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ON THE EXPRESSION OF DELAUNAY'S l, g, h, IN TERMS OF HIS FINALLY ADOPTED CONSTANTS.

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We have in Delaunay's lunar theory,

l, the mean anomaly of the Moon,

g, the mean distance of perigee from ascending node,

h, the mean longitude of ascending node,

quantities which vary directly as the time, the coefficients of t, or values of $\frac{dl}{dt}$, $\frac{dg}{dt}$, $\frac{dh}{dt}$, being given in his *Théorie du Mouvement de la Lune*, vol. II. pp. 237, 238. But these values are not expressed in terms of his constants a (or n), e, γ, finally adopted as explained p. 800, and it seems very desirable to obtain the expressions of l, g, h, in terms of these finally adopted constants: I have accordingly effected this transformation (which I found less laborious than I had anticipated). It will be convenient to imagine the a, n, e, γ of pp. 237, 238 replaced by A, N, E, Γ respectively. This being so, and writing m for the $\frac{n}{n'}$ of p. 800 we have, p. 800,

$$
\begin{aligned}
A = a \Big\{ 1 &+ \left[-\tfrac{3}{4} m^2 - \tfrac{825}{256} m^3\right] \frac{a^2}{a'^2} \\
&+ \left(-\tfrac{2}{3} + 3\gamma^2 - \tfrac{3}{4} e^2 - e'^2 - 2\gamma^2 + \tfrac{5}{2}\gamma^2 e^2 + \tfrac{9}{2}\gamma^2 e'^2 - \tfrac{1}{16} e^4 - \tfrac{9}{8} e^2 e'^2 - \tfrac{5}{4} e'^4\right) m^2 \\
&+ \left(-\tfrac{9}{4}\gamma^2 - \tfrac{225}{16} e^2 + \tfrac{45}{8}\gamma^4 + \tfrac{81}{2}\gamma^2 e^2 - \tfrac{23}{4}\gamma^2 e'^2 + \tfrac{675}{128} e^4 - \tfrac{825}{16} e^2 e'^2\right) m^3 \\
&+ \left(\tfrac{1705}{288} - \tfrac{1529}{64}\gamma^2 - \tfrac{14639}{256} e^2 + \tfrac{7469}{192} e'^2\right) m^4 \\
&+ \left(\tfrac{787}{48} - \tfrac{9323}{256}\gamma^2 - \tfrac{227555}{1024} e^2 + \tfrac{7083}{32} e'^2\right) m^5 \\
&+ \tfrac{5887}{162} m^6 \\
&+ \tfrac{29809}{432} m^7,
\end{aligned}
$$

and hence calculating N from the formula $N^2A^3 = n^2a^3$, we find

$$\begin{aligned} N = n \Big\{ 1 &+ [\tfrac{9}{8} m^2 + \tfrac{2475}{512} m^3] \frac{a^2}{a'^2} \\ &+ (1 - \tfrac{9}{2}\gamma^2 + \tfrac{9}{8} e^2 + \tfrac{3}{2} e'^2 + 3\gamma^4 - \tfrac{15}{4}\gamma^2 e^2 - \tfrac{27}{4}\gamma^2 e'^2 + \tfrac{3}{32} e^4 + \tfrac{27}{16} e^2 e'^2 + \tfrac{15}{8} e'^4)\, m \\ &+ (\tfrac{27}{8}\gamma^2 + \tfrac{675}{32} e^2 - \tfrac{135}{16}\gamma^4 - \tfrac{243}{4}\gamma^2 e^2 + \tfrac{69}{8}\gamma^2 e'^2 - \tfrac{2025}{256} e^4 + \tfrac{2475}{32} e^2 e'^2)\, m^3 \\ &+ (-\tfrac{515}{64} + \tfrac{3627}{128}\gamma^2 + \tfrac{44877}{512} e^2 - \tfrac{7149}{128} e'^2)\, m^4 \\ &+ (-\tfrac{787}{32} + \tfrac{30849}{512}\gamma^2 + \tfrac{754665}{2048} e^2 - \tfrac{21249}{64} e'^2)\, m^5 \\ &+ (-\tfrac{13183}{192})\, m^6 \\ &+ (-\tfrac{20807}{144})\, m^7, \end{aligned}$$

$= n(1+Q)$ suppose.

