477.

ON THE GRAPHICAL CONSTRUCTION OF A SOLAR ECLIPSE.

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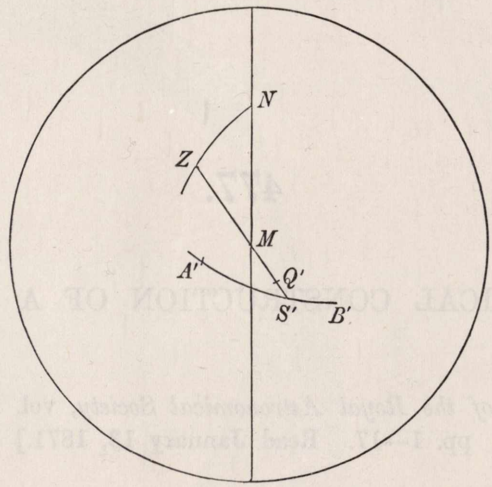
THE present Memoir contains the explanation of a Graphical Construction of a Solar Eclipse, which (it appears to me) is at once easy, and susceptible of considerable accuracy: I think that if made on the suggested scale (radius $=12$ inches) we might by means of it construct a diagram such as the eclipse-diagrams of the *Nautical Almanac*, with at least as much accuracy as could be exhibited in a diagram on that scale.

Article Nos. 1 to 9. *General Explanation of the Construction.*

1. We may imagine the celestial sphere as seen from the centre of the Earth stereographically projected at each instant during the eclipse—the radius of the bounding circle of the projected hemisphere being a given length, say twelve inches, which is taken as unity—in such wise that the centre of the Moon is always at the centre of the projection, say M, and the pole (suppose the north pole, say N) of the Earth on a given radius: its position on this radius will in strictness be variable, viz. distance from centre = projection of Moon's N. P. D. $= \tan\frac{1}{2}\Delta$. Suppose, for a moment, that the position at each instant of the Sun's centre were also laid down on the projection, so as to obtain the projection of the Sun's relative orbit; this will be a terminated short line $A'B'$ (fig. 1), nearly straight, and lying near the centre of the projection (this relative orbit is not to be actually laid down, but it is replaced, as will presently be explained, by a relative orbit on a very enlarged scale); if at any instant the position of the Sun on the relative orbit be denoted by S', then the straight line MS' is the projection of the arc of great circle through the centres of the Moon and Sun, so that E being the angular distance of the centres, the length of the line MS' is $= \tan\frac{1}{2}E$, or (E being small) it is $= \frac{1}{2}E$.

2. Produce $S'M$ through the centre M to a point Z, and consider Z as representing a point on the Earth's surface: to determine the geographical position of Z, we must consider the projected meridian NZ which passes through Z: the arc NZ,

Fig. 1.



regarded as a projection, represents the N. P. D. or colatitude of Z, and the actual angle at N which the tangent of NZ makes with the line NM is equal to the celestial angle ZNM which is = Moon's hour-angle from Z, or what is the same thing = difference of Moon's hour-angle from Greenwich and of the longitude of Z (as the figure is drawn, $\angle\, ZNM$ = Moon's hour-angle E. of Greenwich, *less* E. longitude of Z).

3. Now, considering the Moon and Sun as seen from Z, we may disregard the parallactic depression of the Sun, and attribute to the Moon a displacement equal to the difference of the parallactic displacements of the Moon and Sun; that is, regarding the zenith distances ZM, ZS' as equal, we may consider the Moon's centre as depressed by parallax in the direction of the arc MS' through an arc MQ', $= \sin^{-1} P' \sin ZM$, where $P' = \cdot 99837\,(\sigma' - \pi')$ is the quantity thus designated in the Appendix to the *Nautical Almanac* for 1836, viz. it is $= \sigma' - \pi'$, the difference of the equatoreal horizontal parallaxes at the time of the eclipse, multiplied by a factor $\cdot 99837$, which answers to a distance of Z from the Earth's centre = Earth's radius for latitude 45°. And if we take Q' such that its angular distance from S' = sum of angular semidiameters of the Sun and Moon, the locus of Q' is very nearly a circle about the centre S', and the corresponding positions of Z give the positions on the Earth where the limbs are in exterior contact, or, what is the same thing, give the penumbral curve on the Earth's surface for the position S' of the Sun.

4. Instead of

$$\text{Arc } MQ' = \sin^{-1} P' \sin ZM,$$

we may write

$$\text{Arc } MQ' = \quad P' \sin ZM,$$

or, using ρ to denote the linear distance ZM in the projection, we have $\rho = \tan \frac{1}{2} ZM$, and therefore $\sin ZM = \frac{2\rho}{\rho^2+1}$, hence

$$\text{Arc } MQ' = P' \frac{2\rho}{\rho^2+1},$$

and the linear distance MQ' in the projection is $= \tan \frac{1}{2} \text{ arc } MQ'$, say this is $= \frac{1}{2} \text{ arc } MQ'$, or calling this linear distance r' we have

$$r' = P' \frac{\rho}{\rho^2+1}.$$

5. Hence, if instead of the original representation of the Sun's relative orbit we consider an enlarged representation thereof and of the depressed positions Q' of the Moon, obtained by increasing the several distances from the centre of the projection in the ratio $\frac{1}{2}P'$ to 1, and if instead of A', B', S', Q', we use A, B, S, Q, as referring to this enlarged representation, then representing by r the linear distance MQ, we have $r = \frac{2}{P'} r'$, and consequently

$$r = \frac{2\rho}{\rho^2+1}.$$

We have here r representing the parallactic depression corresponding to the zenith distance ZM, where $\rho = \tan \frac{1}{2} ZM$; that is, $ZM = 90°$, $\rho = 1$, and therefore $r = 1$; but for $ZM = 90°$ the parallactic depression is $= P'$; that is, the scale of the enlarged representation of the Sun's relative orbit, or say simply the scale of the relative orbit (for on the original scale it was never actually constructed at all) is such that we have P' (= about 60′) represented by the radius of the bounding circle of the projected hemisphere, = 12 inches.

