956.

ON RICHELOT'S INTEGRAL OF THE DIFFERENTIAL EQUATION $\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0.$

[From the *Messenger of Mathematics*, vol. XXIII. (1894), pp. 42—47.]

In the Memoir "Einige neue Integralgleichungen des Jacobischen Systems Differentialgleichungen," *Crelle*, t. XXV. (1843), pp. 97—118, Richelot, working with the more general problem of a system of $n-1$ differential equations between n variables, obtains a result which in the particular case $n=2$ (that is, for the differential equation

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0,\quad X=a+bx+cx^2+dx^3+ex^4,$$

and Y the same function of y), is in effect as follows: an integral is

$$\left\{\frac{\sqrt{X}(\theta-y)-\sqrt{Y}(\theta-x)}{x-y}\right\}^2=\square\,(\theta-x)(\theta-y)+\Theta+e(\theta-x)^2(\theta-y)^2,$$

where $\square$, θ are arbitrary constants, and Θ denotes the quartic function

$$a+b\theta+c\theta^2+d\theta^3+e\theta^4;$$

viz. this is theorem 3, p. 107 (*l. c.*), taking therein $n=2$, and writing θ, $\square$ for Richelot's α and const.

The peculiarity is that the integral contains apparently *two* arbitrary constants, and it is very interesting to show how these really reduce themselves to a single arbitrary constant.

Observe that, on the right-hand side, there are terms in θ^4, θ^3 whereas no such terms present themselves on the left-hand side. But by changing the constant $\square$, we can get rid of these terms, and so bring each side to contain only terms in θ^2, θ, 1; viz. writing

$$\square = -2e\theta^2 - d\theta - c + C,$$

where C is a new arbitrary constant, the equation becomes

$$\begin{aligned}\left\{\frac{\sqrt{X}(\theta-y)-\sqrt{Y}(\theta-x)}{x-y}\right\}^2 = \;& \theta^2\,[\quad e(x+y)^2 \quad + d(x+y) + C \quad]\\ & +\theta\,[-2e\,xy(x+y) - d\,xy \quad -(C-c)(x+y)+b]\\ & +\quad [\quad e\,x^2y^2 \quad +(C-c)\,xy \quad +a],\end{aligned}$$

which still contains the two arbitrary constants θ, C.

But this gives the three equations

$$\frac{(\sqrt{X}-\sqrt{Y})^2}{(x-y)^2} = e(x+y)^2 + d(x+y) + C,$$

$$-2\frac{(\sqrt{X}-\sqrt{Y})(y\sqrt{X}-x\sqrt{Y})}{(x-y)^2} = -2e\,xy(x+y) - d\,xy - (C-c)(x+y) + b,$$

$$\frac{(y\sqrt{X}-x\sqrt{Y})^2}{(x-y)^2} = e\,x^2y^2 + (C-c)\,xy + a.$$

The first of these is Lagrange's integral containing the arbitrary constant C; and it is necessary that the three equations shall be one and the same equation; viz. the second and third equations must be each of them a mere transformation of the first, equation.

It is easy to verify that this is so. Starting from the first equation, we require, first the value of

$$-2\frac{(\sqrt{X}-\sqrt{Y})(y\sqrt{X}-x\sqrt{Y})}{(x-y)^2}, = \Omega,$$

for a moment.

We form a rational combination, or combination without any term in $\sqrt{XY}$; this is

$$(x+y)\frac{(\sqrt{X}-\sqrt{Y})^2}{(x-y)^2} - 2\frac{(\sqrt{X}-\sqrt{Y})(y\sqrt{X}-x\sqrt{Y})}{(x-y)^2} = e(x+y)^3 + d(x+y)^2 + C(x+y) + \Omega,$$

where the left-hand side is

$$\frac{(x-y)(X-Y)}{(x-y)^2}, = \frac{X-Y}{x-y},$$

which is

$$= e(x^3+x^2y+xy^2+y^3) + d(x^2+xy+y^2) + c(x+y) + b,$$

and we thence have for

$$\Omega, = -2\frac{(\sqrt{X}-\sqrt{Y})(y\sqrt{X}-x\sqrt{Y})}{(x-y)^2},$$

the value given by the second equation.

Secondly, starting again from the first equation, and proceeding in like manner to find the value of

$$\frac{(y\sqrt{X}-x\sqrt{Y})^2}{(x-y)^2},\ =\Omega,$$

for a moment, we form a rational combination

$$-xy\frac{(\sqrt{X}-\sqrt{Y})^2}{(x-y)^2}+\frac{(y\sqrt{X}-x\sqrt{Y})^2}{(x-y)^2}=-exy(x+y)^2-dxy(x+y)-Cxy+\Omega,$$

where the left-hand side is

$$\frac{(x-y)(-yX+xY)}{(x-y)^2},\ =\frac{-yX+xY}{x-y},$$

which is

$$=-exy(x^2+xy+y^2)-dxy(x+y)-cxy+a;$$

and we thence have for

$$\Omega,\ =\frac{(y\sqrt{X}-x\sqrt{Y})^2}{(x-y)^2},$$

the value given by the third equation.

