# 952.

## ON THE KINEMATICS OF A PLANE, AND IN PARTICULAR ON THREE-BAR MOTION: AND ON A CURVE-TRACING MECHANISM.

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THE first part of the present paper, On the Kinematics of a Plane, and on Three-bar Motion, is purely theoretical: the second part contains a brief description of a Curve-tracing Mechanism, which at my suggestion has been constructed by Prof. Ewing in the workshops of the Engineering Laboratory, Cambridge.

### PART I.

1. The theory of the motion of a plane, when two given points thereof describe given curves, has been considered by Mr S. Roberts in his paper, "On the motion of a plane under given conditions," *Proc. Lond. Math. Soc.* t. III. (1871), pp. 286—318, and he has shown that, if for the given curves the order, class, number of nodes, and of cusps, are $(m, n, \delta, \kappa)$ and $(m', n', \delta', \kappa')$ respectively $(n = m^2 - m - 2\delta - 3\kappa,\ n' = m'^2 - m' - 2\delta' - 3\kappa')$, then for the curve described by any fixed point of the plane:

$$\begin{aligned}
&\text{order} &&= 2mm',\\
&\text{class} &&= 2\,(mm' + mn' + nm'),\\
&\text{number of nodes} &&= mm'\,(2mm' - m - m') + 2\,(m\delta' + m'\delta),\\
&\text{number of cusps} &&= 2\,(m\kappa' + m'\kappa);
\end{aligned}$$

but he remarks that these formulæ require modification when the directrices or either of them pass through the circular points at infinity. And he has considered the case where the two directrices become one and the same curve.

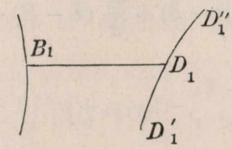
2. It will be convenient to speak of the line joining the two given points as the link; the two given points, say $B$ and $D$, are then the extremities of the link; and I take the length of the link to be $= c$, and the two directrices to be $b$ and $d$; we have thus the link $c = BD$ moving in suchwise that its extremity $B$ describes the curve $b$ of the order $m$, and its extremity $D$ the curve $d$ of the order $m'$: in Mr Roberts' problem, the locus is that described by a point $P$ rigidly connected with the link, or say by a point $P$ the vertex of the triangle $PBD$.

3. The points $B$, $D$ describe of course the directrices $b$, $d$ respectively: taking on $b$ a point $B_1$ at pleasure, then if $B$ be at $B_1$, the corresponding positions of $D$ are the intersections of $d$ by the circle centre $B_1$ and radius $c$, viz. there are thus $2m'$ positions of $D$: and similarly taking on $d$ a point $D_1$ at pleasure, then if $D$ be at $D_1$, the corresponding positions of $B$ are the intersections of $b$ by the circle centre $D_1$ and radius $c$, viz. there are thus $2m$ positions of $B$. The motion thus establishes a $(2m, 2m')$ correspondence between the points of the directrices $b$ and $d$, viz. to a given point on $b$ there correspond $2m'$ points on $d$, and to a given point on $d$ there correspond $2m$ points on $b$. Of course, for a given point on either directrix, the corresponding points on the other directrix may be any or all of them imaginary; and thus it may very well be that for either directrix not the whole curve but only a part or detached parts thereof will be actually described in the course of the motion. In saying that a part is described, we mean described by a continuous motion; say that the point $B$ (the point $D$ remaining always on a part of $d$) is capable of describing continuously a part of $b$; it may very well happen that the point $B$ (the point $D$ remaining always on a different part of $d$) is capable of describing continuously a different part of $b$, but that it is not possible for $B$ to pass from the one to the other of these parts of $b$ without removing $D$ from the one part and placing it on the other part of $d$, and thus that we have on $b$ detached parts each of them continuously described by $B$; and similarly, we may have on $d$ detached parts each of them continuously described by $D$.

4. But dropping for the moment the question of reality, to a given position of $B$ on $b$ there correspond as was mentioned $2m'$ positions of $D$ on $d$, or say $2m'$ positions of the link $c$: in the entire motion of the link it must assume each of these $2m'$ positions, and for each of them the point $B$ comes to assume the position in question on $b$; the directrix $b$ is thus described $2m'$ times, that is, the locus described by $B$ will be the directrix $b$ repeated $2m'$ times, or say a curve of the order $m \times 2m'$, $= 2mm'$. Similarly, the locus described by $D$ will be the directrix $d$ repeated $2m$ times, or say a curve of the order $m' \times 2m$, $= 2mm'$.

