$Q = \frac{1}{2}(B, M)^2$	$(\quad)^8(\quad)^0$	$(\iota\tau)^2$	25
$R = -\frac{1}{2}(B, M)$	$(\quad)^8(\quad)^2$	$(\tau\iota)$	*92
$S = -96(D, M) + 16BO - 7GK$	$(\quad)^9(\quad)^3$	$(j\tau)$	*93
$T = -(J, M)$	$(\quad)^{11}(\quad)^1$	$\gamma = (\tau\alpha)$	*94
$U = \frac{1}{18}(J, O) + \frac{1}{9}GQ$	$(\quad)^{12}(\quad)^0$	$((\iota\alpha), \alpha)$	29
$V = -(B, T)$	$(\quad)^{13}(\quad)^1$	$(\iota\gamma)$	*95
$W = -\frac{1}{6}(O, T)$	$(\quad)^{18}(\quad)^0$	$((\iota\alpha), \gamma)$	29A

338. The Table exhibits the generation of the several covariants; viz. $(A,\ B)$ denotes $\partial_x A \,.\, \partial_y B - \partial_y A \,.\, \partial_x B$, $(A,\ B)^2$ denotes $\partial_x{}^2 A \,.\, \partial_y{}^2 B - 2\partial_x\partial_y A \,.\, \partial_x\partial_y B + \partial_y{}^2 A \,.\, \partial_x{}^2 B$, &c. (see *post,* No. 348). The column f, $\iota = (ff)^4$, &c. shows Gordan's notation, and the generation of his 23 forms $\big((ff)^4$ written as with him for $(f,\ f)^4$, &c.$\big)$: it will be observed that the forms are not identical; if the calculations had been made *de novo,* I should have adopted his values, simply omitting numerical factors of the several forms (thus every term of ι, $= (ff)^4$ contains the factor $2\,.\,(120)^2$, $= 28800$): of course the presence of these numerical factors renders the f, ι, ϕ, &c. as they stand inconvenient for the expression of results; and the numerical fixation of the values was no part of Gordan's object. But by reason of the existing Tables the change of notation is in fact more than this; thus H instead of being a submultiple of $(B,\ C)^2$, that is, of p, is in fact $= -\tfrac{1}{5}(B,\ C)^2 + \tfrac{2}{5}B^2$; and so in other cases. If the occasion for it arises, there is no difficulty in expressing any one of the forms f, ι, ϕ, &c. in terms of the $(A,\ B,\ C\,..\,V,\ W)$; thus in the instance just referred to, $p = (\phi\iota)^2$, we have

$$\phi = (ff)^2 = (A,\ A)^2 = 800\,C,$$

and

$$\iota = (ff)^4 = (A,\ A)^4 = 28800\,B,$$

whence $p = 2304000\,(B,\ C)^2$; also $(B,\ C)^2 = -\,5H + 2B^2$; and therefore, finally,

$$p = -\,11520000\,H + 4608000\,B^2.$$

339. I remark upon the value $S = -\,96\,(D,\ M) + 16BO - 7GK$, that S is the complete value of a covariant $(\quad)^9\,(\quad)^3$, the leading coefficient of which is given in Table No. 86 of my Eighth Memoir; the form $(D,\ M)$, omitting a numerical factor (if any), would have had smaller numerical coefficients, but there is in the form actually adopted the advantage that it vanishes for $a = 0$, $b = 0$, that is, when the quintic has two equal roots, [see *post,* No. 346].

340. I now form the following Table No. 88, viz. this is the Table No. 82 of my Eighth Memoir, carried as far as a^8, but with the composite covariants expressed by means of the foregoing letters A, B, $C, \ldots,$ W; instead of giving the syzygies as in Table No. 82, I transfer them to a separate Table, No. 89. In all other respects the arrangement is as explained, Eighth Memoir, No. 253; but in place of N, S, S' I have written $*$, Σ, Σ' to denote new covariant, new syzygy, derived syzygy, respectively; and I have, as to the terms a^8x^{14}, a^8x^{20} respectively, introduced the new symbol σ to denote an interconnexion of syzygies, as appearing by the Table No. 89, and as will be further explained.

Table No. 88.

[In this Table and the subsequent Table 89, I have for convenience used, instead of capitals, the small italic letters $a, b, c, \dots w$ to denote the 23 irreducible covariants of the quintic.]

