961.

A TRIGONOMETRICAL IDENTITY.

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THE following was proposed as a Senate House Problem: Given the equations

$$a\cos(\beta+\gamma)+b\cos(\beta-\gamma)+c=0,$$
$$a\cos(\gamma+\alpha)+b\cos(\gamma-\alpha)+c=0,$$
$$a\cos(\alpha+\beta)+b\cos(\alpha-\beta)+c=0,$$

it is to be shown that $a^2+2bc-b^2=0$.

Assume

$$\cos\alpha+i\sin\alpha,\ \cos\beta+i\sin\beta,\ \cos\gamma+i\sin\gamma=x,\ y,\ z,$$

then the equations are

$$a\left(yz+\frac{1}{yz}\right)+b\left(\frac{y}{z}+\frac{z}{y}\right)+2c=0,$$
$$a\left(zx+\frac{1}{zx}\right)+b\left(\frac{z}{x}+\frac{x}{z}\right)+2c=0,$$
$$a\left(xy+\frac{1}{xy}\right)+b\left(\frac{x}{y}+\frac{y}{x}\right)+2c=0,$$

that is,

$$a(1+y^2z^2)+b(y^2+z^2)+2cyz=0,$$
$$a(1+z^2x^2)+b(z^2+x^2)+2czx=0,$$
$$a(1+x^2y^2)+b(x^2+y^2)+2cxy=0,$$

the second and third equations give

$$a : b : 2c = x(x^2-yz) : x(-1+x^2yz) : (1-x^4)(y+z);$$

or, say a, b, $2c$ are equal to these values; and then, substituting in the first equation, we have

$$x(1+y^2z^2)(x^2-yz)+x(y^2+z^2)(-1+x^2yz)+(1-x^4)(y^2z+yz^2)=0,$$

which is

$$(x-y)(x-z)\{x+y+z-(yz+zx+xy)(x+y+z)\}=0,$$

viz. our relation between x, y, z is

$$x+y+z-(yz+zx+xy)\,xyz=0,$$

or, what is the same thing,

$$\frac{1}{yz}+\frac{1}{zx}+\frac{1}{xy}-(yz+zx+xy)=0.$$

Then

$$\begin{aligned} a+b &= x(-1+x^2)(1+yz),\\ a-b &= x(\;\;1+x^2)(1-yz),\\ 2c\;\; &= (\;\;1-x^4)(y+z), \end{aligned}$$

$$a^2-b^2=x^2(1-x^4)(y^2z^2-1),\quad 2bc=x(-1+x^2yz)(1-x^4)(y+z).$$

The equation to be verified is

$$x(y^2z^2-1)=(1-x^2yz)(y+z),$$

that is,

$$x+y+z-(yz+zx+xy)\,xyz=0,$$

as it should be.

A somewhat different form of the proof is as follows:—We have *identically*

$$\begin{vmatrix} \cos(\beta+\gamma), & \cos(\beta-\gamma), & 1 \\ \cos(\gamma+\alpha), & \cos(\gamma-\alpha), & 1 \\ \cos(\alpha+\beta), & \cos(\alpha-\beta), & 1 \end{vmatrix}$$

$$=4\sin\tfrac{1}{2}(\beta-\gamma)\sin\tfrac{1}{2}(\gamma-\alpha)\sin\tfrac{1}{2}(\alpha-\beta)\{\sin(\beta+\gamma)+\sin(\gamma+\alpha)+\sin(\alpha+\beta)\},$$

and therefore the relation between the angles is

$$\sin(\beta+\gamma)+\sin(\gamma+\alpha)+\sin(\alpha+\beta)=0.$$

From the second and third equations,

$$a:b:c=\sin\{\tfrac{1}{2}(\beta+\gamma)-\alpha\}:-\sin\{\tfrac{1}{2}(\beta+\gamma)+\alpha\}:2\sin\alpha\cos\alpha\cos\tfrac{1}{2}(\beta-\gamma),$$

or say

$$\begin{aligned} a &= \;\;\sin\tfrac{1}{2}(\beta+\gamma)\cos\alpha-\cos\tfrac{1}{2}(\beta+\gamma)\sin\alpha,\\ b &= -\sin\tfrac{1}{2}(\beta+\gamma)\cos\alpha-\cos\tfrac{1}{2}(\beta+\gamma)\sin\alpha,\\ c &= \;\;2\sin\alpha\cos\alpha\cos\tfrac{1}{2}(\beta-\gamma), \end{aligned}$$

therefore

$$a^2-b^2=-4\sin\alpha\cos\alpha\sin\tfrac{1}{2}(\beta+\gamma)\cos(\beta+\gamma)=-2\sin\alpha\cos\alpha\sin(\beta+\gamma),$$

$$\begin{aligned} bc &= 2\sin\alpha\cos\alpha\{-\cos\tfrac{1}{2}(\beta-\gamma)\sin\tfrac{1}{2}(\beta+\gamma)\cos\alpha-\cos\tfrac{1}{2}(\beta-\gamma)\cos\tfrac{1}{2}(\beta+\gamma)\sin\alpha\},\\ &= 2\,.\,\tfrac{1}{2}\sin\alpha\cos\alpha\{-(\sin\beta+\sin\gamma)\cos\alpha-(\cos\beta+\cos\gamma)\sin\alpha\},\\ &= \sin\alpha\cos\alpha\{-\sin(\gamma+\alpha)-\sin(\alpha+\beta)\}=\sin\alpha\cos\alpha\sin(\beta+\gamma), \end{aligned}$$

whence therefore

$$a^2 - b^2 + 2bc = 0,$$

which is the required relation.

The equation to be proved may also be written

$$\begin{vmatrix} \cos(\beta+\gamma), & \cos(\beta-\gamma), & 1 \\ \cos(\gamma+\alpha), & \cos(\gamma-\alpha), & 1 \\ \cos(\alpha+\beta), & \cos(\alpha-\beta), & 1 \end{vmatrix}$$

$$= 4 \sin \tfrac{1}{2}(\beta-\gamma) \sin \tfrac{1}{2}(\gamma-\alpha) \sin \tfrac{1}{2}(\alpha-\beta) \{\sin(\beta+\gamma) + \sin(\gamma+\alpha) + \sin(\alpha+\beta)\},$$

or putting

$$\beta+\gamma = a, \quad b-c = \gamma-\beta,$$
$$\gamma+\alpha = b, \quad c-a = \alpha-\gamma,$$
$$\alpha+\beta = c, \quad a-b = \beta-\alpha,$$

this becomes

$$\begin{vmatrix} \cos a, & \cos(b-c), & 1 \\ \cos b, & \cos(c-a), & 1 \\ \cos c, & \cos(a-b), & 1 \end{vmatrix}$$

$$= -4 \sin \tfrac{1}{2}(b-c) \sin \tfrac{1}{2}(c-a) \sin \tfrac{1}{2}(a-b)(\sin a + \sin b + \sin c),$$

an identity which may be proved without difficulty.