The values of E, Γ are given p. 800, but for the present purpose we only require E^2, and Γ^2 to the fifth order, viz. the values of these are at once found to be

$$E^2 = e^2\,(1 + \tfrac{81}{64} m^2 - \tfrac{2595}{128} m^3),$$
$$\Gamma^2 = \gamma^2\,(1 + \tfrac{57}{64} m^2 - \tfrac{129}{128} m^3),$$

whence also $E^4 = e^4$ and $\Gamma^4 = \gamma^4$.

The formulæ of pp. 237—238 now give

$$\begin{aligned} l = nt \Big\{ 1 &+ [-\tfrac{81}{32} m^2 - \tfrac{2475}{512} m^3] \frac{a^2}{a'^2} + Q \\ &+ \left\{ \begin{aligned} &(-\tfrac{7}{4} + \tfrac{21}{2}\gamma^2 - \tfrac{3}{4} e^2 - \tfrac{21}{8} e'^2 + \tfrac{33}{4}\gamma^4 - \tfrac{39}{8}\gamma^2 e^2 + \tfrac{63}{4}\gamma^2 e'^2 - \tfrac{9}{8} e^2 e'^2 - \tfrac{105}{32} e'^4)\, m^2 \\ &+ (\tfrac{1197}{128}\gamma^2 - \tfrac{243}{256} e^2)\, m^4 \\ &+ (-\tfrac{2709}{256}\gamma^2 + \tfrac{7785}{512} e^2)\, m^5 \end{aligned} \right\} (1+Q)^{-1} \\ &+ \left\{ \begin{aligned} &(-\tfrac{225}{32} + \tfrac{81}{4}\gamma^2 - \tfrac{675}{64} e^2 - \tfrac{825}{32} e'^2 - \tfrac{243}{4}\gamma^4 + \tfrac{1863}{32}\gamma^2 e^2 + \tfrac{629}{8}\gamma^2 e'^2 + \tfrac{2025}{256} e^4 - \tfrac{2475}{64} e^2 e'^2)\, m^3 \\ &\quad + (\tfrac{4617}{256}\gamma^2 - \tfrac{54675}{4096} e^2)\, m^5 \end{aligned} \right\} (1+Q)^{-2} \\ &+ (-\tfrac{3265}{128} + \tfrac{3345}{32}\gamma^2 - \tfrac{7089}{256} e^2 - \tfrac{48225}{256} e'^2)\, m^4 (1+Q)^{-3} \\ &+ (-\tfrac{243925}{2048} + \tfrac{175425}{256}\gamma^2 - \tfrac{167835}{2048} e^2 - \tfrac{1502265}{1024} e'^2)\, m^5 (1+Q)^{-4} \\ &+ (-\tfrac{12626759}{24576})\, m^6 (1+Q)^{-5} \\ &+ (-\tfrac{1365131021}{589824})\, m^7 (1+Q)^{-6} \end{aligned}$$

(Observe that writing herein $Q = 0$, and omitting the terms in m^4 and m^5 in the coefficient of $(1+Q)^{-1}$, and the term in m^5 in the coefficient of $(1+Q)^{-2}$, we have the original formula of p. 237)

$$\begin{aligned} g = nt \Big\{ &[\tfrac{45}{16} m^2 + \tfrac{585}{32} m^3] \frac{a^2}{a'^2} \\ &+ \left\{ \begin{aligned} &(\tfrac{3}{2} - \tfrac{15}{2}\gamma^2 + \tfrac{9}{8} e^2 + \tfrac{9}{4} e'^2 - \tfrac{45}{4}\gamma^4 + 15\gamma^2 e^2 - \tfrac{45}{4}\gamma^2 e'^2 - \tfrac{27}{64} e^4 + \tfrac{27}{16} e^2 e'^2 + \tfrac{45}{16} e'^4)\, m^2 \\ &\quad + (-\tfrac{855}{128}\gamma^2 + \tfrac{729}{512} e^2)\, m^4 \\ &\quad + (\tfrac{1935}{256}\gamma^2 - \tfrac{23355}{1024} e^2)\, m^5 \end{aligned} \right\} (1+Q)^{-1} \end{aligned}$$