6. The process is, construct the relative orbit on the scale P' = radius of bounding circle: take S for the position at any given instant of the Sun in the relative orbit, and with centre S and radius $= s + \sigma$ (sum of the angular semidiameters, of course on the same scale) describe a circle. The positions A and B of the Sun at the beginning and end of the eclipse respectively are such that this circle just touches the bounding circle externally, viz. the distances of A and B from the centre of the projection are each = radius of bounding circle $+ s + \sigma$. At any intermediate instant the circle, radius $s + \sigma$, lies wholly or partly within the bounding circle; in the latter case we attend only to the arc thereof which lies within the bounding circle. Taking then Q any point whatever on the circle or arc in question, we join Q with the centre M of the projection, and produce this line through M to a point Z, such that the distances MQ, MZ, being r, ρ respectively, we have as above

$$r = \frac{2\rho}{\rho^2+1},$$

or, what is the same thing, writing θ in place of z, and regarding this angle θ as a variable parameter, the relation between r, ρ, may be expressed by means of the two equations, $\rho = \tan \frac{1}{2}\theta$, $r = \sin\theta$.

7. Practically the construction may be performed very easily by means of a straight edge twenty-four inches long, graduated from the centre, one half of it for the values of r, and the other half for the corresponding values of ρ (that is, the first half is graduated for $\sin\theta$, and the second half for $\tan\frac{1}{2}\theta$): we have thus, corresponding to the circle or arc of circle which is the locus of Q, a closed curve, or arc thereof terminated each way at the bounding circle, for the locus of Z: which curve or arc of a curve is the penumbral curve on the Earth's surface for the position S of the Sun in the relative orbit.

8. The north pole of the Earth occupies in the projection a given position, viz. it is situate on a given radius at a distance $=\tan\frac{1}{2}$ Moon's N.P.D.; which N.P.D. may be considered as being throughout the eclipse constant, and equal to its value at the middle of the eclipse. But in order to arrive at the geographical signification of the figure it is necessary to lay down on the projection the position of the meridian of Greenwich; which position, it will be remembered, varies according to the position of S. Supposing this done, we could of course (at least theoretically) draw the whole series of meridians and parallels, and thereby determine the latitudes and longitudes of the several points of the penumbral curve, or (if need is) transfer it to a different projection of the Earth's surface. The actual description of the meridians and parallels would, however, be very laborious, and fortunately it can be avoided by means of a single blank projection and a slight modification of the foregoing process, as will be explained.

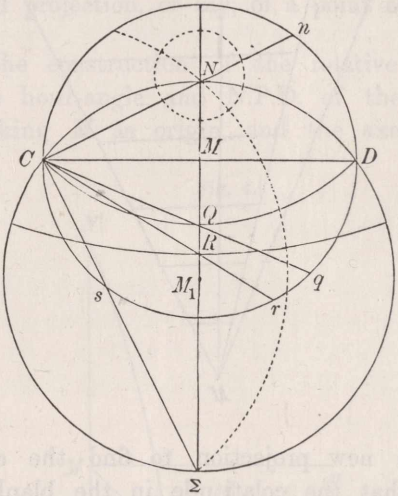
9. But before considering how this is, it is proper to remark that constructing as above a figure of the penumbral curves corresponding to the several positions of the Sun: by what precedes these different curves may indeed be considered as belonging to the same position of the north pole in the projection, but they belong to different positions of the meridian of Greenwich; and thus they do not constitute a representation of the penumbral curves each in its proper terrestrial position, but only a representation in which the penumbral curves are affected each of them by a different displacement in longitude.

Article Nos. 10 to 13. *Modification in order to the Applicability of a Single Blank Projection.*

10. Imagine a stereographic projection of the meridians and parallels *on the plane of a meridian*, radius of this meridian, that is of the bounding circle of the projected hemisphere, being $=12$ inches as before; and the poles N, Σ being of course opposite points on the circumference of the bounding circle—the meridians and parallels are, however, to be produced outside the bounding circle; say this is the "blank projection," and let its centre be denoted by M_1. Then, if at any point M on the radius MN, we draw the chord CD at right angles to M_1N, and on CD as diameter describe a circle, this will cut out from the blank projection a new projection having the last-mentioned circle for its bounding circle, and in which N is the north pole; viz. the meridians of the blank projection will be meridians, and the parallels of the blank projection will be parallels, in this new projection. And, moreover, if the longitudes are reckoned from the meridian NMM_1, then the meridian of a given longitude in the blank projection will in the new projection be the meridian of the *same*

longitude—but the parallel of a given colatitude c in the blank projection will, in the new projection, be the parallel of a different colatitude c',—the relation of c, c' being, however, a very simple one, as presently explained.

Fig. 2.

