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ON THE GAUSSIAN THEORY OF SURFACES.

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IN the Memoir, Bour, "Théorie de la déformation des surfaces" (*Jour. de l'Éc. Polyt.*, Cah. 39 (1862), pp. 1—148), the author, working with the form $ds^2 = dv^2 + g^2du^2$ as a special case of Gauss's formula $ds^2 = Edp^2 + 2Fdpdq + Gdq^2$, obtains (p. 29) the following equations which he calls *fundamental*:—

$$[\text{IV.}] \ldots\ldots \begin{cases} \dfrac{1}{g}\dfrac{dg_1}{dv} = T^2 - HH_1, \\ \dfrac{dT}{du} + \dfrac{d\,.\,Hg}{dv} - H_1 g_1 = 0, \\ \dfrac{d\,.\,Tg^2}{dv} + g\dfrac{dH_1}{du} \quad = 0, \end{cases}$$

where g_1 is written to denote $\dfrac{dg}{dv}$, and where (see p. 26)

H is the curvature of the normal section containing the tangent to the curve $v = \text{constant}$,

H_1 is the curvature of the normal section at right angles to the preceding, containing the tangent to the (geodesic) curve $u = \text{constant}$,

T is the torsion of the same geodesic curve;

or, what is the same thing (see p. 25), the quadric equation for the determination of the principal radii of curvature at the point of the surface is

$$\left(\frac{1}{\rho} - H\right)\left(\frac{1}{\rho} - H_1\right) - T^2 = 0.$$

Writing for greater convenience K in place of the suffixed letter H_1, also V instead of g, so that the differential formula is $ds^2 = dv^2 + V^2du^2$, the equations become

$$\begin{cases} \dfrac{1}{V}\dfrac{d^2V}{dv^2} = T - HK, \\ \dfrac{dT}{du} + \dfrac{d\,.\,HV}{dv} - K\dfrac{dV}{dv} = 0, \\ \dfrac{d\,.\,TV^2}{dv} + V\dfrac{dK}{du} \qquad = 0\,; \end{cases}$$

or, if we use the suffix 1 to denote differentiation in regard to v, and the suffix 2 to denote differentiation in regard to u, then the equations are

$$\frac{V_{11}}{V} = T^2 - HK,$$

$$T_2 + (HV)_1 - KV_1 = 0,$$

$$(TV^2)_1 + K_2V = 0,$$

or, what is the same thing,

$$\begin{cases} V_{11} = V(T^2 - HK), \\ T_2 + H_1V + (H-K)\,V_1 = 0, \\ T_1V + 2TV_1 + K_2 \qquad = 0. \end{cases}$$

I wish to show how these formulæ connect themselves with formulæ belonging to the general form $ds^2 = Edp^2 + 2Fdpdq + Gdq^2$. These involve not only Gauss's coefficients E, F, G, but also the coefficients E', F', G' belonging to the inflexional tangents; and, for convenience, I quote the system of definitions, Salmon's *Geometry of Three Dimensions*, 3rd ed., 1874, p. 251, viz.

$$dx,\ dy,\ dz = adp + a'dq,\quad bdp + b'dq,\quad cdp + c'dq\,;$$

$$d^2x = \alpha dp^2 + 2\alpha' dpdq + \alpha'' dq^2,$$

$$d^2y = \beta dp^2 + 2\beta' dpdq + \beta'' dq^2,$$

$$d^2z = \gamma dp^2 + 2\gamma' dpdq + \gamma'' dq^2\,;$$

$$A,\ B,\ C = bc' - b'c,\quad ca' - c'a,\quad ab' - a'b\,;\quad V^2 = EG - F^2\,;$$

$$E' = A\alpha + B\beta + C\gamma,\quad F' = A\alpha' + B\beta' + C\gamma',\quad G' = A\alpha'' + B\beta'' + C\gamma'',$$

so that E', F', G' are, in fact, the determinants

$$\begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ \alpha, & \beta, & \gamma \end{vmatrix},\quad \begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ \alpha', & \beta', & \gamma' \end{vmatrix},\quad \begin{vmatrix} a, & b, & c \\ a', & b', & c' \\ \alpha'', & \beta'', & \gamma'' \end{vmatrix}.$$