5. In general, if $B_1D_1$ be any position of the link and if $B$ moves from $B_1$ along $b$ in a determinate sense, then $D$ will move from $D_1$ along $d$ in a determinate sense; and if $B$ moves from $B_1$ along $b$ in the opposite sense, then also $D$ will move from $D_1$ along $d$ in the opposite sense. Or what is the same thing, we may have $B$ moving in a determinate sense through $B_1$, and $D$ moving in a determinate sense through $D_1$; and reversing the sense of $B$'s motion, we reverse also the sense of $D$'s motion. But there are certain critical positions of the link, viz. we have a critical position when

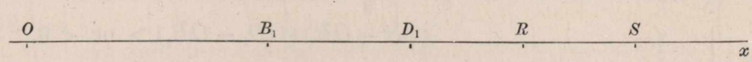
the link is a normal at $B_1$ to the directrix $b$, or a normal at $D_1$ to the directrix $d$. Say first the link is a normal at $B_1$ to the directrix $b$. The infinitesimal element at $B_1$ may be regarded as a straight line at right angles to the link; hence if for a moment $D_1$ is regarded as a fixed point, the link may rotate in either direction round $D_1$, that is, $B$ may move from $B_1$ along $b$ in either of the two opposite senses, say $B_1$ is a "two-way point." But if on $d$ we take on opposite sides of $D_1$ the consecutive points $D_1'$ and $D_1''$, say $D_1'D_1$ cuts $D_1B_1$ at an acute angle and $D_1''D_1$ cuts it at an




obtuse angle, then $D_1'$ will be nearer to $b$ than was $D_1$, and thus the circle centre $D_1'$ and radius $c$ will cut $b$ in two real points $B_1'$ and $B_1''$ near to and on opposite sides of $B_1$; or as $D$ moves to $D_1'$, $B$ will move from $B_1$ indifferently to $B_1'$ or $B_1''$. Contrariwise, $D_1''$ is further from $b$ than was $D_1$, and thus the circle centre $D_1''$ and radius $c$, will not meet $b$ in any real point near to $B_1$, and hence $D$ is incapable of moving from $D_1$ in the sense $D_1D_1''$. Or what is the same thing, the described portion of $d$, which includes a point $D_1'$, will terminate at $D_1$, or say $D_1$ is a "summit" on the directrix $d$. We have thus a summit on $d$, corresponding to the two-way point on $b$. And of course in like manner, if the link is a normal at $D_1$ to the directrix $d$, then $D_1$ is a two-way point on $d$, and the corresponding point $B_1$ is a summit on $b$.

6. If the link is at the same time a normal at $B_1$ to $b$ and at $D_1$ to $d$, then each of the points $B_1$, $D_1$ is a two-way point and also a summit; or more accurately, each of them is a two-way point and also a pair of coincident summits.

But the case requires further investigation. Considering the position $B_1D_1$ as given, we may take the axis of $x$ coincident with this line, and the origin $O$ in suchwise




that $OB_1$, $OD_1$ are each positive and $OD_1 > OB_1$; say we have $OD_1 = \delta$, $OB_1 = \beta$, and therefore $\delta - \beta = c$. The equation of the curve $b$ in the neighbourhood of $B_1$ is $y^2 = 2\rho(x - \beta)$, where $\rho$ is the radius of curvature at $B_1$, assumed to be positive when the curve is convex to $O$, or what is the same thing when the centre of curvature $R$ lies to the right of $B_1$ ($OR - OB_1 = +$); and similarly the equation of $d$ in the neighbourhood of $D_1$ is $y^2 = 2\sigma(x - \delta)$, where $\sigma$ is the radius of curvature at $D_1$ assumed to be positive when the curve is convex to $O$, or what is the same thing when the centre of curvature $S$ lies to the right of $D_1$ ($OS - OD_1 = +$).