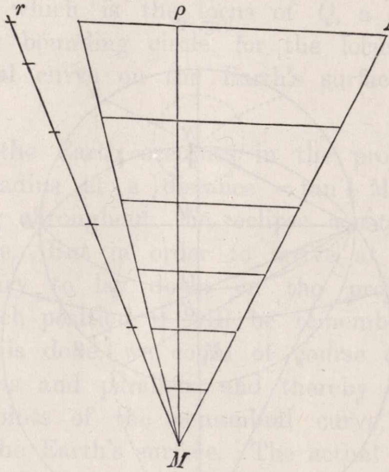


11. The blank projection thus at once gives a projection in which the north pole N has any assumed position whatever; and it is easy to see that in order that its distance MN from the centre of the projection may represent a given angle Δ, we have only to take $M_1M = \cos\Delta$ (that is $= 12$ inches $\times \cos\Delta$), the corresponding value of MC being $MC = \sin\Delta$ (that is $= 12$ inches $\times \sin\Delta$). Hence Δ denoting the Moon's N.P.D. at the middle of the eclipse, we can by means of the blank projection construct a projection such as that above referred to, only the radius of its bounding circle, instead of being unity (12 inches), is in the reduced ratio of $1 : \sin\Delta$.

12. The figure of the penumbral curves as originally constructed requires, therefore, to be reduced in the ratio $1 : \sin\Delta$, viz. each of the distances from the centre M should be reduced in this ratio; this could of course be done easily enough with a pair of proportional compasses; but by means of a different graduation of the straight edge we may, *in the first instance*, construct the penumbral curves on the proper reduced scale; viz. assuming that we have on the proper scale a proportional-scale figure such as is here shown, the line Mr ($= 12$ inches) being graduated for $\sin\theta$, and the line MA (also $= 12$ inches) for $\tan\frac{1}{2}\theta$, and a set of parallel lines being drawn through the last-mentioned graduations—then taking the distance $M\rho = \sin\Delta$, that is $= 12$ inches $\times \sin\Delta$, and drawing the line $M\rho$, this line will, it is clear, be graduated for $\sin\Delta\tan\frac{1}{2}\theta$: so that we may from the figure graduate the straight edge, the one half of it by means of the line Mr, and the other half of it by means of the line $M\rho$; and with the straight edge thus graduated, at once lay down the penumbral curve on the scale now in question. And we thus obtain a figure containing as well

the penumbral curves, as the meridians and parallels which serve to fix their terrestrial position.

Fig. 3.



13. It remains in the new projection to find the colatitude belonging to any given parallel. Supposing that the colatitude in the blank projection is $=c'$, then it may be shown that the colatitude c of the same parallel in the reduced projection is given by means of the equation

$$\tan \tfrac{1}{2} c = \cot \tfrac{1}{2} \Delta \tan \tfrac{1}{2} c',$$

from which c might be calculated numerically: but the required value may also be obtained graphically. In fact, considering the parallel which cuts $N\Sigma$ (see fig. 2) in a point R, then, if by lines drawn from C as a centre we project N, R, Σ, on the circumference of the bounding circle of the new projection—say the projections of these points are n, r, s, respectively, the arc ns is a semicircle, and the arcs nr, sr, are respectively the N.P.D. and the S.P.D. of the parallel in question. It may be added that in the new projection the equator is represented by the parallel through the points C, D; so that if this cuts $N\Sigma$ in Q, and the point Q be in like manner projected on the bounding circle—say its projection is q, then the arcs nq, sq, will be each of them a quadrant, and the arc qr will be the latitude of the parallel in question.

Article Nos. 14 to 18. *As to the Construction of the Relative Orbits.*

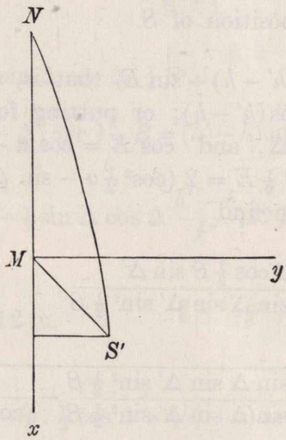
14. It is convenient to notice that if e, e′, be the values of the equation of time at the preceding and following Greenwich Mean Noons (viz. e or e′ = G.M.T. of apparent Noon) then that the Sun's hour-angle E. of Greenwich at the Greenwich mean time t is

$$h' = \mathrm{e} + t\left(1 + \frac{\mathrm{e}' - \mathrm{e}}{24^{\mathrm{h}}}\right)$$

and that if a, a', are the R.A.'s of the Moon and Sun respectively, then $h' - h = a' - a$, which is also of the form $A + Bt$. In the reduced projection, the Moon is always at the centre M; by means of the values of $h' - h$ we lay down at any instant the Sun's position in R.A. and then by means of the values of h', the position of the meridian of Greenwich; and we thus at any instant read off the terrestrial longitude of any point of the reduced projection, or say, of a point on the penumbral curve.