In conclusion, I give what is in effect the process by which Richelot obtained his integral. The integral is $v=\square$, where

$$v=\frac{-\Theta}{\theta-x\,.\,\theta-y}-e(\theta-x\,.\,\theta-y)+(\theta-x\,.\,\theta-y)\,\Omega^2,$$

if, for shortness,

$$\Omega=\frac{\sqrt{X}}{\theta-x\,.\,x-y}+\frac{\sqrt{Y}}{\theta-y\,.\,y-x},$$

and it is required thence to show that

$$\frac{dx}{\sqrt{X}}+\frac{dy}{\sqrt{Y}}=0,$$

or, what is the same thing, to show that v satisfies the partial differential equation

$$\sqrt{X}\frac{dv}{dx}-\sqrt{Y}\frac{dv}{dy}=0.$$

We have

$$\frac{dv}{dx}=\frac{-\Theta}{(\theta-x)^2(\theta-y)}+e(\theta-y)-(\theta-y)\,\Omega^2+2(\theta-x)(\theta-y)\,\Omega\frac{d\Omega}{dx},$$

$$\frac{dv}{dy}=\frac{-\Theta}{(\theta-x)(\theta-y)^2}+e(\theta-x)-(\theta-x)\,\Omega^2+2(\theta-x)(\theta-y)\,\Omega\frac{d\Omega}{dy},$$

and thence, attending to the value of Ω,

$$\sqrt{X}\frac{dv}{dx}-\sqrt{Y}\frac{dv}{dy}=\frac{-\Theta}{\theta-x\,.\,\theta-y}(x-y)\,\Omega+(e-\Omega^2)(\theta-x)(\theta-y)(x-y)\,\Omega$$
$$+2(\theta-x)(\theta-y)\,\Omega\left(\sqrt{X}\frac{d\Omega}{dx}-\sqrt{Y}\frac{d\Omega}{dy}\right),$$

or say

$$-\frac{\left(\sqrt{X}\frac{dv}{dx}-\sqrt{Y}\frac{dv}{dy}\right)}{(\theta-x)(\theta-y)(x-y)\,\Omega}=\frac{\Theta}{(\theta-x)^2(\theta-y)^2}-e+\Omega^2-\frac{2}{x-y}\left(\sqrt{X}\frac{d\Omega}{dx}-\sqrt{Y}\frac{d\Omega}{dy}\right);$$

and it is consequently to be shown that the function on the right-hand side is $=0$. We have

$$\sqrt{X}\frac{d\Omega}{dx}=\frac{\tfrac{1}{2}X'}{(\theta-x)(x-y)}+\frac{X}{(\theta-x)^2(x-y)}-\frac{X}{(\theta-x)(x-y)^2}+\frac{\sqrt{(XY)}}{(\theta-y)(x-y)^2},$$

$$\sqrt{Y}\frac{d\Omega}{dy}=\frac{\tfrac{1}{2}Y'}{(\theta-y)(y-x)}+\frac{Y}{(\theta-y)^2(y-x)}-\frac{Y}{(\theta-y)(x-y)^2}+\frac{\sqrt{(XY)}}{(\theta-x)(x-y)^2},$$

and thence

$$\sqrt{X}\frac{d\Omega}{dx}-\sqrt{Y}\frac{d\Omega}{dy}=\frac{\tfrac{1}{2}X'}{(\theta-x)(x-y)}-\frac{\tfrac{1}{2}Y'}{(\theta-y)(y-x)}$$
$$+\left\{\frac{X}{(\theta-x)^2}+\frac{Y}{(\theta-y)^2}\right\}\frac{1}{x-y}$$
$$-\left(\frac{X}{\theta-x}-\frac{Y}{\theta-y}\right)\frac{1}{(x-y)^2}$$
$$-\frac{\sqrt{(XY)}}{(\theta-x)(\theta-y)(x-y)}.$$

Multiplying by $\dfrac{2}{x-y}$, we may put the result in the form

$$\frac{2}{x-y}\left(\sqrt{X}\frac{d\Omega}{dx}-\sqrt{Y}\frac{d\Omega}{dy}\right)=\frac{1}{\theta-x}\frac{d}{dx}\frac{X}{(x-y)^2}+\frac{1}{\theta-y}\frac{d}{dy}\frac{Y}{(x-y)^2}$$
$$+\frac{2X}{(\theta-x)^2(x-y)^2}+\frac{2Y}{(\theta-x)^2(x-y)^2}-\frac{2\sqrt{(XY)}}{(\theta-x)(\theta-y)(x-y)^2};$$

and the equation to be verified thus is

$$0=\frac{\Theta}{(\theta-x)^2(\theta-y)^2}-e+\Omega^2$$
$$-\frac{1}{\theta-x}\frac{d}{dx}\frac{X}{(x-y)^2}-\frac{2X}{(\theta-x)^2(x-y)^2}$$
$$-\frac{1}{\theta-y}\frac{d}{dy}\frac{Y}{(x-y)^2}-\frac{2Y}{(\theta-x)^2(x-y)^2}$$
$$+\frac{2\sqrt{(XY)}}{(\theta-x)(\theta-y)(x-y)^2}.$$

But decomposing the first term into simple fractions, we have

$$\begin{aligned}\frac{\Theta}{(\theta-x)^2(\theta-y)^2}=&+e\\ &+\frac{1}{\theta-x}\frac{d}{dx}\frac{X}{(x-y)^2}+\frac{X}{(\theta-x)^2(x-y)^2}\\ &+\frac{1}{\theta-y}\frac{d}{dy}\frac{Y}{(x-y)^2}+\frac{Y}{(\theta-y)^2(x-y)^2}.\end{aligned}$$

Also for the third term, we have

$$\begin{aligned}\Omega^2=&\quad\frac{X}{(\theta-x)^2(x-y)^2}\\ &+\frac{Y}{(\theta-y)^2(x-y)^2}\\ &-\frac{2\sqrt{(XY)}}{(\theta-x)(\theta-y)(x-y)^2},\end{aligned}$$

and substituting these values the several terms destroy each other, so that the right-hand side is $=0$, as it should be.