Ind. a.	Ind. x.	Coeff.			
1	5	1	a	.	*
	3	0		.	
	1	0		.	
2	10	1	a^2	.	
	8	0		.	
	6	1	c	.	*
	4	0		.	
	2	1	b	.	*
	0	0		.	
3	15	1	a^3	.	
	13	0		.	
	11	1	ac	.	
	9	1	f	.	*
	7	1	ab	.	
	5	1	e	.	*
	3	1	d	.	*
	1	0		.	
4	20	1	a^5	.	
	18	0		.	
	16	1	a^2c	.	
	14	1	af	.	
	12	2	a^2b, c^2	.	
	10	1	ae	.	
	8	2	ad, bc	.	
	6	1	i	.	*
	4	2	b^2, h	.	*
	2	0		.	
	0	1	g	.	*

Table No. 88 (continued).

Ind. a.	Ind. x.	Coeff.			
5	25	1	a^5		
	23	0		.	
	21	1	a^3c	.	
	19	1	a^2f	.	
	17	2	$a^3b,\ ac^2$	.	
	15	2	$a^2e,\ cf$	.	
	13	2	$a^2d,\ abc$	.	
	11	2	$ai,\ bf,\ ce$	.	Σ
	9	3	$ab^2,\ ah,\ cd$	.	
	7	2	$be,\ l$	.	*
	5	2	$ag,\ bd$	.	
	3	1	k	.	*
	1	1	j	.	*
6	30	1	a^6	.	
	28	0		.	
	26	1	a^4c	.	
	24	1	a^3f	.	
	22	2	$a^4b,\ a^2c^2$	.	
	20	2	$a^3e,\ acf$	.	
	18	3	$a^3d,\ a^2bc,\ c^3,\ f^2$	.	Σ
	16	2	$a^2i,\ abf,\ ace$	.	Σ
	14	4	$a^2b^2,\ a^2h,\ acd,\ bc^2,\ ef$	.	Σ
	12	3	$abe,\ al,\ ci,\ df$	.	Σ
	10	4	$a^2g,\ abd,\ b^2c,\ ch,\ e^2$	.	Σ
	8	2	$ak,\ bi,\ de$	.	Σ
	6	4	$aj,\ b^3,\ bh,\ cg,\ d^2$	.	Σ
	4	1	n	.	*
	2	2	$bg,\ m$	.	*
	0	0		.	
7	35	1	a^7	.	
	33	0		.	
	31	1	a^5c	.	
	29	1	a^4f	.	
	27	2	$a^5b,\ a^3c^2$	.	
	25	2	$a^4e,\ a^2cf$	.	
	23	3	$a^4d,\ a^3bc,\ ac^3,\ af^2$	.	Σ′
	21	3	$a^3i,\ a^2bf,\ a^2ce,\ c^2f$	.	Σ′
	19	4	$a^3b^2,\ a^3h,\ a^2cd,\ abc^2,\ aef$	.	Σ′
	17	4	$a^2be,\ a^2l,\ aci,\ adf,\ bcf,\ c^2e$	.	2Σ′
	15	5	$a^3g,\ a^2bd,\ ab^2c,\ ach,\ ae^2,\ c^2d,\ fi$	.	Σ′, Σ
	13	4	$a^2k,\ abi,\ ade,\ b^2f,\ bce,\ cl,\ fh$	.	Σ
	11	5	$a^2j,\ ab^3,\ abh,\ acg,\ ad^2,\ bcd,\ ei$	.	Σ
	9	4	$an,\ b^2e,\ bl,\ ck,\ di,\ eh,\ fg$	.	3Σ
	7	4	$abg,\ am,\ b^2d,\ cj,\ dh$	.	Σ
	5	3	$bk,\ p,\ eg$	.	*
	3	2	$bj,\ dg$	.	
	1	1	o	.	*

Table No. 88 (concluded).

Ind. a.	Ind. x.	Coeff.			
8	40	1	a^8	.	
	38	0		.	
	36	1	a^6c	.	
	34	1	a^5f	.	
	32	2	$a^6b,\ a^4c^2$	.	
	30	2	$a^5e,\ a^3cf$	.	
	28	3	$a^5d,\ a^4bc,\ a^2c^3,\ a^2f^2$	.	Σ'
	26	3	$a^4i,\ a^3bf,\ a^3ce,\ ac^2f$	.	Σ'
	24	5	$a^4b^3,\ a^4h,\ a^3cd,\ a^2bc^2,\ a^2ef,\ cf^2,\ c^4$	.	Σ'
	22	4	$a^3be,\ a^3l,\ a^2ci,\ a^2df,\ abcf,\ ac^2e$	.	$2\Sigma'$
	20	6	$a^4g,\ a^3bd,\ a^2b^2c,\ a^2ch,\ a^2e^2,\ ac^2d,\ afi,\ bc^3,\ bf^2,\ cef$	.	σ
	18	5	$a^3k,\ a^2bi,\ a^2de,\ ab^2f,\ abce,\ acl,\ afh,\ c^2i,\ cdf$	.	$4\Sigma'$
	16	7	$a^3j,\ a^2b^3,\ a^2bh,\ a^2cg,\ a^2d^2,\ abcd,\ aei,\ b^2c^2,\ bef,\ c^2h,\ ce^2,\ fl$	.	$5\Sigma'$
	14	5	$a^2n,\ ab^2e,\ abl,\ ack,\ adi,\ aeh,\ afg,\ bci,\ bdf,\ cde$	.	σ
	12	7	$a^2bg,\ a^2m,\ ab^2d,\ acj,\ adh,\ b^3c,\ bch,\ be^2,\ c^2g,\ cd^2,\ el,\ fk,\ i^2$	.	3Σ
	10	5	$abk,\ aeg,\ ap,\ b^2i,\ bde,\ cn,\ dl,\ fj,\ hi$	.	3Σ
	8	6	$abj,\ adg,\ b^4,\ b^2h,\ bcg,\ bd^2,\ cm,\ ek,\ h^2$	.	2Σ
	6	3	$ao,\ bn,\ dk,\ ej,\ gi$	.	2Σ
	4	4	$b^2g,\ bm,\ dj,\ gh$	.	
	2	1	r	.	*
	0	2	$g^2,\ q$	.	*