15. With regard to the construction of the relative orbit, it is to be observed that if at any instant the hour-angle and N.P.D. of the Moon are h, Δ, and those of the Sun, h', Δ', then taking M as origin, and the axes Mx, My, in the direction

Fig. 4.



of NM produced, and perpendicular hereto to the right (or eastwards), then the rectangular coordinates of S' are approximately $x = \frac{1}{2}(\Delta' - \Delta)$, $y = \frac{1}{2}(h' - h)\sin\Delta$, where $h' - h$ is equal to the difference of R.A. of the Sun and Moon. Hence, in the adopted relative orbit, the coordinates of S would be

$$x = \frac{\Delta' - \Delta}{P'}\ 12 \text{ in.} \qquad y = \frac{h' - h}{P'} \sin\Delta \,.\, 12 \text{ in.}$$

where, P' being reckoned in minutes, $\Delta' - \Delta$ and $h' - h$ are also reckoned in minutes.

16. Moreover, Δ may be considered as constant during the eclipse: and the relative orbit, assumed to be a straight line, will be determined by means of two points thereof; viz. knowing the values of $\Delta' - \Delta$, and $h' - h$ at about the time of the beginning and at about the time of the end of the eclipse, we construct by these formulæ two points of the orbit, and joining them by a straight line, we have the orbit. Also the position at any instant of the Sun in this relative orbit will be obtained by considering its motion therein as being uniform. I think there is no advantage in the adoption of a more accurate construction: for although we may for any given instant use the accurate values of h, Δ, h', Δ', and so construct the position in the relative orbit, and the corresponding penumbral curve, yet if in the deter-

mination of the geographical significance thereof, we were to use for each curve a different value of Δ, the simplicity of the construction would disappear; and it is, moreover, doubtful whether the trifling corrections would not be within the limits of the necessary errors of the drawing.

17. But if MS' be $=E$, and $\angle xMS'=\theta$, the accurate values for the coordinates of S' are $x=\tan\frac{1}{2}E.\cos\theta$, $y=\tan\frac{1}{2}E.\sin\theta$, and the values for the coordinates of S are $x=\frac{2}{P'.\text{arc }1'}\tan\frac{1}{2}E.\cos\theta.12\text{ in.}$, $y=\frac{2}{P'.\text{arc }1'}\tan\frac{1}{2}E.\sin\theta.12\text{ in.}$, where P' is still reckoned in minutes, and of course $\text{arc }1'=\frac{\pi}{10800}$. As the scale is considerable, it is worth while to inquire whether the employment of the accurate formulæ would produce an appreciable difference in the position of S.

We have $\sin\theta\div\sin\Delta'=\sin(h'-h)\div\sin E$, that is, $\sin E\sin\theta=\sin(h'-h)\sin\Delta'$, and $\cos E=\cos\Delta\cos\Delta'+\sin\Delta\sin\Delta'\cos(h'-h)$; or putting for shortness $\Delta'-\Delta=\alpha$, $h'-h=\beta$, we have $\sin E\sin\theta=\sin\beta\sin\Delta'$, and $\cos E=\cos\alpha-\sin\Delta\sin\Delta'.2\sin^2\frac{1}{2}\beta$. Hence, attending to the equations $\cos^2\frac{1}{2}E=2(\cos^2\frac{1}{2}\alpha-\sin\Delta\sin\Delta'\sin^2\frac{1}{2}\beta)$ and $\sin^2\frac{1}{2}E=2(\sin^2\frac{1}{2}\alpha+\sin\Delta\sin\Delta'\sin^2\frac{1}{2}\beta)$, we find

$$\tan\tfrac{1}{2}E\sin\theta=\frac{\sin\frac{1}{2}\beta\cos\frac{1}{2}\beta\sin\Delta'}{\cos^2\frac{1}{2}\alpha-\sin\Delta\sin\Delta'\sin^2\frac{1}{2}\beta},$$

and

$$\tan\tfrac{1}{2}E\cos\theta=\sqrt{\frac{\sin^2\frac{1}{2}\alpha+\sin\Delta\sin\Delta'\sin^2\frac{1}{2}\beta}{\cos^2\frac{1}{2}\alpha-\sin\Delta\sin\Delta'\sin^2\frac{1}{2}\beta}-\frac{\sin^2\frac{1}{2}\beta\cos^2\frac{1}{2}\beta\sin^2\Delta'}{(\cos^2\frac{1}{2}\alpha-\sin\Delta\sin\Delta'\sin^2\frac{1}{2}\beta)^2}},$$

whence, considering α, β as small quantities of the same order, and neglecting terms of the third order, we have $\tan\frac{1}{2}E\sin\theta=\sin\frac{1}{2}\beta\cos\frac{1}{2}\beta\sin\Delta'$, or what is the same thing, $=\sin\frac{1}{2}\beta\sin\Delta$, or finally, $=\frac{1}{2}\beta\sin\Delta$, that is $\frac{1}{2}(h'-h)\sin\Delta$, which is the foregoing approximate value, and thus in the adopted orbit $y=\frac{h'-h}{P'}\,12\text{ in.}=\text{approx. value}$. As regards the expression for $\tan\frac{1}{2}E\cos\theta$, writing for a moment $\Omega=\sec^2\frac{1}{2}\alpha\sin\Delta\sin\Delta'\sin^2\frac{1}{2}\beta$, the quantity under the radical sign is

$$\frac{\tan^2\frac{1}{2}\alpha+\Omega}{1-\Omega}-\frac{\sin^2\frac{1}{2}\beta\cos^2\frac{1}{2}\beta\sin^2\Delta'}{\cos^4\frac{1}{2}\alpha.(1-\Omega)^2},$$