Consider now $(x_1, y_1)$ the coordinates of a point on $b$ in the neighbourhood of $B_1$, $y_1^2 = 2\rho(x_1 - \beta)$, and taking $B$ at this point, let $(x_2, y_2)$ be the coordinates of the corresponding point $D$ on $d$ in the neighbourhood of $D_1$, $y_2^2 = 2\sigma(x_2 - \delta)$. We have

$$c^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2,$$

and here

$$x_1 = \beta + \frac{y_1^2}{2\rho}, \quad x_2 = \delta + \frac{y_2^2}{2\sigma},$$

whence

$$x_1^2 = \beta^2 + \frac{\beta y_1^2}{\rho}, \quad x_1 x_2 = \beta\delta + \tfrac{1}{2}\frac{\delta y_1^2}{\rho} + \tfrac{1}{2}\frac{\beta y_2^2}{\sigma}, \quad x_2^2 = \delta^2 + \frac{\delta y_2^2}{\sigma}.$$

The equation thus becomes

$$(\delta - \beta)^2 + \frac{y_1^2}{\rho}(\beta - \delta) + \frac{y_2^2}{\sigma}(\delta - \beta) + (y_1 - y_2)^2 = c^2,$$

that is,

$$y_1^2\left(1 + \frac{\beta - \delta}{\rho}\right) - 2y_1 y_2 + y_2^2\left(1 + \frac{\delta - \beta}{\sigma}\right) = 0,$$

a quadric equation between $y_1$ and $y_2$. Evidently if we had taken $D$ a point on $d$, coordinates $(x_2, y_2)$, in the neighbourhood of $D_1$ and had sought for the coordinates $(x_1, y_1)$ of the corresponding point $B$ on $b$ in the neighbourhood of $B_1$, we should have found the same equation between $y_1$ and $y_2$.

7. The equation will have real roots if

$$1 > \left(1 + \frac{\beta - \delta}{\rho}\right)\left(1 + \frac{\delta - \beta}{\sigma}\right);$$

viz. $\rho$, $\sigma$ having the same sign, this is

$$\rho\sigma > (\rho + \beta - \delta)(\sigma + \delta - \beta):$$

but $\rho$, $\sigma$ having opposite signs, then

$$\rho\sigma < (\rho + \beta - \delta)(\sigma + \delta - \beta).$$

These conditions may be written

$$(OR - OB_1)(OS - OD_1) - (OS - OB_1)(OR - OD_1) > \text{ or } < 0,$$

that is,

$$(OS - OR)(OD_1 - OB_1) > \text{ or } < 0.$$

But we have $OD_1 - OB_1 = +$, and therefore, $\rho$, $\sigma$ having the same sign, the condition of reality is $OS > OR$, i.e. $S$ to the right of $R$; but $\rho$, $\sigma$ having opposite signs, the condition of reality is $OS < OR$, i.e. $S$ to the left of $R$. Observe that, $S$ lying to the left of $R$, we cannot have $\rho = -$, $\sigma = +$, and that the second alternative thus is $\rho = +$, $\sigma = -$, then $OS < OR$, or $S$ lies to the left of $R$.

The condition was investigated as above in order to exhibit more clearly the geometrical signification: but of course the original form, or say the equation

$$1 - \left(1 + \frac{\beta - \delta}{\rho}\right)\left(1 + \frac{\delta - \beta}{\sigma}\right) > 0,$$

gives at once

$$\frac{\delta - \beta}{\rho\sigma}(\delta + \sigma - \beta - \rho) > 0.$$

8. Writing the quadric equation in the form

$$y_1^2\left(1-\frac{c}{\rho}\right)-2y_1y_2+\left(1+\frac{c}{\sigma}\right)y_2^2=0,$$

we have

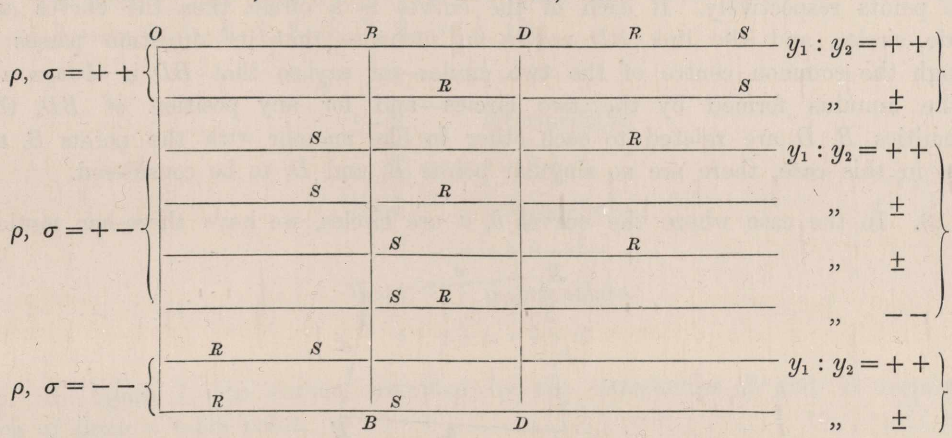
$$\left(1-\frac{c}{\rho}\right)y_1=\left\{1\pm\sqrt{\frac{c}{\rho\sigma}(c+\sigma-\rho)}\right\}y_2;$$