$$+\left\{\begin{array}{l}(\tfrac{27}{4}-\tfrac{351}{16}\gamma^2-\tfrac{297}{64}e^2+\tfrac{401}{16}e'^2+\tfrac{135}{2}\gamma^4-\tfrac{1053}{32}\gamma^2e^2-\tfrac{1297}{16}\gamma^2e'^2+\tfrac{675}{256}e^4-\tfrac{1079}{64}e^2e'^2)\,m^3\\ \qquad +(-\tfrac{20007}{1024}\gamma^2-\tfrac{24057}{4096}e^2)\,m^5\end{array}\right\}(1+Q)^{-2}$$

$$+(\tfrac{1995}{64}-\tfrac{7989}{64}\gamma^2-\tfrac{9969}{256}e^2+\tfrac{29535}{128}e'^2)\,m^4(1+Q)^{-3}$$

$$+(\tfrac{17709}{128}-\tfrac{376653}{512}\gamma^2-\tfrac{440787}{2048}e^2+\tfrac{883245}{512}e'^2)\,m^5(1+Q)^{-4}$$

$$+\ \tfrac{2431349}{4096}\,m^6(1+Q)^{-5}$$

$$+\ \tfrac{62329307}{24576}\,m^7(1+Q)^{-6}$$

(where writing $Q=0$, and omitting the terms in m^4 and m^5 in the coefficient of $(1+Q)^{-1}$, and the term in m^5 in the coefficient of $(1+Q)^{-2}$, we have the original formula of p. 237). And

$$h=nt\left\{[-\tfrac{45}{32}m^2-\tfrac{1935}{512}m^3]\frac{a^2}{a'^2}\right.$$

$$+\left\{\begin{array}{l}(-\tfrac{3}{4}+\tfrac{3}{2}\gamma^2-\tfrac{3}{2}e^2-\tfrac{9}{8}e'^2-\tfrac{51}{8}\gamma^2e^2+\tfrac{9}{4}\gamma^2e'^2+\tfrac{21}{64}e^2-\tfrac{9}{4}e^2e'^2-\tfrac{45}{32}e'^4)\,m^2\\ \qquad +(\tfrac{171}{128}\gamma^2-\tfrac{243}{128}e^2)\,m^4\\ \qquad +(-\tfrac{387}{256}\gamma^2+\tfrac{7785}{256}e^2)\,m^5\end{array}\right\}(1+Q)^{-1}$$

$$+\left\{\begin{array}{l}(\tfrac{9}{32}-\tfrac{27}{16}e^2-\tfrac{189}{32}\gamma^2+\tfrac{23}{32}e'^2+\tfrac{27}{16}\gamma^4+\tfrac{567}{16}\gamma^2e^2-\tfrac{99}{16}\gamma^2e'^2-\tfrac{675}{256}e^4-\tfrac{349}{16}e^2e'^2)\,m^3\\ \qquad +(-\tfrac{1539}{1024}\gamma^2-\tfrac{15309}{2048}e^2)\,m^5\end{array}\right\}(1+Q)^{-2}$$

$$+(\tfrac{177}{128}-\tfrac{195}{64}\gamma^2-\tfrac{699}{32}e^2+\tfrac{2685}{256}e'^2)\,m^4(1+Q)^{-3}$$

$$+(\tfrac{10949}{2048}-\tfrac{6369}{512}\gamma^2-\tfrac{133839}{1024}e^2+\tfrac{75759}{1024}e'^2)\,m^5(1+Q)^{-4}$$

$$+\ \tfrac{467977}{24576}\,m^6(1+Q)^{-5}$$

$$+\ \tfrac{26983045}{589824}\,m^7(1+Q)^{-6}$$

(where writing $Q=0$, and omitting the terms in m^4 and m^5 in the coefficient of $(1+Q)^{-1}$, and the term in m^5 in the coefficient of $(1+Q)^{-2}$, we have the original formula of p. 238). We hence have

$$\begin{aligned}l&=nt\,\{A+(1+B)\,Q+CQ^2\},\\ &=nt\,\{A+Q+BQ+CQ^2\},\\ g&=nt\,\{A'+B'Q+C'Q^2\},\\ h&=nt\,\{A''+B''Q+C''Q^2\},\end{aligned}$$