and, taking this to the third order, it is

$$=\tan^2\alpha+\Omega(1+\tan^2\tfrac{1}{2}\alpha)-\frac{\sin^2\frac{1}{2}\beta\cos^2\frac{1}{2}\beta\sin^2\Delta'}{\cos^4\frac{1}{2}\alpha},$$

which, substituting for Ω its value, is

$$=\tan^2\tfrac{1}{2}\alpha+\frac{\sin^2\frac{1}{2}\beta\sin\Delta'}{\cos^4\frac{1}{2}\alpha}(\sin\Delta-\sin\Delta'\cos^2\tfrac{1}{2}\beta),$$

where $\sin\Delta-\sin\Delta'\cos^2\frac{1}{2}\beta=\sin\Delta-\sin(\Delta+\alpha)\cos^2\frac{1}{2}\beta$, or neglecting herein terms of the second order, this is

$$= \sin\Delta - (\sin\Delta + \sin\alpha\cos\Delta)\cos^2\tfrac{1}{2}\beta,$$

$$= -\sin\alpha\cos\Delta, = -2\tan\tfrac{1}{2}\alpha\cos^2\tfrac{1}{2}\alpha\cos\Delta,$$

so that to the third order the quantity under the radical sign is

$$= \tan^2\tfrac{1}{2}\alpha - \frac{2\tan\frac{1}{2}\alpha\sin^2\frac{1}{2}\beta\sin\Delta\cos\Delta}{\cos^2\frac{1}{2}\alpha};$$

and to the second order, that is finally neglecting terms of the third order,

$$\tan\tfrac{1}{2}E\cos\theta = \tan\tfrac{1}{2}\alpha - \frac{\sin^2\frac{1}{2}\beta\sin\Delta\cos\Delta}{\cos^2\frac{1}{2}\alpha},$$

or, what is the same thing,

$$= \tfrac{1}{2}\alpha - \sin\Delta\cos\Delta\,.\,\tfrac{1}{4}\beta^2.$$

18. Hence, writing $\alpha = (\Delta' - \Delta)\operatorname{arc} 1'$, $\beta = (h' - h)\operatorname{arc} 1'$, and passing to the adopted orbit, we have

$$x = \frac{\Delta' - \Delta}{P'}\,12\text{ in.} - \tfrac{1}{2}\sin\Delta\cos\Delta\,\frac{h' - h}{P'}\,12\text{ in.} \times (h' - h)\operatorname{arc} 1',$$

viz. putting

$$y = \frac{h' - h}{P'}\sin\Delta\,.\,12\text{ in.}$$

we have

$$x = \frac{\Delta' - \Delta}{P'}\,12\text{ in.} - y\,.\,\tfrac{1}{2}\cos\Delta \times (h' - h)\operatorname{arc} 1',$$

or say

$$= \frac{\Delta' - \Delta}{P'}\,12\text{ in.} - y\,.\,\tfrac{1}{2}\cos\Delta\,(h' - h)\,\frac{\pi}{10800}.$$

The value of the second term may amount to about $\frac{1}{10}$ of an inch, and thus be sensible, but there is no difficulty in taking account of it.

Article No. 19. *As to the Equation* $r = \dfrac{2\rho}{\rho^2 + 1}$.

19. It may be remarked that the equation $r = \dfrac{2\rho}{\rho^2 + 1}$, which served for the graduation of the straight edge, was in effect obtained from the equations

$$r = \frac{2}{P'}\tan\tfrac{1}{2}E,\quad P'\sin z = \sin E,\quad \rho = \tan\tfrac{1}{2}z$$

by assuming therein $\tan\frac{1}{2}E = \frac{1}{2}E$ and $\sin E = E$ respectively. But the elimination of E and z can be effected without this assumption, viz. we have $\sin E = \dfrac{2\tan\frac{1}{2}E}{1 + \tan^2\frac{1}{2}E} = \dfrac{P'r}{1 + \frac{1}{4}P'^2 r^2}$, and then as before, $\sin z = \dfrac{2\tan\frac{1}{2}z}{1+\tan^2\frac{1}{2}z} = \dfrac{2\rho}{1+\rho^2}$, whence the relation between r and ρ is found to be

$$\frac{r}{1+\frac{1}{4}P'^2r^2} = \frac{2\rho}{1+\rho^2},$$

which however assumes that P' is reckoned in parts of the radius; reckoning it as before in minutes, we must, instead of P', write $P' \operatorname{arc} 1' = \dfrac{P'\pi}{10800}$, viz. the numerical value is about $\frac{1}{60}$, and taking it to be this number, the formula is

$$\frac{r}{1+\dfrac{1}{14400}r^2} = \frac{2\rho}{1+\rho^2},$$

where r, ρ are reckoned in parts of the radius ($=12$ inches). Supposing that r_1 is calculated from the formula $r_1 = \dfrac{2\rho}{1+\rho^2}$, then we have very nearly $r = r_1\left(1+\dfrac{r_1^2}{14400}\right)$, and r_1 being $=1$ at most, the correction is inappreciable: if however this were not the case, the more accurate formula might have been used; the only difference being that the making of the graduation would have been more laborious.