The equation for the determination of the principal radii of curvature is

$$(E'\rho - EV)(G'\rho - GV) - (F'\rho - FV)^2 = 0,$$

which, in the particular case $F=0$ (and therefore $V^2=EG$), becomes

$$(E'\rho - EV)(G'\rho - GV) - F'^2\rho^2 = 0,$$

or, as this may be written,

$$\left(\frac{1}{\rho} - \frac{E'}{EV}\right)\left(\frac{1}{\rho} - \frac{G'}{GV}\right) - \frac{F'^2}{EGV^2} = 0,$$

an equation which corresponds with Bour's form

$$\left(\frac{1}{\rho} - K\right)\left(\frac{1}{\rho} - H\right) - T^2 = 0,$$

and becomes identical with it, if

$$E' = EVK, \quad G' = GVH, \quad F' = -V^2T.$$

But, making p, q correspond to Bour's variables, p to v, and q to u, it is necessary to show that the foregoing values (and not the interchanged values $E' = GVH$, $G' = EVK$) are the correct ones. We have, Salmon, p. 254,

$$\left\| \begin{matrix} dq, & \rho E' - VE, & \rho F' - VF \\ -dp, & \rho F' - VF, & \rho G' - VG \end{matrix} \right\| = 0;$$

or, putting herein $F=0$, the equations may be written

$$\frac{dq}{-dp} = \frac{E'}{F'}\left(1 - \frac{VE}{\rho E'}\right) = \frac{F'}{G'} \div \left(1 - \frac{VG}{\rho G'}\right);$$

or, we see that to $dq=0$ corresponds the value $\frac{1}{\rho} = \frac{E'}{EV}$, and to $dp=0$ the value $\frac{1}{\rho} = \frac{G'}{GV}$. Hence the former of these values of $\frac{1}{\rho}$ corresponds to Bour's $du=0$, that is, to his $\frac{1}{\rho} = K$; and the latter to Bour's $dv=0$, that is, to his $\frac{1}{\rho} = H$; or the values are, as stated,

$$E' = EVK, \quad G' = GVH.$$

The formula $ds^2 = Edp^2 + 2Fdpdq + Gdq^2$ agrees with Bour's $ds^2 = dv^2 + g^2du^2$, if $p=u$, $q=v$, $E=1$, $F=0$, $G=g^2$. With these values, $V^2 = EG - F^2 = g^2$, or say $g = V$, and Bour's equation is, as it was before written, $ds^2 = dv^2 + V^2du^2$. And we have to find the three equations which, putting therein $p=u$, $q=v$, $E=1$, $F=0$, $G=V^2$, $E' = VK$, $F' = -V^2T$, $G' = V^3H$, reduce themselves to Bour's equations.

The first of these is nothing else than the equation for the measure of curvature, viz. Salmon, p. 262 (but, using the suffixes 1 and 2 to denote differentiation in regard to p and q respectively), this is

$$\begin{aligned} 4(E'G' - F'^2) = \; & E(E_2G_2 - 2F_1G_2 + G_1^2) \\ & + F(E_1G_2 - E_2G_1 - 2E_2F_1 + 4F_1F_2 - 2F_1G_1) \\ & + G(E_1G_1 - 2E_1F_2 + E_2^2) \\ & - 2(EG - F^2)(E_{22} - 2F_{12} + G_{11}). \end{aligned}$$

In fact, writing herein $E=1$, $F=0$, and therefore the differential coefficients of E and F each $=0$, the equation becomes

$$4(E'G'-F'^2)=G_1^2-2GG_{11},$$

which is

$$4V^4(HK-T^2)=(2VV_1)^2-2V^2(2V_1^2+2VV_{11}),\ =-4V^3V_{11};$$

or finally it is

$$V_{11}=V(T^2-HK).$$

The other two of Bour's equations are derived from equations which give respectively the values of $E_2'-F_1'$ and $F_2'-G_1'$; viz. starting from the equations