where $\left(\text{omitting the terms in } \dfrac{a^2}{a'^2}\right)$

$$\begin{aligned}A=1&+(-\tfrac{7}{4}+\tfrac{21}{2}\gamma^2-\tfrac{3}{4}e^2-\tfrac{21}{8}e'^2+\tfrac{33}{4}\gamma^4-\tfrac{39}{8}\gamma^2e^2+\tfrac{63}{4}\gamma^2e'^2-\tfrac{9}{8}e^2e'^2-\tfrac{109}{32}e'^4)\,m^2\\ &+(-\tfrac{225}{32}+\tfrac{81}{4}\gamma^2-\tfrac{675}{64}e^2-\tfrac{825}{32}e'^2-\tfrac{243}{4}\gamma^4+\tfrac{1863}{32}\gamma^2e^2+\tfrac{629}{8}\gamma^2e'^2+\tfrac{2025}{256}e^4-\tfrac{105}{32}e'^4)\,m^3\\ &+(-\tfrac{3265}{128}+\tfrac{14577}{128}\gamma^2-\tfrac{1833}{64}e^2-\tfrac{48225}{256})\,m^4\\ &+(-\tfrac{243925}{2048}+\tfrac{177333}{256}\gamma^2-\tfrac{328065}{4096}e^2-\tfrac{1502265}{1024}e'^2)\,m^5\\ &+(-\tfrac{12626759}{24576})\,m^6\\ &+(-\tfrac{1365131021}{589824})\,m^7.\end{aligned}$$

$$B = \left(\tfrac{7}{4} - \tfrac{21}{2}\gamma^2 + \tfrac{3}{4}e^2 + \tfrac{21}{8}e'^2\right)m^2$$
$$+ \left(\tfrac{225}{16} - \tfrac{81}{2}\gamma^2 + \tfrac{675}{32}e^2 + \tfrac{825}{16}e'^2\right)m^3$$
$$+ \tfrac{9795}{128}m^4$$
$$+ \tfrac{243925}{512}m^5.$$

$$C = -\tfrac{7}{4}m^2$$
$$- \tfrac{675}{32}m^3.$$

$$A' = \left(\tfrac{3}{2} - \tfrac{15}{2}\gamma^2 + \tfrac{9}{8}e^2 + \tfrac{9}{4}e'^2 - \tfrac{45}{4}\gamma^4 + 15\gamma^2e^2 - \tfrac{45}{4}\gamma^2e'^2 - \tfrac{27}{16}e^4 + \tfrac{27}{16}e^2e'^2 + \tfrac{45}{16}e'^4\right)m^2$$
$$+ \left(\tfrac{27}{4} - \tfrac{351}{16}\gamma^2 - \tfrac{297}{64}e^2 + \tfrac{401}{16}e'^2 + \tfrac{135}{2}\gamma^4 - \tfrac{1053}{32}\gamma^2e^2 - \tfrac{1297}{16}\gamma^2e'^2 + \tfrac{675}{256}e^4 - \tfrac{1079}{64}e^2e'^2\right)m^3$$
$$+ \left(\tfrac{1995}{64} - \tfrac{16833}{128}\gamma^2 - \tfrac{19209}{512}e^2 + \tfrac{29535}{128}e'^2\right)m^4$$
$$+ \left(\tfrac{17709}{128} - \tfrac{765573}{1024}\gamma^2 - \tfrac{999051}{4096}e^2 + \tfrac{883245}{512}e'^2\right)m^5$$
$$+ \tfrac{2431349}{4096}m^6$$
$$+ \tfrac{62329307}{24576}m^7.$$

$$B' = \left(-\tfrac{3}{2} + \tfrac{15}{2}\gamma^2 - \tfrac{9}{8}e^2 - \tfrac{9}{4}e'^2\right)m^2$$
$$+ \left(-\tfrac{27}{2} + \tfrac{351}{8}\gamma^2 + \tfrac{297}{32}e^2 - \tfrac{401}{8}e'^2\right)m^3$$
$$- \tfrac{5985}{64}m^4$$
$$- \tfrac{17709}{32}m^5.$$