Article No. 20. *Remark as to the Geometrical Theory of the Projection of the Penumbral Curve.*

20. The stereographic projection of the penumbral curve on the Earth's surface (assumed to be spherical) is, as I have elsewhere shown, a bicircular quartic. It may be shown that the stereographic projection, as given by the foregoing approximate method, is a bicircular quartic: we have, in fact a circle, the equation of which in the polar coordinates r, θ is

$$(r\cos\theta - \alpha)^2 + r^2\sin^2\theta = \beta^2,$$

and where (θ being unaltered) r is changed into ρ, where $r = \dfrac{2\rho}{\rho^2+1}$, that is $\dfrac{1}{r} = \frac{1}{2}\left(\rho + \dfrac{1}{\rho}\right)$. The equation of the circle is

$$r^2 - 2\alpha r\cos\theta + \alpha^2 - \beta^2 = 0,$$

or say

$$1 - \frac{2\alpha\cos\theta}{r} + \frac{\alpha^2-\beta^2}{r^2} = 0,$$

and the transformed equation is therefore

$$1 - \alpha\cos\theta\left(\rho+\frac{1}{\rho}\right) + \tfrac{1}{4}(\alpha^2-\beta^2)\left(\rho+\frac{1}{\rho}\right)^2 = 0,$$

that is

$$(\alpha^2-\beta^2)(\rho^2+1)^2 - 4\alpha\cos\theta\rho(\rho^2+1) + 4\rho^2 = 0,$$

or in rectangular coordinates

$$\rho^4 + 2\rho^2 + 1 - \frac{4\alpha}{\alpha^2-\beta^2}x(\rho^2+1) + \frac{4}{\alpha^2-\beta^2}\rho^2 = 0,$$

that is

$$\rho^4 - \frac{4\alpha}{\alpha^2 - \beta^2} x\rho^2 + \left(2 + \frac{4}{\alpha^2 - \beta^2}\right)\rho^2 - \frac{4\alpha}{\alpha^2 - \beta^2} x + 1 = 0,$$

where $\rho^2 = x^2 + y^2$; the form of the equation shows that the curve is a bicircular quartic. Writing for shortness $\frac{2}{\alpha^2 - \beta^2} = m$, the equation is

$$\rho^4 - 2m\alpha x\rho^2 + (2 + 2m)\rho^2 + 1 - 2m\alpha x = 0,$$

that is

$$\{\rho^2 - m(\alpha x - 1) + 1\}^2 - m^2(\alpha x - 1)^2 - 2m = 0,$$

or, what is the same thing,

$$\{(x - \tfrac{1}{2}m\alpha)^2 + y^2 - \tfrac{1}{4}m^2\alpha^2 + m + 1\}^2 - m^2(\alpha x - 1)^2 - 2m = 0,$$

which putting $x + \frac{1}{2}m\alpha$ for x is

$$(x^2 + y^2 - \tfrac{1}{4}m^2\alpha^2 + m + 1)^2 - m^2(\alpha x + \tfrac{1}{2}m\alpha^2 - 1)^2 - 2m = 0,$$

viz. the terms of the fourth order being $(x^2 + y^2)^2$, and there being no terms of the third order, the curve represented by this equation is a bicircular quartic.

Article Nos. 21 to 30. *Practical Details and Application to Eclipse of December* 21–22, 1870.

21. There are some practical details which it is proper to explain, using to fix the ideas the eclipse of December 21–22, 1870: the constant value of Δ (see *infrà*) is taken to be $+90° + 22° 35'$[1].

I have a blank projection (radius 12 in. as above) with the meridians and parallels each at intervals of 5°. And also another blank form which has on it merely a circle, radius 12 in., graduated as to one quadrant thereof with lines about $1\frac{1}{2}$ in. long, inwards towards the centre. It contains also in a corner the foregoing proportional-scale figure.

22. On the blank projection I measure off, downwards from the centre, a distance $M_1M = 12 \sin 22° 35' (= 4{\cdot}61)$, distances all in inches; and then with the centre M and radius $MC = 12 \cos 22° 35' (= 11{\cdot}08)$, describe a circle which is the bounding circle of the reduced projection. With this same radius I describe on the second form, concentric with the 12-inch circle and above the horizontal diameter thereof, a semicircle: and, cutting out the included area, replace it with tracing cloth. The form thus prepared is placed over the blank projection, so that the semicircle shall coincide with the corresponding semicircle on the blank projection, and the two sheets are fixed together by their lower edges, and by folding down the remaining sides. We have thus the upper half of the reduced projection, represented by the semicircle, with the

[1] See Plate, which exhibits in dotted lines the blank projection under the other blank form; the part within the red semicircle, as seen through the tracing cloth, the rest really hidden.

meridians and parallels marked out thereon by lines seen through the tracing cloth. See the Plate; the dotted line shows a paper scale afterwards affixed to the second form or upper sheet. Observe that so far the only eclipse-datum made use of is the value $\Delta = 90° + 22° \; 35'$.