the two values of $y_1 : y_2$ will have the same sign or opposite signs according as $1-\frac{c}{\rho}$ and $1+\frac{c}{\sigma}$ have the same sign or opposite signs, and in the case where these have the same sign, then this is also the sign of each of the two values of $y_1 : y_2$. Or what is the same thing, if $1-\frac{c}{\rho}$ and $1+\frac{c}{\sigma}$ are each of them positive, then the two values of $y_1 : y_2$ are each of them positive; if $1-\frac{c}{\rho}$ and $1+\frac{c}{\sigma}$ are each of them negative, then the two values of $y_1 : y_2$ are each of them negative; and if $1-\frac{c}{\rho}$ and $1+\frac{c}{\sigma}$ have opposite signs, then the two values of $y_1 : y_2$ have opposite signs. Considering the different cases $\rho,\ \sigma = ++,\ +-,\ --$, we find

$\rho,\ \sigma = ++$, then values of $y_1 : y_2$ are $++$ or $--$, according as $DR$, $BS$ are $++$ or $--$,
$\rho,\ \sigma = +-$ „ „ „ „ „ $DR$, $SB$ „ „
$\rho,\ \sigma = --$ „ „ „ „ „ $RD$, $SB$ „ „

and in each case the values of $y_1 : y_2$ are $+-$, if the two distances referred to have opposite signs: $DR = +$ means that $R$ is to the right of, or beyond, $D$, and so in other cases.

9. The different cases, two real roots as above, are

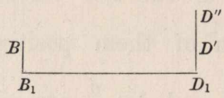



Obviously the cases $\rho,\ \sigma = --$, correspond exactly to the cases $\rho,\ \sigma = +, +$; the only difference is that the concavities, instead of the convexities, of the two curves are turned towards the point $O$.

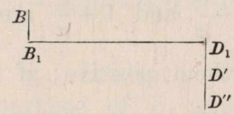
10. If the two roots of the quadratic equation are imaginary, then $B_1D_1$ is a conjugate or isolated position of the link, and $B_1$, $D_1$ are isolated points on the curves $b$ and $d$ respectively.

11. If the roots are real, then the three cases $y_1 : y_2 = + +$, $- -$ and $+ -$, may be delineated as in the annexed figures, viz. taking in each case $y_1$ as positive, that is, imagining $B$ to move upwards from $B_1$ through an infinitesimal arc of $b$, then $D$ moves from $D_1$ through either of two infinitesimal arcs of $d$, both upwards, both downwards, or the one upwards and the other downwards, as shown in the figures

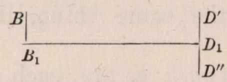
$y_1 : y_2 = + +$




$y_1 : y_2 = - -$




$y_1 : y_2 = \pm$

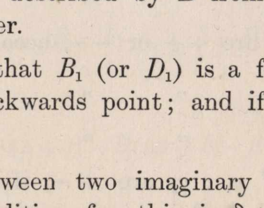



and where it is to be observed that, reversing the sense of the motion of $B$ from $B_1$, we reverse also the senses of the motion of $D$ from $D_1$: moreover that, considering $D$ as moving through an infinitesimal arc of $d$ from $D_1$, we have the like relations thereto of the two infinitesimal arcs of $b$ described by $B$ from $B_1$. Thus the points $B_1$ and $D_1$ are singular points of like character.

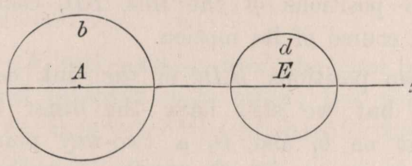
If $y_1 : y_2 = + +$, we may say that $B_1$ (or $D_1$) is a for-forwards point; if $y_1 : y_2 = - -$, then that $B_1$ (or $D_1$) is a back-backwards point; and if $y_1 : y_2 = \pm$, then that $B_1$ (or $D_1$) is a back-forwards point.

12. The separating case between two imaginary roots and two real roots is that of two equal real roots: the condition for this is $\delta + \sigma = \beta + \rho$, that is, $OS = OR$, or the two centres of curvature are coincident; the characters of the points $B_1$ and $D_1$ would in this case depend on the aberrancies of curvature of the curves $b$ and $d$ at these points respectively. If each of the curves is a circle, then the curves are concentric circles, and the link $BD$ moves in suchwise that its direction passes always through the common centre of the two circles—or say so that $BD$ is always a radius of the annulus formed by the two circles—and for any position of $BD$, the two extremities $B$, $D$ are related to each other in like manner with the points $B_1$ and $D_1$. Thus, in this case, there are no singular points $B_1$ and $D_1$ to be considered.