341. The syzygies and interconnexions of syzygies are given in

Table No. 89.

[See *ante* Table No. 88.]

(5, 11)	$ai + bf - ce = 0$
(6, 18)	$a^3d - a^2bc + 4c^3 + f^2 = 0$
(6, 14)	$a^2h - 6acd - 4bc^2 - ef = 0$
(6, 12)	$al - 2ci + 3df = 0$
(6, 10)	$a^2g - 12abd - 4b^2c - e^2 = 0$
(6, 8)	$ak + 2bi - 3de = 0$
(6, 6)	$aj - b^3 + 2bh - cg - 9d^2 = 0$
(7, 15)	$a^2bd - ab^2c + ach - 6c^2d - fi = 0$
(7, 13)	$a^2k - abi - 3b^2f + 6cl + 3fh = 0$
(7, 11)	$a^2j - ab^3 + abh - 9ad^2 - 6bcd - ei = 0$
(7, 9)	$an - b^2e - 6di + 2eh - fg = 0$
	$2bl + 6di - eh + fg = 0$
	$2ck - 12di + eh - fg = 0$
(7, 7)	$am + 2b^2d + cj - 3dh = 0$

Table No. 89 (continued).

σ, (8, 20)	$0\,.\,a^2(a^2g-12abd-4b^2c-e^2)$		*suprà* (6, 10)
	$-a\,(a^2bd-abc^2+ach-6c^2d-fi)$		„ (7, 15)
	$+b\,(a^3d-a^2bc+4c^3+f^2)$		„ (6, 18)
	$+c\,(a^2h-6acd-4bc^2-ef)$		„ (6, 14)
	$-f\,(ai+bf-ce)$	$=0$	„ (5, 11)
σ, (8, 14)	$0\,.\,a\,(an-b^2e-6di+2eh-fg)$		*suprà* (7, 9)
	$+\ a\,(2bl+6di-eh+fg)$		„ („)
	$+\ a\,(2ck-12di+eh-fg)$		„ („)
	$-2b\,(al-2ci+3df)$		„ (6, 12)
	$-2c\,(ak+2bi-3de)$		„ (6, 8)
	$+6d\,(ai+bf-ce)$	$=0$	„ (5, 11)
(8, 12)	$ab^2d-b^3c+2bch-c^2g+i^2=0$		
	$-3adh-2bch+2c^2g+18cd^2+fk-2i^2=0$		
	$el+fk-2i^2=0$		
(8, 10)	$abk-cn-6dl-2fj+hi=0$		
	$ap+2cn+fj=0$		
	$b^2i+cn+3dl+fj-2hi=0$		
(8, 8)	$abj-b^4+4b^2h-9bd^2+12cm-ek-3h^2=0$		
	$adg+2b^2h-12bd^2+8cm-ek-2h^2=0$		
(8, 6)	$ao+6dk-3ej+2gi=0$		
	$bn+3dk-ej+gi=0$		

342. In illustration take any one of the lines of Table No. 88, for instance [resuming the notation by capital letters] the line

$$(7,\ 17)\ |\ 4\ |\quad A^2BE,\ A^2L,\ ACI,\ ADF,\ BCF,\ C^2E\quad |\ 2\Sigma'\ |$$

there are here 6 composite covariants, but the number of asyzygetic covariants is $=4$; there must therefore be $6-4, =2$ syzygies; we have however (see Table No. 89) two derived syzygies of the right form, viz. these are

$$A\,(AL-2CI+3DF)=0,$$
$$C\,(AI+BF-CE)=0,$$

which are designated as $2\Sigma'$, and there is consequently no new syzygy Σ.