23. We have for the eclipse in question, taking t for the G.M.T. in hours, positive or negative according as the time is after or before G.M. Noon, Dec. 22, and h' also in hours,

$$h' = 0^{\text{h}}{\cdot}02 + t \cdot 9996,$$

and then taking the values of α, α', Δ, Δ' from the N.A. we have as follows:

G.M.T. 1870, Dec.	$h'+2^{\text{h}}=$	$\alpha=$	$\alpha'=$	$\Delta=90°+$	$\Delta'=90°+$	$h+2^{\text{h}}=$ $h'+2^{\text{h}}+\alpha'-\alpha$	$\Delta'-\Delta$ in Minutes of Arc.	$h'-h$ in ditto.
d h	h m s	h m s	h m s	° ′ ″	° ′ ″	h m s	′	′
21 22	0 1 13·42	17 56 1·84	18 1 48·72	22 27 8·5	23 27 17·1	0 ,7 0·30	60·143	− 86·62
22 3	5 1 7·37	18 9 26·74	18 2 44·27	22 43 12·5	23 27 13·9	4 54 24·90	44·023	+ 100·63

and moreover

$$\begin{array}{lrl} \text{Moon's Parallax} & \sigma' = & 60' \; 38''{\cdot}6 \\ \text{Sun's ditto} & \pi' = & \; 9{\cdot}1 \\ \hline & \sigma' - \pi' = & 60 \;\; 29{\cdot}5 = 60'{\cdot}49 \\ & P' & = 60{\cdot}39 \\ \text{Moon's Semidiam.} & s = & 16' \; 33''{\cdot}2 \\ \text{Sun's ditto} & s' = & 16 \;\; 17{\cdot}9 \\ \hline & s + s' = & 32 \;\; 51 \;\; {\cdot}2 = 32'{\cdot}85 \end{array}$$

whence

$$\frac{s+s'}{P'} \, 12 \text{ in.} = \quad 6{\cdot}53$$

viz. this is the radius of the circles used in the construction of the penumbral curves.

24. We have for x, y the formulæ

$$x = \frac{\Delta' - \Delta}{P'} \, 12 \text{ in.} + y\,(h' - h)\, 0{\cdot}00006,$$

$$y = \frac{h' - h}{P'} \, 12 \text{ in.} \times \sin \Delta,$$

viz. I find Dec. 21, 22^{h},

$$x = 11{\cdot}95 - {\cdot}06 = \quad 11{\cdot}89,$$

$$y \qquad\qquad\qquad - 15{\cdot}80,$$

and Dec. 22—3^h,

$$x = 8{\cdot}75 + {\cdot}11 = 8{\cdot}85,$$

$$y = +18{\cdot}45,$$

where I have taken account of the small corrections to the approximate values of x: it may be added that, using the conjunction-value $52' \, 9''{\cdot}4$ of $\Delta' - \Delta$, we have at conjunction,

$$x = 10{\cdot}36, \quad y = 0.$$

25. We thus lay down on the relative orbit the two points 22^h and 3^h, and the point of conjunction or intersection with the axis of x; the three points are found to be sensibly in a straight line: the distance between the extreme points is about 34 inches, representing 5 hours, so that the scale is nearly 7 inches to an hour: the line is then graduated to quarters of an hour. We then, by means of the distance $12 + 6{\cdot}53 = 18{\cdot}53$, mark off on the relative orbit, the points B, E, which correspond to the beginning and end of the eclipse respectively: the times as read off from my figure, and compared with the true times given in the N. A. are

	from figure	N. A.
Beginning	22^h 12^m·5	22 13·6
End	2 40 ·5	2 41·1

26. With centre B describing a circle radius 6·53 this will of course just touch the 12-inch circle, and the penumbral curve will be a mere point, viz., this is the point B' on the bounding circle, opposite to the point of contact. And so with centre E describing a circle of the same radius 6·53, that will just touch the 12-inch circle, and the penumbral curve will be a mere point, viz., the point E' on the bounding circle, opposite the point of contact.

27. I draw the circles corresponding to the times 22^h 30^m, 45^m, 23^h 0^m, viz., so much of each as lies within the 12-inch circle. Each of these is then transformed into a penumbral curve, drawn in the upper semicircle on the tracing cloth. For this purpose we construct a straight edge of paper, the one half graduated for $12 \sin \theta$, the other half for $11{\cdot}08 \tan \frac{1}{2} \theta$, by means of the proportional-scale figure, as already explained: $\theta = 0°$ to $90°$ at intervals of $5°$, is quite sufficient; the points on any particular penumbral curve are laid down in pairs with the utmost facility, and the curve is traced by hand from 4 or 5 pairs of points.

28. We then graduate for latitude; viz., we see through the tracing cloth, the equator cutting the vertical radius in Q, and a parallel cutting the same radius, say in R; drawing lines from C, we refer these to the points q, r on the bounding circle, viz., on the quadrant thereof which is graduated by means of the graduation-lines of the 12-inch circle; and we thus read off the latitude of the parallel in question; this latitude is then marked for each parallel on the vertical radius from Q up to the bounding circle, viz., not on the tracing cloth, but on the paper affix; and we then on this same radius (on the paper affix) interpolate the positions where this would be intersected by the parallels for the colatitudes, 5°, 10°, 15°, &c. Or

(what is perhaps better) we may without marking the latitudes of the parallels of the blank form, construct directly the last-mentioned graduations; viz., marking off on the bounding circle from the point q, equal intervals of 5°, and from any such mark drawing to C, a line to meet the vertical radius, the point of intersection is the point belonging to the parallel, latitude equal to the corresponding multiple of 5°.