13. In the case where the curves $b$, $d$ are circles, we have three-bar motion: say




the figure is as here shown; I take in it $b$, $d$ for the radii of the two circles respectively and $a$ for the distance of their centres; viz. we have the link $BD = c$, pivoted at its extremities to the arms or radii $AB = b$, and $ED = d$, which rotate about the fixed centres $A$, $E$ at a distance from each other $= a$. Here $a$, $b$, $c$, $d$ are each of them positive; $a$, $b$, $d$ may have any values, but then $c$ is at most $= a + b + d$, and if $a > b + d$ then $c$ is at least $= a - b - d$; but if $a =$ or $< b + d$, then $c$ may be $= 0$, viz. it may have any value from 0 to $a + b + d$. And in either case there will be critical values of $c$. The cases are very numerous. To make an exhaustive enumeration, we may assume $d$ at most $= b$, and in each of the two cases $d < b$ and $d = b$, considering the centre of the circle $d$ as moving from the right of the centre of the circle $b$ towards this centre, we may in the first instance divide as follows:




$d < b$

⊙ $d$ exterior to ⊙ $b$,
„ touches it externally,
„ cuts it,
„ touches it internally,
„ lies within it,
„ is concentric with it,

$d = b$

⊙ $d$ exterior to ⊙ $b$,
„ touches it externally,
„ cuts it,
„ is concentric and thus coincident with it;

and then, in each of these cases, give to the length $c$ of the link its different admissible values.

14. Considering the case $d < b$, then we have (see Plate I., at p. 516), exterior series, the figures 1, 1—2, 2, 2—3, 3, 3—4, 4, viz.

fig. 1, $c = a - b - d$,
1—2, „ intermediate,
2, $c = a - b + d$,
2—3, „ intermediate,
3, $c = a + b - d$,
3—4, „ intermediate,
4, $c = a + b + d$.

15. In figure 1, the curves described by the extremities $B$ and $D$ respectively are each of them a mere point.

In figure 1—2, we have $a + d > b + c$ and $a + b > d + c$. Hence in the course of the motion the arms $b$, $c$ come into a right line, giving a position $B_1D_1'$ of the link, where $B_1$ is a two-way point on $b$ and $D_1'$ a summit on $d$; or rather, there are two such positions symmetrically situate on opposite sides of the axis $Ax$. And again, in the course of the motion, the arms $d$, $c$ come into a right line, giving a position $B_1'D_1$, where $D_1$ is a two-way point on $d$ and $B_1'$ a summit on $b$; or rather, there are two such positions symmetrically situate on opposite sides of the axis $Ax$. Only an arc of the circle $b$ is described, viz. the arc adjacent to $d$ included between the two summits $B_1'$ on $b$; and in like manner, only an arc of the circle $d$ is described, viz. the arc adjacent to $b$ included between the two summits $D_1'$ on $d$. The described portions on $b$ and $d$ respectively are to be regarded each of them as a double line or indefinitely thin bent oval: and it is to be observed that for a given position of $B$ (or $D$) there are two positions of the link $BD$, each of these positions being assumed by the link in the course of its motion.

16. In figure 2, the two positions $B_1D_1'$ of the link come to coincide together in a single axial position $BD$, but we still have the other two positions $B_1'D_1$ of the link, where $B_1'$ is a summit on $b$, and $D_1$ a two-way point on $d$. As regards $BD$, this is the configuration $\rho$, $\sigma = --$, $R$, $B$, $S$, $D$; $y_1 : y_2 = \pm$, and thus each of the axial points $B$, $D$ is a back-and-forwards point. Thus only the arc $B'B_1'$ of the circle $b$ is described by the point $B$, but the whole circumference of the circle $d$ is described by the point $D$. If we further examine the motion it will appear that, as $B$ moves from the axial point $B$ say to the upper summit $B_1'$ and returns to $B$, then $D$ starting from the axial point $D$ may describe (*and that in either sense*, viz. $y_1 = +$, then we have $y_2 = \pm$) the entire circumference of $d$, returning to the axial point $D$; and similarly, as $B$ moves from the axial point $B$ to the lower summit $B_1'$ and returns to $B$, then $D$ starting as before from the axial point $D$ may describe (*and that in either sense*, viz. $y_1 = -$, then we have $y_2 = \pm$) the entire circumference of $d$, returning to the axial point $D$. It is thus not the entire arc $B_1'B_1'$ but each of the half-arcs $BB_1'$ which corresponds, and that in either of two ways, to the circumference of $d$.