But in the line

$$(7,\ 15)\ |\ 5\ |\quad A^3G,\ A^2BD,\ AB^2C,\ ACH,\ AE^2,\ C^2D,\ FI\quad |\ \Sigma',\ \Sigma\ |$$

there are 7 composite covariants, but the number of asyzygetic covariants is $=5$; there must therefore be $7-5, =2$ syzygies. One of these is the derived syzygy

$$A\,(A^2G-E^2-12ABD-4B^2C)=0,$$

which is designated by Σ'; the other is a new syzygy (see Table No. 89),

$$A^2BD - ABC^2 + ACH - 6C^2D - FI = 0,$$

designated by Σ.

343. Take now the line

$(8, 20) \mid 6 \mid \quad A^4G,\ A^3BD,\ A^2B^2C,\ A^2CH,\ A^2E^2,\ AC^2D,\ AFI,\ BC^3,\ BF^2,\ CEF \quad \mid 5\Sigma',\ \sigma \mid$;

there are here 10 composite covariants, but the number of irreducible covariants is $= 6$; there should therefore be $10 - 6$, $= 4$ syzygies. There are, however, the 5 derived syzygies

$$A^2(A^2G - 12ABD - 4B^2C - E^2) = 0, \text{ \&c. (see Table No. 89)}$$

designated by $5\Sigma'$; since these are equivalent to 4 syzygies only there must be 1 identical relation between them (designated by σ), viz. this is the equation $0 = 0$ obtained by adding the several syzygies, multiplied each by the proper numerical factor as shown Table No. 89.

344. Again, for the line

$(8, 14) \mid 5 \mid \quad A^2N,\ AB^2E,\ ABL,\ ACK,\ ADI,\ AEH,\ AFG,\ BCI,\ BDF,\ CDE \quad \mid 6\Sigma',\ \sigma \mid$

there are here 10 composite covariants, but only 5 irreducible covariants; there should therefore be $10 - 5$, $= 5$ syzygies; we have in fact the 6 derived syzygies

$$A(AN - B^2E - 6DI + 2EH - FG) = 0 \text{ \&c. (see Table No. 89)}$$

designated by $6\Sigma'$; these must therefore be connected by 1 identical relation (designated by σ), viz. this is the equation $0 = 0$ obtained by adding the several syzygies, each multiplied by the proper numerical factor as shown Table No. 89.

345. These two cases (σ) are in fact the instances which present themselves where a correction is required to my original theory. The two identical relations in question were disregarded in my original theory, and this accordingly gave the two non-existent irreducible covariants $(a, ..)^8(x, y)^{14}$ and $(a, ..)^8(x, y)^{20}$. And reverting to No. 336, these give in the denominator of $A(x)$ the factors $(1 - a^8x^{20})(1 - a^8x^{14})$. In virtue hereof, writing $x = 1$, we have in $A(x)$ the factor $\dfrac{(1 - a^8)^{10}}{(1 - a^8)^4}$, $= (1 - a^8)^6$, agreeing with the function $(1 -)^{-1}(1 - a)^{-2} \dots (1 - a^8)^6 \dots$. And we thus see that the denominator factors of $A(x)$ do not all of them refer to irreducible covariants; viz. we have

$$ax^5,\ a^2x^6,\ a^2x^2,\ a^3x^9,\ a^3x^5,\ a^3x^3,\ a^4x^6,\ a^4x^4,\ a^4,\ a^5x^7,\ a^5x^3,\ a^5x,\ a^6x^4,\ a^6x^2,\ a^7x^3,\ a^7x,\ a^8x^2,\ a^8,$$

each referring to an irreducible covariant, but a^8x^{20} and a^8x^{14} each referring to an identical relation (σ) or interconnexion of syzygies. And we thus understand how, consistently with the number of the irreducible covariants being finite, the expression for $A(x)$ may be as above the quotient of two infinite products; viz. there will be in the denominator a finite number of factors each referring to an irreducible covariant, but the remaining infinite series of denominator factors will refer each factor to an identical relation or interconnexion of syzygies. But I do not see how we can by the theory distinguish between the two classes of factors, so as to determine the number of the irreducible covariants, or even to make out affirmatively that the number of them is finite.

346. The new covariants O, P, R, S, T, V are as follows:

[Table No. 90 (Covariant O),
Table No. 91 (Covariant P),
Table No. 92 (Covariant R),
Table No. 93 (Covariant S),
Table No. 94 (Covariant V),

printed in the paper 143, "Tables of the Covariants M to W of the Binary Quintic; from the second, third, fifth, eighth, ninth and tenth Memoirs on Quantics" with the insertion as therein mentioned of the terms with zero coefficients. The covariant $S, = -96(D, M) + 16BO - 7GK$, of the present Memoir is there called S', and there is given the more simple form $S = (D, M)$, of this covariant.]