29. Finally, we must (not on the tracing cloth but on the paper affix) graduate an arc of the equator for the position of the meridian of Greenwich, that is for h. We have

$$\text{At } 22^{h} \quad h = -2^{h} + 0^{h}\; 7^{m}\; 0^{s}{\cdot}30 = -1^{h}\; 52^{m}\; 52^{s}{\cdot}30 = -28^{\circ}\; 13'{\cdot}08$$

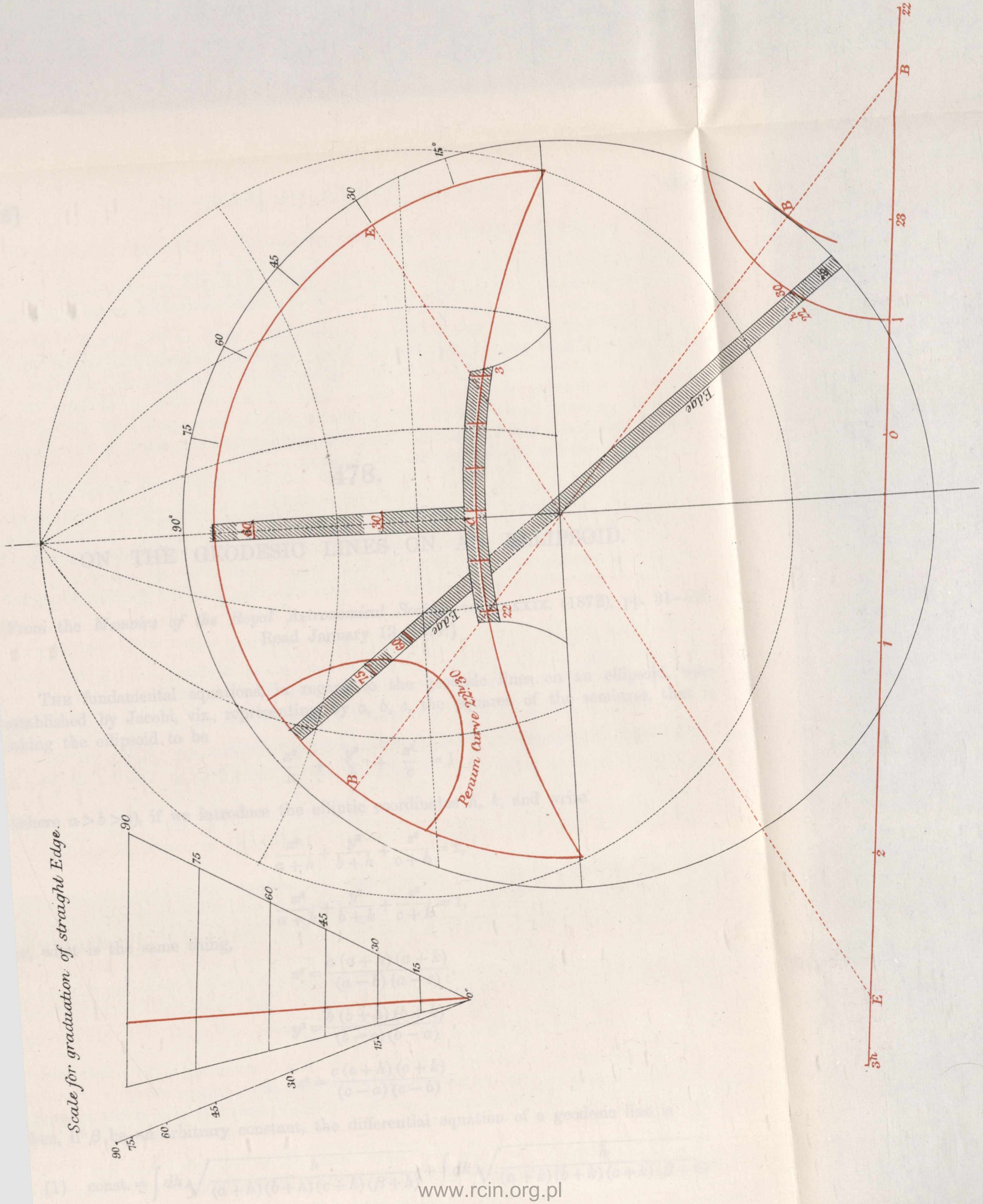
$$\text{At } 3^{h} \quad h = -2^{h} + 4^{h}\; 54^{m}\; 24^{s}{\cdot}90 = +2^{h}\; 54^{m}\; 24^{s}{\cdot}90 = +43^{\circ}\; 36'{\cdot}20$$

The equator is already graduated in longitude by means of the meridians of the blank projection: hence we lay down the marks for 22^{h} and 3^{h} in the positions belonging to $-28^{\circ}\,13'$, and $+43^{\circ}\,36'$ respectively. And then since the interval of 5 hours answers to $71^{\circ}\,49'$, that of 1 hour will answer to $14^{\circ}\,22'$, so that, measuring off these intervals of longitude, we have the marks for the intermediate times 23^{h}, 0^{h}, 1^{h}, 2^{h}; or it might be proper to find in this way the marks corresponding to each interval of 20^{m} of time, answering to about 5° of longitude; the further subdivisions would be proportional to the intervals of time.

30. I have in this way read off the positions of the points B' and E' belonging to the beginning and end of the eclipse; the values, as compared with the true ones, are

		From Figure	N. A.
		°	° ′
B'	latitude N.	34	35 37
	longitude W.	47	45 44
E'	latitude N.	26	26 5
	longitude W.	$38\frac{1}{2}$	37 16

I remark that my figure, although drawn carefully, is not drawn with anything like the degree of accuracy which would be easily attainable; and I think that far better results might be obtained. I merely from a scale lay down tenths and estimate hundredths of an inch, but certainly fiftieths might be laid down from a scale.


Scale for graduation of straight Edge
Edge
Edge
Penum Curve 22h.30
E
B