$$C' = \tfrac{3}{2}m^2$$
$$+ \tfrac{81}{4}m^3.$$

$$A'' = \left(-\tfrac{3}{4} + \tfrac{3}{2}\gamma^2 - \tfrac{3}{2}e^2 - \tfrac{9}{8}e'^2 - \tfrac{51}{8}\gamma^2e^2 + \tfrac{9}{4}\gamma^2e'^2 + \tfrac{21}{64}e^4 - \tfrac{9}{4}e^2e'^2 - \tfrac{45}{32}e'^4\right)m^2$$
$$+ \left(\tfrac{9}{32} - \tfrac{27}{16}\gamma^2 - \tfrac{189}{32}e^2 + \tfrac{23}{32}e'^2 + \tfrac{27}{16}\gamma^4 + \tfrac{567}{16}\gamma^2e^2 - \tfrac{99}{16}\gamma^2e'^2 - \tfrac{675}{256}e^4 - \tfrac{349}{16}e^2e'^2\right)m^3$$
$$+ \left(\tfrac{177}{128} - \tfrac{219}{128}\gamma^2 - \tfrac{3039}{128}e^2 + \tfrac{2685}{256}e'^2\right)m^4$$
$$+ \left(\tfrac{10949}{2048} - \tfrac{15825}{1024}\gamma^2 - \tfrac{220707}{2048}e^8 + \tfrac{75759}{1024}e'^2\right)m^5$$
$$+ \tfrac{467977}{24576}m^6$$
$$+ \tfrac{26983045}{589824}m^7.$$

$$B'' = +\left(\tfrac{3}{4} - \tfrac{3}{2}\gamma^2 + \tfrac{3}{2}e^2 + \tfrac{9}{8}e'^2\right)m^2$$
$$+ \left(-\tfrac{9}{16} + \tfrac{27}{8}\gamma^2 + \tfrac{189}{16}e^2 - \tfrac{23}{16}e'^2\right)m^3$$
$$- \tfrac{531}{128}m^4$$
$$- \tfrac{10949}{512}m^5.$$

$$C'' = -\tfrac{3}{4}m^2$$
$$+ \tfrac{27}{32}m^3.$$

And in terms BQ, $B'Q$, $B''Q$, we have

$$\begin{aligned} Q = & \ (1 - \tfrac{9}{2}\gamma^2 + \tfrac{9}{8}e^2 + \tfrac{3}{2}e'^2)\, m^2 \\ & + (\tfrac{27}{8}\gamma^2 + \tfrac{675}{32}e^2)\, m^3 \\ & - \tfrac{515}{64} m^4 \\ & - \tfrac{787}{32} m^5, \end{aligned}$$

and in the terms CQ^2, $C'Q^2$, $C''Q^2$, simply $Q^2 = m^4$. Hence finally the required values of l, g, h, are

$$\begin{aligned} l = nt \Big\{ & 1 + [-\tfrac{45}{32}m^2 - \tfrac{7425}{512}m^3]\, \frac{a^2}{a'^2} \\ & + (-\tfrac{3}{4} + 6\gamma^2 + \tfrac{3}{8}e^2 - \tfrac{9}{8}e'^2 + \tfrac{45}{4}\gamma^4 - \tfrac{69}{8}\gamma^2 e^2 + 9\gamma^2 e'^2 + \tfrac{3}{32}e^4 + \tfrac{9}{16}e^2 e'^2 - \tfrac{45}{32}e'^4)\, m^2 \\ & + (-\tfrac{225}{32} + \tfrac{189}{8}\gamma^2 + \tfrac{675}{64}e^2 - \tfrac{825}{32}e'^2 - \tfrac{1107}{16}\gamma^4 - \tfrac{81}{32}\gamma e^2 + \tfrac{349}{4}\gamma^2 e'^2 + \tfrac{2475}{64}e^2 e'^2)\, m^3 \\ & + (-\tfrac{4071}{128} + \tfrac{15852}{128}\gamma^2 + \tfrac{31605}{512}e^2 - \tfrac{61179}{256}e'^2)\, m^4 \\ & + (-\tfrac{265493}{2048} + \tfrac{335403}{512}\gamma^2 + \tfrac{1483665}{4096}e^2 - \tfrac{1767840}{1024}e'^2)\, m^5 \\ & + (-\tfrac{12822631}{24576})\, m^6 \\ & + (-\tfrac{1273925965}{589824})\, m^7, \end{aligned}$$