Article Nos. 347 to 365. *Sketch of Professor* GORDAN'S *proof for the finite Number*, $= 23$, *of the Covariants of a Binary Quintic.*

347. I propose to reproduce the leading points of Professor Gordan's proof that the binary quintic $(a, b, c, d, e, f \between x, y)^5$ has a finite system of 23 covariants, viz. a system such that every other covariant whatever is a rational and integral function of these 23 covariants.

348. DERIVATION. Consider for a moment any two binary quantics ϕ, ψ of the same or different orders, and which may be either independent quantics, or they may be both or one of them covariants, or a covariant, of a binary quantic f. We may form the series of *derivatives*

$$(\phi, \psi)^0 = \phi\psi,$$

$$(\phi, \psi)^1 = \overline{12} \ \phi_1\psi_2 = \partial_x\phi \,.\, \partial_y\psi - \partial_y\phi \,.\, \partial_x\psi,$$

$$(\phi, \psi)^2 = \overline{12}^2 \ \phi_1\psi_2 = \partial_x^2\phi \,.\, \partial_y^2\psi - 2\partial_x\partial_y\phi \,.\, \partial_x\partial_y\psi + \partial_y^2\phi \,.\, \partial_x^2\psi,$$

$$\vdots$$

where, however, there is no occasion to use the notation $(\phi, \psi)^0$ (as this is simply the product $\phi\psi$), and the succeeding derivatives may (when there is no risk of ambiguity) be written more shortly $(\phi\psi)$, $(\phi\psi)^2$, $(\phi\psi)^3$, &c.; in all that follows the word "derivative" (Gordan's *Uebereinanderschiebung*) is to be understood in this special sense.

349. The degree of the derivative $(\phi\psi)^k$ is the sum of the degrees of the constituents ϕ, ψ; the order of the derivative is the sum of the orders *less* $2k$; it being understood throughout that the word degree refers to the coefficients, and the

word order to the variables. In speaking generally of the covariants or of all the covariants of a quantic f, or of the covariants or all the covariants of a given degree or order, we of course exclude from consideration covariants linearly connected with other covariants (for otherwise the number of terms would be infinite); but unless it is expressly so stated, we do not carry this out rigorously so as to make the system to consist of asyzygetic covariants; viz. it is assumed that the system is complete, but not that it is divested of superfluous terms.

350. THEOREM A. The covariants of a quantic f of a given degree m can be all of them obtained by derivation from f and the covariants of the next inferior degree $(m-1)$.

In particular for the degree 1 the only covariant is the quantic f itself; for the degree 2 the covariants are $(ff)^0$, $(ff)^2$, $(ff)^4, \ldots$: using for a moment β to denote each of these in succession, the covariants of the third degree are $(\beta f)^0$, $(\beta f)^1$, $(\beta f)^2, \ldots$; and so on.

351. Suppose that the covariants of the second degree $(ff)^0$, $(ff)^2$, $(ff)^4 \ldots$ are in this order represented by β_1, β_2, $\beta_3 \ldots$, then the covariants of the third degree written in the order

$$(\beta_1 f)^0,\ (\beta_1 f),\ (\beta_1 f)^2,\ \ldots (\beta_2 f)^0,\ (\beta_2 f),\ (\beta_2 f)^2,\ \ldots (\beta_3 f)^0,\ (\beta_3 f),\ (\beta_3 f)^2 \ldots$$

may be represented by γ_1, γ_2, $\gamma_3, \ldots$, the covariants of the fourth degree written in the order

$$(\gamma_1 f)^0,\ (\gamma_1 f),\ (\gamma_1 f)^2,\ \ldots (\gamma_2 f)^0,\ (\gamma_2 f),\ (\gamma_2 f)^2,\ \ldots (\gamma_3 f)^0,\ (\gamma_3 f),\ (\gamma_3 f)^2 \ldots$$

may be represented by δ_1, δ_2, $\delta_3 \ldots$, and so on: we thus obtain in a definite order the covariants of a given degree m; say, these are μ_1, μ_2, μ_3, $\mu_4, \ldots$: any term μ_3 is said to be a *later* term than the preceding terms μ_1, μ_2, and an *earlier* term than the following ones, μ_5, μ_6, &c.

Observe that each term μ_r is a derivative $(\lambda_q f)^k$, the derivatives of an earlier λ are earlier than those of a later λ; and as regards the derivatives of the same λ, the derivative with a less index of derivation is earlier than that with a greater index of derivation, or, what is the same thing, those are earlier which are of the higher order.

