# 758.

## SOLUTION OF A SENATE-HOUSE PROBLEM.

[From the *Messenger of Mathematics*, vol. XI. (1882), pp. 23—25.]

PROVE that, if $a+b+c=0$ and $x+y+z=0$, then

$$\begin{aligned}&4(ax+by+cz)^3\\&-3(ax+by+cz)(a^2+b^2+c^2)(x^2+y^2+z^2)\\&-2(b-c)(c-a)(a-b)(y-z)(z-x)(x-y)\\&-54abcxyz=0.\end{aligned}$$

I do not know the origin of this identity, nor do I see any very simple way of proving it: that which seems the most straightforward way is to transform the third line, which, omitting the factor $-2$, is

$$\begin{vmatrix}1, & 1, & 1\\ a, & b, & c\\ a^2, & b^2, & c^2\end{vmatrix}\cdot\begin{vmatrix}1, & 1, & 1\\ x, & y, & z\\ x^2, & y^2, & z^2\end{vmatrix},$$

$$=\begin{vmatrix}3, & a+b+c, & a^2+b^2+c^2\\ x+y+z, & ax+by+cz, & a^2x+b^2y+c^2z\\ x^2+y^2+z^2, & ax^2+by^2+cz^2, & a^2x^2+b^2y^2+c^2z^2\end{vmatrix};$$

and therefore when $a+b+c=0$ and $x+y+z=0$, is

$$\begin{aligned}=\;&3(ax+by+cz)(a^2x^2+b^2y^2+c^2z^2)\\&-3(a^2x+b^2y+c^2z)(ax^2+by^2+cz^2)\\&-(ax+by+cz)(a^2+b^2+c^2)(x^2+y^2+z^2);\end{aligned}$$

or, as this may be written,

$$\begin{aligned}
= \quad & 6\,(ax + by + cz)\,(a^2x^2 + b^2y^2 + c^2z^2) \\
- \quad & (ax + by + cz)\,(a^2 + b^2 + c^2)\,(x^2 + y^2 + z^2) \\
- & 3\,(ax + by + cz)\,(a^2x^2 + b^2y^2 + c^2z^2) \\
- & 3\,(a^2x + b^2y + c^2z)\,(ax^2 + by^2 + cz^2).
\end{aligned}$$

Here the third and fourth lines, omitting the factor $-3$, are

$$2\,(a^3x^3 + b^3y^3 + c^3z^3) + (ab^2 + a^2b)\,(xy^2 + x^2y) + (ac^2 + a^2c)\,(xz^2 + x^2z) + (bc^2 + b^2c)\,(yz^2 + y^2z),$$

where, in virtue of the two relations, each of the last three product-terms is $= abcxyz$, and the whole is thus

$$\begin{aligned}
= \quad & 2\,(a^3x^3 + b^3y^3 + c^3z^3) \\
& + 3abcxyz.
\end{aligned}$$

The product of the two determinants is thus

$$\begin{aligned}
= \quad & 6\,(ax + by + cz)\,(a^2x^2 + b^2y^2 + c^2z^2) \\
- \quad & (ax + by + cz)\,(a^2 + b^2 + c^2)\,(x^2 + y^2 + z^2) \\
- & 6\,(a^3x^3 + b^3y^3 + c^3z^3) \\
- & 9\ abcxyz;
\end{aligned}$$

and this being so the identity to be verified is

$$\begin{aligned}
& 4\ \ (ax + by + cz)^3 \\
+(-3 + 2 =) -1 \ \ & (ax + by + cz)\,(a^2 + b^2 + c^2)\,(x^2 + y^2 + z^2) \\
& -12\,(ax + by + cz)\,(a^2x^2 + b^2y^2 + c^2z^2) \\
& +12\,(a^3x^3 + b^3y^3 + c^3z^3) \\
+(18 - 54 =) -36abcxyz \quad & = 0.
\end{aligned}$$

We have here the terms

$$\begin{aligned}
& 12\,(a^3x^3 + b^3y^3 + c^3z^3 - 3abcxyz), \\
= {} & 12\,(ax + by + cz)\,(a^2x^2 + b^2y^2 + c^2z^2 - bcyz - cazx - abxy),
\end{aligned}$$

so that the left-hand side is now divisible by $ax + by + cz$, and throwing out this factor the equation becomes

$$\begin{aligned}
& 4\ \ (ax + by + cz)^2 \\
- \quad & (a^2 + b^2 + c^2)\,(x^2 + y^2 + z^2) \\
- & 12\,(a^2x^2 + b^2y^2 + c^2z^2) \\
+ & 12\,(a^2x^2 + b^2y^2 + c^2z^2 - bcyz - cazx - abxy) = 0;
\end{aligned}$$

or, as this may be written,

$$4(a^2x^2+b^2y^2+c^2z^2-bcyz-cazx-abxy)$$
$$-\quad(a^2+b^2+c^2)(x^2+y^2+z^2)=0,$$

which under the assumed relations $a+b+c=0$, $x+y+z=0$ may be verified without difficulty. It may be remarked that we have identically

$$\begin{aligned}&8(a^2x^2+b^2y^2+c^2z^2-bcyz-cazx-abxy)\\&-2(a^2+b^2+c^2)(x^2+y^2+z^2)\\&=(x+y+z)\left\{\begin{array}{l}x(\ \ 3a^2-\ b^2-\ c^2+2bc-2ca-2ab)\\+y(-\ a^2+3b^2-\ c^2-2bc+2ca-2ab)\\+z(-\ a^2-\ b^2+3c^2-2bc-2ca+2ab)\end{array}\right\}\\&+(a+b+c)\left\{\begin{array}{l}a(\ \ 3x^2-\ y^2-\ z^2+2yz-2zx-2xy)\\+b(-\ x^2+3y^2-\ z^2-2yz+2zx-2xy)\\+c(-\ x^2-\ y^2+3z^2-2yz-2zx+2xy)\end{array}\right\},\end{aligned}$$

which is a more complete form of the last-mentioned theorem.