17. In figure 2—3, there are four critical positions $B_1'D_1$ (forming two pairs, those of the same pair situate symmetrically on opposite sides of the axis $Ax$) where, as before, $B_1'$ is a summit on $b$, and $D_1$ a two-way point on $d$. The described portions of $b$ are the detached arcs $B_1'B_1'$ between the two upper summits, and $B_1'B_1'$ between the two lower summits: the described portion of $d$ is the whole circumference. In fact, attending to one of the arcs on $b$, say the upper arc $B_1'B_1'$, as $B$ moves from one of the summits, say the left-hand summit $B_1'$, and then returns to the left-hand summit $B_1'$, then $D$, starting from the corresponding two-way point $D_1$, may describe, *and that in either sense*, the entire circumference of $d$, returning to the same point $D_1$; and similarly, as $B$ describes the lower arc $B_1'B_1'$, starting from and returning to a summit, then $D$, starting from the corresponding two-way point $D_1$, may describe, *and that in either sense*, the entire circumference of $d$, returning to the same two-way point $D_1$.

18. In figure 3, two of the positions $B_1'D_1$ have come to coincide together in the axial position $BD$: but we still have the other two positions $B_1'D_1$, where $B_1'$ is a summit on $b$, and $D_1$ a two-way point on $d$. As regards the axial points $B$, $D$, this is the configuration $\rho$, $\sigma = ++$; $B$, $R$, $D$, $S$; $y_1 : y_2 = \pm$, viz. each of the points $B$, $D$ is a back-and-forwards point. The two detached arcs $B_1'B_1'$ of $b$ have united themselves into a single arc $B_1'B_1'$, which is the described portion of $b$; the described portion of

$d$ is, as before, the entire circumference. It is to be observed (as in fig. 2) that properly it is not the entire arc $B_1'B_1'$, but each of the half-arcs $BB_1'$, which corresponds to the entire circumference of $d$.

19. The figure 3—4 closely corresponds to fig. 1—2, the only difference being that the arcs $B_1'B_1'$ and $D_1'D_1'$, which are the described portions of $b$ and $d$ respectively, (instead of being the nearer portions, or those with their convexities facing each other) are the further portions, or those with their concavities facing each other, of the two circles respectively.

Finally, in fig. 4, the described portions of the two circles reduce themselves to the axial points $B$ and $D$ respectively.

20. Still assuming $d < b$, and passing over the case of external contact, we come to that in which the circles intersect each other; but this case has to be subdivided. Since the circles intersect, we have $b + d > a$, consistently herewith we may have:—

| | |
|---|---|
| $b$, $d$ each $< a$, | $A$, $E$ each outside the lens common to the two circles, |
| $b = a$, $d < a$, | $A$ outside, $E$ on boundary of the lens, |
| $b > a$, $d < a$, | $A$ outside, $E$ inside the lens, |
| $b > a$, $d = a$, | $A$ on boundary of, $E$ inside the lens, |
| $b$, $d$ each $> a$, | $A$, $E$, each inside the lens; |

and in each case we have to consider the different admissible values of $c$. I omit the discussion of all these cases.

21. Still assuming $d < b$, and passing over the case of internal contact, we come to that of the circle $d$ included within the circle $b$: we have here again a subdivision of cases; viz. we may have $d > a$, that is, $A$ inside $d$; $d = a$, that is, $A$ on the circumference of $d$; or $d < a$, that is, $A$ outside $d$. The critical values of $c$, arranged in order of increasing magnitude in these three cases respectively, are:—

| $d > a$ | $d = a$ | $d < a$ |
|---|---|---|
| $b - d - a$, | $b - 2d$, | $b - d - a$, |
| $b - d + a$, | $b$, | $b + d - a$, |
| $b + d - a$, | $b$, | $b - d + a$, |
| $b + d + a$, | $b + 2d$, | $b + d + a$. |

I attend only to the first case; we have here (see Plate II., at p. 516), interior series, the figures 1, 1—2, 2, 2—3, 3, 3—4, 4, viz.