352. The series μ_1, μ_2, μ_3, $\mu_4 \ldots$ is not asyzygetic; we make it so, by considering in succession whether the several terms μ_2, $\mu_3, \ldots$ respectively are expressible as linear functions of the earlier terms, and by omitting every term which is so expressible. The reduced series thus obtained is called T_1, T_2, $T_3, \ldots$. Observe that not every μ is a T, but that every T is a μ; every T therefore arises from a derivation upon f and a certain term λ; which term λ (supposing the λ series reduced in like manner to S_1, S_2, $S_3, \ldots$) is a linear function of certain of the S's. Each later T is derived from later S's, or it may be from the same S's as an earlier T; viz. if the later T is derived from $(S_1, S_2, \ldots S_\theta)$, then the earlier T is derived, it may be, from $(S_1, S_2, \ldots S_\theta)$, or from $(S_1, S_2, \ldots S_{\theta-k})$, but so that there is not in the series any term later than S_θ.

And if, considering any T as thus derived from certain of the S's, and in like manner each of these S's as derived from certain of the R's, and so on, we descend to any preceding series,

$$M_1,\ M_2,\ M_3 \ldots$$

it will appear that the T is derived from a certain number $(M_1,\ M_2, \ldots M_\phi)$ of the terms of this series.

353. The quadricovariants $(ff)^0$, $(ff)^2$, $(ff)^4, \ldots$ are of different orders, and consequently asyzygetic. They form therefore a series such as the T-series, and they may be represented by

$$B_1,\ B_2,\ B_3, \ldots.$$

Supposing f to be of the order n, B_1 is of the order $2n$, B_2 of the order $2n-4$, B_3 of the order $2n-8$, and so on. Those terms which are of an order greater than n, are said to be of the form W (agreeing with a subsequent more general definition of W); those which are of an order equal to or less than n, are said to be of the form χ; so that the earlier terms of the B series are W, and the later terms are χ; viz. the χ terms taken in order, beginning with the earliest, are $\chi_1, \chi_2, \chi_3, \ldots$.

354. By what precedes any particular T is derived from certain terms $B_1, B_2, \ldots B_\theta$, of the B series. This series, $B_1, B_2, \ldots B_\theta$, may stop short of the terms χ, or it may include a certain number of them, say $\chi_1, \chi_2, \ldots \chi_r$. The terms derived from the χ's are in the sequel denoted by P_χ.

355. Every covariant whatever is a form or sum of forms such as

$$\overline{12}^{\alpha}\, \overline{13}^{\beta}\, \overline{23}^{\gamma} \ldots f_1 f_2, \ldots f_m;$$

writing in regard to any such expression

$$\Sigma \text{ ind. } 1 = i,\ \Sigma \text{ ind. } 2 = j, \ldots$$

(viz. i is the sum of all those indices α, β, &c. which belong to a term containing the symbolic number 1, j the sum of all the indices α, γ, &c. which belong to a term containing the symbolic number 2, and so on) then each of the numbers $i, j, \ldots$ is at most $=n$, that is $n-i, n-j, \ldots$ may be any of them $=0$, but they cannot be any of them negative; the degree of the function is $=m$, and its order is $=mn-i-j\ldots$ It is to be further observed that the form is a function of the differential coefficients of f of the orders $n-i$, $n-j$, &c. respectively. It follows that if $n-i, n-j, \ldots$ are none of them $=0$, the form in question may be obtained from a like form belonging to a quantic f' of the next inferior order $n-1$ by replacing therein the coefficients $a', b', \ldots$ by $ax+by$, $bx+cy$, &c. respectively: for example, if f denote the cubic function $(a, b, c, d \between x, y)^3$, then the Hessian hereof is $\overline{12}^2 f_1 f_2$; the like form in regard to the quadric $f' = (a', b', c' \between x, y)^2$ is $\overline{12}^2 f_1' f_2'$, which is $= a'c' - b'^2$; and substituting herein $ax+by$, $bx+cy$, $cx+dy$ for a', b', c' respectively, we have the Hessian $\overline{12}^2 f_1 f_2$ of the cubic. A covariant of f derivable in this manner from a covariant of the next inferior quantic f' is said to be a special covariant.

356. Reverting to the form

$$\overline{12}^{\alpha}\,\overline{13}^{\beta}\,\overline{23}^{\gamma}\ldots f_1 f_2 \ldots f_m;$$

if, as before, $n-1$, $n-j$, &c. are each of them >0; if there is at least one index i which is $=$ or $<\frac{1}{2}n$ (that is, for which $n-i>\frac{1}{2}n$), and if the order $mn-i-j\ldots$ be $>n$, then the form, or any sum of such forms, is said to be a form or covariant W. Every covariant W is thus a special covariant, but not conversely. In the particular case $m=2$, the form is

$$\overline{12}^{\alpha} f_1 f_2,$$

which will be a form W if $n-\alpha>\frac{1}{2}n$, or, what is the same thing, $2n-2\alpha>n$, that is if the order be $>n$. Hence, as already mentioned, the covariants T of the degree 2 are W, or else χ, according as the order is greater than n, or as it is equal to or less than n.

