463.

NOTE ON A DIFFERENTIAL EQUATION.

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THE following investigation was suggested to me by Mr Harley's "Remarks on the Theory of the Transcendental Solution of Algebraic Equations," communicated to the Society at the Meeting of the 4th of February.

Mr Harley's equation

$$y^n - ny + (n-1)\,x = 0$$

may be written

$$y = \frac{n-1}{n}\,x + \frac{1}{n}\,y^n;$$

or putting

$$\frac{n-1}{n}\,x = u,\ \frac{1}{n} = a,$$

it becomes

$$y = u + ay^n,$$

which equation may be considered instead of the original equation; and it is to be shown that y, regarded as a function of u, satisfies a certain linear differential equation of the order $n-1$. In fact, expanding y by Lagrange's theorem, we have

$$y = u + au^n + \frac{a^2}{1\,.\,2}\,(u^{2n})' + \frac{a^3}{1\,.\,2\,.\,3}\,(u^{3n})'' + \&c.,$$

$$= u + au^n + \frac{a^2}{1\,.\,2}\,2n\,.\,u^{2n-1} + \frac{a^3}{1\,.\,2\,.\,3}\,3n\,(3n-1)\,u^{3n-2} + \&c.,$$

the law whereof is obvious, and using the ordinary notation of factorials, viz. $[n]^r = n(n-1)\ldots(n-r+1)$, we may write

$$y = S_\theta \,.\, \frac{[n\theta]^{\theta-1}}{[\theta]^\theta}\, a^\theta\, u^{(n-1)\,\theta+1},$$

where θ extends from 0 to ∞.

It is now very easy to show that y satisfies the differential equation

$$\left[u\frac{d}{du}\right]^{n-1} y = na\left[\frac{n}{n-1}\,u\frac{d}{du} - \frac{2n-1}{n-1}\right]^{n-1} u^{n-1}\,y.$$

In fact, using on the left-hand side the foregoing value of y, and on the right-hand side the following value of $u^{n-1}y$, obtained from that of y by writing $\theta-1$ in the place of θ, viz.

$$u^{n-1}\,y = S_\theta\,\frac{[n\theta-n]^{\theta-2}}{[\theta-1]^{\theta-1}}\, a^{\theta-1}\, u^{(n-1)\,\theta+1},$$

and observing that in general the symbol $u\dfrac{d}{du}$, as regards u^m, is $=m$, the equation in question will be satisfied, if only

$$\frac{[n\theta]^{\theta-1}}{[\theta]^\theta}\,[(n-1)\,\theta+1]^{n-1} = \frac{n\,[n\theta-n]^{\theta-2}}{[\theta-1]^{\theta-1}}\left[\frac{n}{n-1}\,((n-1)\,\theta+1) - \frac{2n-1}{n-1}\right]^{n-1},$$

where the right-hand side is

$$= \frac{n\,[n\theta-n]^{\theta-2}}{[\theta-1]^{\theta-1}}\,[n\theta-1]^{n-1};$$

and the equation may be written

$$\frac{n\theta\,[n\theta-1]^{\theta-2}}{\theta\,[\theta-1]^{\theta-1}}\,[(n-1)\,\theta+1]^{n-1} = \frac{n\,[n\theta-n]^{\theta-2}}{[\theta-1]^{\theta-1}}\,[n\theta-1]^{n-1},$$

that is,

$$[n\theta-1]^{\theta-2}\,[(n-1)\,\theta+1]^{n-1} = [n\theta-1]^{n-1}\,[n\theta-n]^{\theta-2},$$

which, since each side of the equation is $=[n\theta-1]^{\theta+n-3}$, is obviously true.

The foregoing differential equation is developable in the form

$$\left\{\alpha_0 + \alpha_1 u\frac{d}{du} + \alpha_2 u^2\left(\frac{d}{du}\right)^2 \ldots + \alpha_{n-1}\,u^{n-1}\left(\frac{d}{du}\right)^{n-1}\right\} y = \frac{1}{na}\left(\frac{d}{du}\right)^{n-1} y;$$

but to find the coefficients $\alpha_0, \alpha_1, \ldots \alpha_{n-1}$, I start from this form, and proceed to substitute in the equation the value of y, which on the left-hand side I use in the original form, and on the right-hand side in the form obtained by writing $\theta+1$ in the place of θ, viz.

$$y = S_\theta\,\frac{[n\,(\theta+1)]^\theta}{[\theta+1]^{\theta+1}}\,\alpha^{\theta+1}\,u^{(n-1)\,\theta+n}.$$

The equation to be satisfied is

$$\frac{[n\theta]^{\theta-1}}{[\theta]^{\theta}}\left(\alpha_0+\alpha_1\left[(n-1)\,\theta+1\right]^1+\alpha_2\left[(n-1)\,\theta+1\right]^2\ldots+\alpha_{n-1}\left[(n-1)\,\theta+1\right]^{n-1}\right)$$
$$=\frac{1}{n}\,\frac{[n\,(\theta+1)]^{\theta}}{[\theta+1]^{\theta+1}}\left[(n-1)\,\theta+n\right]^{n-1},$$

or, what is the same thing,

$$\frac{1}{[\theta]^{\theta}}\left(\alpha_0\,[n\theta]^{\theta-1}+\alpha_1\,[n\theta]^{\theta}+\quad\alpha_2\,[n\theta]^{\theta+1}\quad\ldots+\alpha_{n-1}\,[n\theta]^{\theta+n-2}\right)=\frac{1}{n}\,\frac{[n\,(\theta+1)]^{\theta+n-1}}{[\theta+1]^{\theta+1}}.$$

Observing that the right-hand side may be written

$$\frac{1}{n}\cdot\frac{n\,(\theta+1)\,[n\theta+n-1]^{\theta+n-2}}{(\theta+1)\,[\theta]^{\theta}},$$

the equation becomes

$$\alpha_0\,[n\theta]^{\theta-1}+\alpha_1\,[n\theta]^{\theta}+\quad\alpha_2\,[n\theta]^{\theta+1}\quad\ldots+\alpha_{n-1}\,[n\theta]^{\theta+n-2}\quad=[n\theta+n-1]^{\theta+n-2},$$

or, what is the same thing,

$$\alpha_0+\quad\alpha_1[(n-1)\,\theta+1]^1+\alpha_2\,[(n-1)\theta+1]^2\ldots+\alpha_{n-1}[(n-1)\,\theta+1]^{n-1}=[n\theta+n-1]^{n-1};$$

so that α_0, α_1, ... α_{n-1} are the coefficients of the expansion of $[n\theta+n-1]^{n-1}$ (which is a rational and integral function of θ, of the degree $n-1$) in a factorial series, as shown by the left-hand side of the equation.

To determine the actual values, write

$$(n-1)\,\theta+1=\phi,$$

this gives

$$n\theta+n-1=\frac{n\phi+n^2-3n+1}{n-1};$$

and we have therefore

$$\left[\frac{n\phi+n^2-3n+1}{n-1}\right]^{n-1}=\alpha_0+\alpha_1\,[\phi]^1+\alpha_2\,[\phi]^2\ldots+\alpha_{n-1}\,[\phi]^{n-1};$$

and thus the general expression is

$$\alpha_s=\frac{1}{[s]^s}\,\Delta^s\left(\frac{n\phi+n^2-3n+1}{n-1}\right),$$

where Δ denotes the difference in regard to ϕ ($\Delta U_\phi=U_{\phi+1}-U_\phi$), and, after the operation Δ^s is performed, ϕ is to be put equal to zero.