$$\begin{aligned} g = nt \Big\{ & [\tfrac{45}{16}m^2 + \tfrac{585}{32}m^3]\, \frac{a^2}{a'^2} \\ & + (\tfrac{3}{2} - \tfrac{15}{2}\gamma^2 + \tfrac{9}{8}e^2 + \tfrac{9}{4}e'^2 - \tfrac{45}{4}\gamma^4 + 15\gamma^2 e^2 - \tfrac{45}{4}\gamma^2 e'^2 - \tfrac{27}{64}e^4 + \tfrac{27}{16}e^2 e'^2 + \tfrac{45}{16}e'^4)\, m^2 \\ & + (\tfrac{27}{4} - \tfrac{351}{16}\gamma^2 - \tfrac{297}{64}e^2 + \tfrac{401}{16}e'^2 + \tfrac{135}{2}\gamma^4 - \tfrac{1053}{32}\gamma^2 e^2 - \tfrac{1297}{16}\gamma^2 e'^2 + \tfrac{675}{256}e^4 - \tfrac{1079}{64}e^2 e'^2)\, m^3 \\ & + (\tfrac{1899}{64} - \tfrac{15009}{128}\gamma^2 - \tfrac{20649}{512}e^2 - \tfrac{28959}{128}e'^2)\, m^4 \\ & + (\tfrac{15981}{128} - \tfrac{663621}{1024}\gamma^2 - \tfrac{1152843}{4096}e^2 + \tfrac{847213}{512}e'^2)\, m^5 \\ & + \tfrac{2103893}{4096} m^6 \\ & + \tfrac{52802843}{24576} m^7, \end{aligned}$$

$$\begin{aligned} h = nt \Big\{ & [-\tfrac{45}{32}m^2 - \tfrac{1935}{512}m^3)\, \frac{a^2}{a'^2} \\ & + (-\tfrac{3}{4} + \tfrac{3}{2}\gamma^2 - \tfrac{3}{2}e^2 - \tfrac{9}{8}e'^2 - \tfrac{51}{8}\gamma^2 e^2 + \tfrac{9}{4}\gamma^2 e'^2 + \tfrac{21}{64}e^4 - \tfrac{9}{4}e^2 e'^2 - \tfrac{45}{32}e'^4)\, m^2 \\ & + (\tfrac{9}{32} - \tfrac{27}{16}\gamma^2 - \tfrac{189}{32}e^2 + \tfrac{23}{32}e'^2 + \tfrac{27}{16}\gamma^4 + \tfrac{567}{16}\gamma^2 e^2 - \tfrac{99}{16}\gamma^2 e'^2 - \tfrac{675}{256}e^4 - \tfrac{349}{16}e^2 e'^2)\, m^3 \\ & + (\tfrac{273}{128} - \tfrac{843}{128}\gamma^2 - \tfrac{2739}{128}e^2 + \tfrac{3261}{256}e'^2)\, m^4 \\ & + (\tfrac{9797}{2048} - \tfrac{7185}{1024}\gamma^2 - \tfrac{165411}{2048}e^2 + \tfrac{73423}{1024}e'^2)\, m^5 \\ & + \tfrac{199273}{24576} m^6 \\ & + \tfrac{6657733}{589824} m^7, \end{aligned}$$

which values satisfy, as they should do, the equation $l+g+h=nt$. I recall that the precise signification of the constants is as follows: n is the coefficient of t in the expression of the Moon's longitude in terms of the time, a the corresponding elliptic value of the mean distance ($n^2a^3=$ sum of masses), e the eccentricity, such that in the expression of the longitude the coefficient of the leading term of the equation of the centre has its elliptic value

$$=2e-\tfrac{1}{4}e^3+\tfrac{5}{96}e^5$$

and γ the sine of the half-inclination, such that in the expression of the latitude the coefficient of the leading term has its elliptic value

$$=2\gamma-2\gamma e^2-\tfrac{1}{4}\gamma^5+\tfrac{7}{32}\gamma e^4+\tfrac{1}{4}\gamma^5e^2-\tfrac{5}{144}\gamma e^6$$

n', a' are the mean motion and mean distance of the Sun, $m=\dfrac{n'}{n}$, and e' is the eccentricity of the Sun's orbit, considered as constant.