357. *Theorem B.* If any covariant T be expressible as the sum of a form W and of earlier T's than itself, then forming the derivative $(Tf)^k$, either this is not a form T, or being a form T, it is expressible as the sum of a form W and of earlier T's than itself; or, what is the same thing, $(Tf)^k$, if it be a form T, is (like the original T) the sum of a form W and of earlier T's than itself.

Hence also every form T is the sum of a form W, and of forms derived from the functions $\chi_1, \chi_2, \ldots$, say

$$T = W + P_\chi,$$

or, what is the same thing, every covariant whatever is of the form $W+P_\chi$.

358. The proof that for a form f of the order n the number of covariants is finite, depends on the assumption that the number is finite for a form f' of the next inferior order $n-1$: this being so, the number of the special covariants of f will be finite; say these are $A_1, A_2, A_3, \ldots$ (f is itself one of the series, but we may separate it, and speak of the form f and its special covariants): the forms W are functions of the special covariants, and hence every covariant whatever of f is of the form $F(A)+P_\chi$; but it requires still a long investigation to pass from this to the theorem of the existence of a finite number of forms V such that every covariant whatever is $F(V)$. I pass this over, and reproduce only the investigation for the case of the quintic.

359. Starting from the assumed system of forms,

$$f,\ \phi=(ff)^2,\ i=(ff)^4,\ j=(fi)^2,\ \alpha=(ji)^2,\ p=(\phi i)^2,\ \tau=(pi)^2,\ \gamma=(\tau\alpha),$$
$$(f\phi),\ (fp),\ (f\tau),\ (j\tau),$$
$$(fi),\ (\phi i),\ (ji),\ (pi),\ (\tau i),$$
$$(i\alpha),\ (i\gamma),\ (ii)^2,\ ((i\alpha),\ \alpha),\ (i\tau)^2,\ ((i\alpha),\ \gamma),$$

say, the 23 forms U, it is to be shown that every other covariant whatever of the quintic is of the form $F(U)$.

The special covariants are f, ϕ, $(f\phi)$, i, j, which are forms U; the only form χ is i, so that instead of P_χ writing P_i, every covariant whatever of f is

$$=F(U)+P_i;$$

and it remains to show that every form P_i is $F(U)$; or, what is the same thing, that if H be any form $F(U)$ whatever, then that (Hi) and $(Hi)^2$ are each of them $F(U)$.

360. In order to show that every covariant of a degree not exceeding m is $F(U)$, it will be sufficient to show that the several forms (Hi) and $(Hi)^2$ of a degree not exceeding m are each of them $F(U)$: and if for this purpose we assume that it is shown that every covariant of a degree not exceeding $m-1$ is $F(U)$, then in regard to the forms (Hi) and $(Hi)^2$ of the degree m, it will be sufficient to show that any such form is a function of covariants of a degree inferior to m.

361. First for the form (Hi): we have $(PQ,\ i)=P(Qi)+Q(Pi)$; and hence we see that (Hi) will be $F(U)$ if only (Ui) is always $F(U)$.

In forming the derivative of i with the several covariants U, we may omit i itself, and also the four invariants $(ii)^2$, $(i\tau)^2$, $((i\alpha),\ \alpha)$, $((i\alpha),\ \gamma)$, since in each of these cases the derivative is $=0$. We have therefore to consider the derivative of i with

$$f,\ \phi,\ j,\ \alpha,\ p,\ \tau,\ \gamma,\ (f\phi),\ (fp),\ (f\tau),\ (j\tau),\ (fi),\ (\phi i),\ (ji),\ (pi),\ (\tau i),\ (i\alpha),\ (i\gamma),$$

respectively: the first seven of these are each of them U; the remaining eleven are each of them of the form $((PQ),\ i)$. Now $((PQ),\ i)$ is a linear function of $P(Qi)^2$, $Q(Pi)^2$, and $i(PQ)^2$, that is $((PQ),\ i)$ is a function of covariants of a lower degree than itself.

362. Next for the form $(Hi)^2$, we have $(PQ,\ i)^2$, a linear function of $P(Qi)^2$, $Q(Pi)^2$, $i(PQ)^2$; and we hence see that $(Hi)^2$ will be $F(U)$ if only $(Ui)^2$ is always $F(U)$.