$$E'=A\alpha\ +B\beta\ +C\gamma\ ,$$

$$F'=A\alpha'\ +B\beta'\ +C\gamma'\ ,$$

$$G'=A\alpha''+B\beta''+C\gamma'',$$

we see at once that E_2' and F_1' contain, E_2' the terms $A\alpha_2+B\beta_2+C\gamma_2$, and F_1' the terms $A\alpha_1'+B\beta_1'+C\gamma_1'$, which are equal to each other ($\alpha_2=\alpha_1'$ since α and α' are the differential coefficients x_{11}, x_{12} of x, and so $\beta_2=\beta_1'$ and $\gamma_2=\gamma_1'$). Hence

$$E_2'-F_1'=A_2\alpha+B_2\beta+C_2\gamma-A_1\alpha'-B_1\beta'-C_1\gamma';$$

and similarly

$$F_2'-G_1'=A_2\alpha'+B_2\beta'+C_2\gamma'-A_1\alpha''-B_1\beta''-C_1\gamma''.$$

Here, from the values of A, B, C, we have

$$A=bc'-cb';\quad A_1=\beta c'-\gamma b'+b\gamma'-c\beta';\quad A_2=\beta'c'-\gamma'b'+b\gamma''-c\beta'';$$

$$B=ca'-ac';\quad B_1=\gamma a'-\alpha c'+c\alpha'-a\gamma';\quad B_2=\gamma'a'-\alpha'c'+c\alpha''-a\gamma'';$$

$$C=ab'-ba';\quad C_1=\alpha b'-\beta a'+a\beta'-b\alpha';\quad C_2=\alpha'b'-\beta'a'+a\beta''-b\alpha'';$$

and, substituting, we find

$$E_2'-F_1'=\quad 2a'\alpha\alpha'+a\alpha''\alpha,$$

$$F_2'-G_1'=-2a\alpha'\alpha''-a'\alpha''\alpha,$$

if, for shortness, $a'\alpha\alpha'$ denotes the determinant

$$\begin{vmatrix} a', & \alpha, & \alpha' \\ b', & \beta, & \beta' \\ c', & \gamma, & \gamma' \end{vmatrix},$$

and so for the other like symbols. Observe that, with

$$\begin{vmatrix} a, & a', & \alpha, & \alpha', & \alpha'' \\ b, & b', & \beta, & \beta', & \beta'' \\ c, & c', & \gamma, & \gamma', & \gamma'' \end{vmatrix},$$

we have in all 10 determinants, viz. these are $a a' \alpha$, $= E'$; $a a' \alpha'$, $= F'$; $a a' \alpha''$, $= G'$; $\alpha \alpha' \alpha''$; and the six determinants $a \alpha \alpha'$, $a \alpha' \alpha''$, $a \alpha'' \alpha$; $a' \alpha \alpha'$, $a' \alpha' \alpha''$, $a' \alpha'' \alpha$. The foregoing expressions of $E_2' - F_1'$ and $F_2' - G_1'$ respectively, substituting therein for the determinants $a' \alpha \alpha'$, $a \alpha'' \alpha$, $a \alpha' \alpha''$, $a' \alpha'' \alpha$ their values as about to be obtained, are the required two equations. We have

$$\begin{aligned} &aa + bb + cc = E, && aa' + bb' + cc' = F, \\ &a'a + b'b + c'c = F, && a'a' + b'b' + c'c' = G, \\ &\alpha a + \beta b + \gamma c = \tfrac{1}{2}E_1, && \alpha a' + \beta b' + \gamma c' = F_1 - \tfrac{1}{2}E_2, \\ &\alpha' a + \beta' b + \gamma' c = \tfrac{1}{2}E_2, && \alpha' a' + \beta' b' + \gamma' c' = \tfrac{1}{2}G_1, \\ &\alpha'' a + \beta'' b + \gamma'' c = F_2 - \tfrac{1}{2}G_1, && \alpha'' a' + \beta'' b' + \gamma'' c' = \tfrac{1}{2}G_2; \end{aligned}$$

and if from the first five equations, regarded as equations linear in (a, b, c), we eliminate these quantities, and from the second five equations, regarded as linear in (a', b', c'), we eliminate these quantities, we obtain two sets each of five equations,