| | |
|---|---|
| fig. 1 | $c = b - d - a$, |
| 1—2 | „ intermediate, |
| 2 | $c = b - d + a$, |
| 2—3 | „ intermediate, |
| 3 | $c = b + d - a$, |
| 3—4 | „ intermediate, |
| 4 | $c = b + d + a$. |

22. In figure 1, the curves described by the points $B_1D$ are each of them a mere point. In figure 1—2, we have two critical positions $B_1'D_1$ situate symmetrically on opposite sides of the axis, $B_1'$ being a summit on $b$, and $D_1$ a two-way point on $d$, and moreover two critical positions $B_1D_1'$ situate symmetrically on opposite sides of the axis, $B_1$ being a two-way point on $b$, and $D_1'$ a summit on $d$. The described portion of $b$ is the arc $B_1'B_1'$, and the described portion of $d$ is the arc $D_1'D_1'$, these two arcs being thus the nearer portions of the two circles respectively.

23. In figure 2, the four critical positions coalesce all of them in the axial position $BD$; the described portions are thus the entire circumferences of the two circles respectively. This is a remarkable case. The configuration is $\rho$, $\sigma = + +$; $B$, $D$, $R$, $S$; $y_1 : y_2 = + +$. Imagine $D$ to move from the axial point $D$ in a given sense round the circle $d$, say with uniform velocity, then $B$ moves from the axial point $B$ in the same sense *but with either of two velocities* round the circle $b$; one of these velocities is at first small but ultimately increases rapidly, the other is at first large but ultimately decreases rapidly, so that the two revolutions of $B$ from the axial point $B$ round the entire circumference to the axial point $B$ correspond each of them to the revolution of $D$ from the axial point $D$ round the entire circumference to the axial point $D$. And similarly, if we imagine $B$ to move in a given sense from the axial point $B$ round the circle $b$, say with uniform velocity, then $D$ moves from the axial point $D$ in the same sense but with either of two velocities round the circle $d$: one of these velocities is at first small but ultimately increases rapidly, the other is at first large but ultimately decreases rapidly, so that the two revolutions from the axial point $D$ round the entire circumference of $d$ to the axial point $D$ correspond each of them to the revolution from the axial point $B$ round the entire circumference of $b$ to the axial point $B$.

24. In figure 2—3, there are no critical positions; the described portions of the circles $b$, $d$ are the entire circumferences of the two circles respectively, these being described in the same sense, by the points $B$ and $D$ respectively. It is to be observed that, to a given position of $B$ on $b$, there correspond two positions of $D$ on $d$, or say two positions of the link, but the link does not in the course of its motion pass from one of these positions to the other; the motions are separate from each other, and may be regarded as belonging to different configurations of the system. And of course in like manner, to a given position of $D$ on $d$, there correspond two positions of $B$ on $b$, or say two positions of the link: we have thus the same two separate motions.

25. In figure 3, the critical axial position $BD$ of the link makes its appearance: the described portions are still the entire circumferences of the two circles respectively. As the point $D$ is here to the left of the point $B$, we must take the origin $O$ to the right of $B$, and reverse the direction of the axis $Ox$; the configuration is thus $\rho$, $\sigma = + -$, $B$, $S$, $R$, $D$; $y_1 : y_2 = - -$. Everything is the same as in fig. 2 except (the signs of $y_1 : y_2$ being, as just mentioned, $- -$) that the motions in the circles $b$ and $d$ instead of being in the same sense are in opposite sense, viz. as $D$ moves from the axial point $D$ in a given sense round the circle $d$ to the axial point $D$ say with uniform velocity, then $B$ moves from the axial point $B$ round the circle $b$ in the opposite sense, *and with either of two velocities;* and similarly, as $B$ moves from the

axial point $B$ in a given sense round the circle $b$ say with uniform velocity, then $D$ moves from the axial point $D$ round the circle $d$ in the opposite sense, *and with either of two velocities.*

26. In figure 3—4, we have again the two critical positions $B_1'D_1$ symmetrically situate on opposite sides of the axis, $B_1'$ a summit on $b$, $D_1$ a two-way point on $d$: and also the two critical positions $B_1D_1'$ symmetrically situate on opposite sides of the axis, $B_1$ a two-way point on $b$, $D_1'$ a summit on $d$. The described portion of $b$ is the arc $B_1'B_1'$, and the described portion of $d$ the arc $D_1'D_1'$, these arcs being thus the further portions of the two circles respectively.