In forming the second derivative of i with the several covariants U, we may omit as before the four invariants, and also omit the four linear covariants α, $i\alpha$, γ, $i\gamma$; we have therefore to consider the second derivatives of i with

$$f,\ \phi,\ i,\ j,\ p,\ \tau,\ (f\phi),\ (fp),\ (f\tau),\ (j\tau),\ (fi),\ (\phi i),\ (ji),\ (pi),\ (\tau i),$$

respectively: the first six of these are each of them U; the remaining nine are each of the form $((PQ),\ i)^2$. Now $((PQ),\ i)^2$ is a linear function of $((Pi)^2,\ Q)$, $((Qi)^2,\ P)$, $P(Qi)^3$, and $Q(Pi)^3$. The first two of these are terms of the same form; $(Pi)^2$, as a covariant of a lower degree than $((PQ),\ i)^2$, is $F(U)$, and hence $((Pi)^2,\ Q)$ will be $F(U)$ if only $(U,\ Q)$ is $F(U)$; Q being here any one of the functions f, ϕ, i, j, p, τ, and U being any one of the functions

$$f,\ \phi,\ i,\ j,\ p,\ \tau,\ \alpha,\ \gamma,\ (f\phi),\ (fp),\ (f\tau),\ (j\tau),\ (fi),\ (\phi i)\ (ji)\ (pi)\ (\tau i)\ (i\alpha)\ (i\gamma).$$

363. For U equal to any one of the last eleven values, the form is $(Q, S)R$ which is $=R(QS)+S(QR)$, and is thus a function of covariants of a lower degree; there remains only the derivatives formed with two of the functions f, ϕ, i, j, p, τ, or of one of these with α or γ. But these are all U other than the derivatives

$$(fj),\ (\phi j),\ (\phi p),\ (\phi\tau),\ (p\tau);\ (f\alpha),\ (\phi\alpha),\ (j\alpha),\ (p\alpha);\ (f\gamma),\ (\phi\gamma),\ (j\gamma),\ (p\gamma),\ (\tau\gamma),$$

and since $\gamma=(\tau\alpha)$, the derivatives containing γ will depend upon covariants of a lower degree; there remain therefore only (fj), (ϕj), (ϕp), $(\phi\tau)$, $(p\tau)$; $(f\alpha)$, $(\phi\alpha)$, $(j\alpha)$, $(p\alpha)$: each of these can be actually calculated in the form $F(U)$.

Hence finally, assuming that every covariant of a degree inferior to m is $F(U)$, it follows that every covariant of the degree m is $F(U)$; whence every covariant whatever is $F(U)$, viz. it is a rational and integral function of the 23 covariants U.

364. It will be observed that, writing A, B, C for P, Q, i, the proof depends on the theorems

$$\begin{array}{lll} ((AB),\ C), & \text{a linear function of} & A(BC)^2,\ B(CA)^2,\ C(AB)^2, \\ (AB,\ C)^2 & \quad ,, \quad ,, & \text{do.} \quad \text{do.} \quad \text{do.} \\ ((AB),\ C)^2 & \quad ,, \quad ,, & ((AC)^2,\ B),\ ((BC)^2,\ A),\ B(AC)^3,\ C(AB)^3, \end{array}$$

which are theorems relating to any three functions A, B, C whatever.

365. I remark upon the proof that the really fundamental theorem seems to be that which I have called theorem A. As to the forms W it is difficult to see *à priori* why such forms are to be considered, or what the essential property involved in their definition is; and in fact in a more recent paper, "Die simultanen Systeme binären Formen" (*Math. Annalen*, t. II. (1869), see p. 256), Professor Gordan has modified the definition of the forms W by omitting the condition that the order of the function shall exceed n; if it were possible further to omit the condition of at least one index being $=$ or $<\frac{1}{2}n$, and so only retain the conditions $n-i$, $n-j$, &c., each of them >0, then the essential property of the forms W would be that any such form was a rational and integral function of the special covariants formed, as above, by means of the quantic of the next inferior order. And moreover, as regards the theorem B, there seems something indirect and artificial in the employment of such a property; one sees no reason why, when a system of irreducible covariants is once written down, it should not be possible to show that the derivatives of $F(U)$ with the original quantic f are each of them $F(U)$, instead of having to show this in regard to the derivatives of $F(U)$ with the several covariants χ: as regards the quintic, where there is a single covariant χ, the *quadric* function i, there is obviously a great abbreviation in this employment of i in place of f; but for the higher orders, assuming that the proof could be conducted by means of the quantic f itself, it does not appear that there would be even an abbreviation in the employment in its stead of the several covariants χ. The like remarks apply to the proof in the last-mentioned paper. I cannot but hope that a more simple proof of Professor Gordan's theorem will be obtained—a theorem the importance of which, in reference to the whole theory of forms, it is impossible to estimate too highly.