$$\left\| \begin{matrix} a, & a', & \alpha, & \alpha', & \alpha'' \\ b, & b', & \beta, & \beta', & \beta'' \\ c, & c', & \gamma, & \gamma', & \gamma'' \\ E, & F, & \tfrac{1}{2}E_1, & \tfrac{1}{2}E_2, & F_2 - \tfrac{1}{2}G_1 \end{matrix} \right\| = 0, \text{ and } \left\| \begin{matrix} a, & a', & \alpha, & \alpha', & \alpha'' \\ b, & b', & \beta, & \beta', & \beta'' \\ c, & c', & \gamma, & \gamma', & \gamma'' \\ F, & G, & F_1 - \tfrac{1}{2}E_2, & \tfrac{1}{2}G_1, & \tfrac{1}{2}G_2 \end{matrix} \right\| = 0.$$

These may be written,

$$\begin{aligned} F\alpha\alpha'\alpha'' - \tfrac{1}{2}E_1 a'\alpha'\alpha'' - \tfrac{1}{2}E_2 a'\alpha''\alpha - (F_2 - \tfrac{1}{2}G_1)\, a'\alpha\alpha' &= 0, \\ -E\alpha\alpha'\alpha'' + \tfrac{1}{2}E_1 a\alpha'\alpha'' + \tfrac{1}{2}E_2 a\alpha''\alpha + (F_2 - \tfrac{1}{2}G_1)\, a\alpha\alpha' &= 0, \\ Ea'\alpha'\alpha'' - F a\alpha'\alpha'' + \tfrac{1}{2}E_2 G' - (F_2 - \tfrac{1}{2}G_1)\, F' &= 0, \\ Ea'\alpha''\alpha - F a\alpha''\alpha - \tfrac{1}{2}E_1 G' + (F_2 - \tfrac{1}{2}G_1)\, E' &= 0, \\ Ea'\alpha\alpha' - F a\alpha\alpha' + \tfrac{1}{2}E_1 F' - \tfrac{1}{2}E_2 E' &= 0; \end{aligned}$$

and

$$\begin{aligned} G\alpha\alpha'\alpha'' - (F_1 - \tfrac{1}{2}E_2)\, a'\alpha'\alpha'' - \tfrac{1}{2}G_1 a'\alpha''\alpha - \tfrac{1}{2}G_2 a'\alpha\alpha' &= 0, \\ -F\alpha\alpha'\alpha'' + (F_1 - \tfrac{1}{2}E_2)\, a\alpha'\alpha'' + \tfrac{1}{2}G_1 a\alpha''\alpha + \tfrac{1}{2}G_2 a\alpha\alpha' &= 0, \\ Fa'\alpha'\alpha'' - G a\alpha'\alpha'' + \tfrac{1}{2}G_1 G - \tfrac{1}{2}G_2 F' &= 0, \\ Fa'\alpha''\alpha - G a\alpha''\alpha - (F_1 - \tfrac{1}{2}E_2)\, G' + \tfrac{1}{2}G_2 E' &= 0, \\ Fa'\alpha\alpha' - G a\alpha\alpha' + (F_1 - \tfrac{1}{2}E_2)\, F' - \tfrac{1}{2}G_1 E' &= 0. \end{aligned}$$

Attending in each set only to the third, fourth, and fifth equations, and combining these in pairs, we obtain