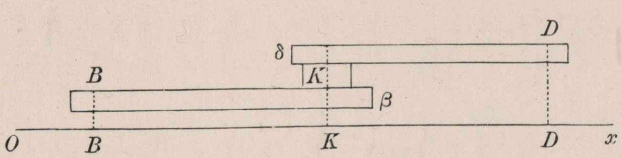
Finally, in figure 4, the described portions reduce themselves to the two points $B$, $D$ respectively.

27. The several forms for $d = b$ can be at once obtained from those for $d < b$; the only difference is that several intermediate forms disappear, and the entire series of divisions is thus not quite so numerous.

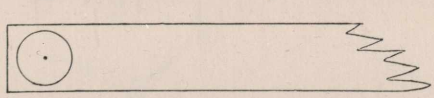
## PART II.

1. The curve-tracing mechanism was devised with special reference to the curves of three-bar motion, viz. the object proposed was that of tracing the curve described by a point $K$ of the link $BD$, the extremities whereof $B$ and $D$ describe given circles respectively, or more generally by a point $K$, the vertex of a triangle $KBD$, whereof the other vertices $B$ and $D$ describe given circles respectively, and that in suchwise that the points $B$ and $D$ might be free to describe the two entire circumferences respectively: but the principle applies to other motions, and I explain it in a general way as follows.

2. Imagine the cranked link $BD$, composed of the bars $B\beta$ and $D\delta$, rigidly attached $B\beta$ to the top and $D\delta$ to the bottom of the cylindrical disk $K$ (this same letter $K$ is used to denote the axis of the disk), and where $B\beta$ and $D\delta$ may be either parallel or inclined to each other at any given angle, so that, referring the points $B$, $K$, $D$ to a horizontal plane, $BKD$ is either a right line, or else $K$ is the vertex of a triangle the other vertices whereof are $B$ and $D$. The disk $K$, with the attached



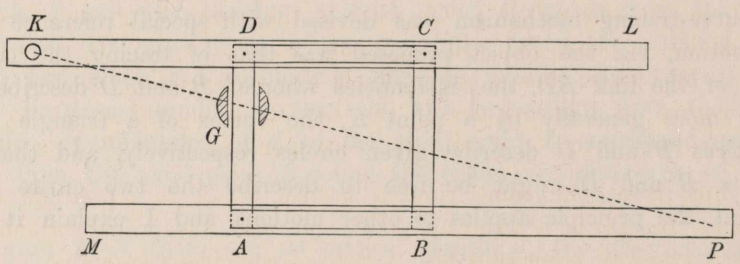

Cranked link with disk: elevation.



Arm of Pentagraph: plan.

bars $B\beta$ and $D\delta$, moves in a horizontal plane: and if the motion of the point $B$ be regulated in any manner by a mechanism lying wholly below $B$ and supported by the bed of the entire mechanism, and similarly if the motion of the point $D$ be regulated in any manner by a mechanism lying wholly above $D$ and supported by a bridge of sufficient length (resting on the bed of the entire mechanism), then the disk $K$ moves in its own horizontal plane unimpeded by other parts of the mechanism: and if we fit the disk $K$ so as to move smoothly within a circular aperture in the arm of a pentagraph, then the pencil of the pentagraph will trace out on a sheet of paper the curve described by the point $K$ on the axis of the disk, or say by the point $K$ of the beam $BKD$. Of course for the three-bar motion, all that is required is that the point $B$ shall describe a circle, viz. it must be pivoted on to an arm $AB$, which is itself pivoted at $A$ to the bed: and that the point $D$ shall describe a circle, viz. it must be pivoted on to an arm $DE$, which is itself pivoted at $E$ to the bridge. Special arrangements are required to enable the variation of the several lengths $AB$, $BK$, $KD$, $DE$ and $ED$, and the mechanism thus unavoidably assumes a form which appears complicated for the object intended to be thereby effected.

3. The form of Pentagraph which I use consists of a parallelogram $ABCD$, pivoted together at the points $A$, $B$, $C$, $D$, the bars $AB$ and $DC$ being above $AD$ and $BC$. There is a cradle $G$, rotating about a fixed centre, and which carries between guides the arm $AD$, which has a sliding motion, so that the lengths $GD$ and $GA$ may be




made to have any given ratio to each other. Above the bar $DC$ and sliding along it, we have the arm $KL$ (where $K$ is the circular aperture which fits on to the disk $K$ of the cranked link): and above $AB$ and sliding along it, we have the arm $MP$ which carries the pencil $P$: of course, in order that the pentagraph may be in adjustment, the points $K$, $G$, $P$ must be *in lineâ*.