$$\begin{aligned} V^2 a\alpha'\alpha'' + (\tfrac{1}{2}FG_1 - FF_2 + \tfrac{1}{2}EG_2)\, F' + (-\tfrac{1}{2}EG_1 + \tfrac{1}{2}FE_2)\, G' &= 0, \\ V^2 a'\alpha'\alpha'' + (\tfrac{1}{2}GG_1 - GF_2 + \tfrac{1}{2}FG_2)\, F' + (-\tfrac{1}{2}FG_1 + \tfrac{1}{2}GE_2)\, G' &= 0; \\ V^2 a\alpha''\alpha + (-\tfrac{1}{2}FE_1 + EF_1 - \tfrac{1}{2}EE_2)\, G' + (-\tfrac{1}{2}FG_1 + FF_2 - \tfrac{1}{2}EG_2)\, E' &= 0, \\ V^2 a'\alpha''\alpha + (-\tfrac{1}{2}GE_1 + FF_1 - \tfrac{1}{2}FE_2)\, G' + (-\tfrac{1}{2}GG_1 + GF_2 - \tfrac{1}{2}FG_2)\, E' &= 0; \\ V^2 a\alpha\alpha' + (\tfrac{1}{2}EG_1 - \tfrac{1}{2}FE_2)\, E' + (\tfrac{1}{2}FE_1 - EF_1 + \tfrac{1}{2}EE_2)\, F' &= 0, \\ V^2 a'\alpha\alpha' + (\tfrac{1}{2}FG_1 - \tfrac{1}{2}GE_2)\, E' + (\tfrac{1}{2}GE_1 - FF_1 + \tfrac{1}{2}FE_2)\, F' &= 0. \end{aligned}$$

We thus obtain

$$E_2' - F_1' = \frac{2}{V^2}\{(\quad -\tfrac{1}{2}FG_1 + \tfrac{1}{2}GE_2)\,E' + (-\tfrac{1}{2}GE_1 + FF_1 \;- \tfrac{1}{2}FF_2)\,F'\}$$

$$+ \frac{1}{V^2}\{(\;\tfrac{1}{2}FE_1 - \;EF_1 + \tfrac{1}{2}EE_2)\,G' + (\;\tfrac{1}{2}FG_1 - FF_2 \;+ \tfrac{1}{2}EG_2)\,E'\},$$

$$F_2' - G_1' = \frac{2}{V^2}\{(\;\tfrac{1}{2}FG_1 - \;FF_2 + \tfrac{1}{2}EG_2)\,F' + (\qquad -\tfrac{1}{2}EG_1 + \tfrac{1}{2}FE_2)\,G'\}$$

$$+ \frac{1}{V^2}\{(-\tfrac{1}{2}GE_1 + \;FF_1 - \tfrac{1}{2}FE_2)\,G' + (-\tfrac{1}{2}GG_1 + GF_2 \;- \tfrac{1}{2}FG_2)\,E'\};$$

or, finally,

$$E_2' - F_1' = \frac{1}{V^2}\{(-\tfrac{1}{2}FG_1 + GE_2 - FF_2 + \tfrac{1}{2}EG_2)\,E'$$

$$+ (-\,GE_1 + 2FF_1 - FF_2)\,F' + (\tfrac{1}{2}FE_1 - EF_1 + \tfrac{1}{2}EE_2)\,G'\},$$

$$F_2' - G_1' = \frac{1}{V^2}\{(-\tfrac{1}{2}GG_1 + GF_2 - \tfrac{1}{2}FG_2)\,E'$$

$$+ (FG_1 - 2FF_2 + EG_2)\,F' + (-\tfrac{1}{2}GE_1 + FF_1 - EG_1 + \tfrac{1}{2}FE_2)\,G'\},$$

which are the required formulæ; and which may, I think, be regarded as new formulæ in the Gaussian theory of surfaces.

Writing herein as before, the first of these becomes

$$(VK)_2 + (V^2T)_1 = \frac{1}{V^2}\{\tfrac{1}{2}(V^2)_2\ VK\},\ = V_2K,$$

that is,

$$V_2K + VK_2 + V^2T_1 + 2VV_1T = V_2K;$$

or finally

$$VT_1 + 2TV_1 + K_2 = 0,$$

which is Bour's third equation. And the second equation becomes

$$-(V^2T)_2 - (V^3H)_1 = \frac{1}{V^2}\{-\tfrac{1}{2}V^2(V^2)_1\ VK + (V^2)_2(-V^2T) - (V^2)_1\ V^3H\},$$

$$= -\,V^2V_1K - 2VV_2T - 2V^2V_1H,$$

that is,

$$-V^2T_2 - 2VV_2T - V^3H_1 - 3V^2V_1H = -\,V^2V_1K - 2VV_2T - 2V^2V_1H;$$

or finally

$$T_2 + VH_1 + (H - K)\,V_1 = 0,$$

which is Bour's second equation.