770.

ON THE 34 CONCOMITANTS OF THE TERNARY CUBIC.

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I HAVE, (by aid of Gundelfinger's formulæ, afterwards referred to), calculated, and I give in the present paper, the expressions of the 34 concomitants of the canonical ternary cubic $ax^3 + by^3 + cz^3 + 6lxyz$, or, what is the same thing, the 34 covariants of this cubic and the adjoint linear function $\xi x + \eta y + \zeta z$: this is the chief object of the paper. I prefix a list of memoirs, with short remarks upon some of them; and, after a few observations, proceed to the expressions for the 34 concomitants; and, in conclusion, exhibit the process of calculation of these concomitants other than such of them as are taken to be known forms. I insert a supplemental table of 6 derived forms.

The list of memoirs (not by any means a complete one) is as follows:

HESSE, Ueber die Elimination der Variabeln aus drei algebraischen Gleichungen vom zweiten Grade mit zwei Variabeln: *Crelle*, t. XXVIII. (1844), pp. 68—96. Although purporting to relate to a different subject, this is in fact the earliest, and a very important, memoir in regard to the general ternary cubic; and in it is established the canonical form, as Hesse writes it, $y_1^3 + y_2^3 + y_3^3 + 6\pi y_1 y_2 y_3$.

ARONHOLD, Zur Theorie der homogenen Functionen dritten Grades von drei Variabeln: *Crelle*, t. XXXIX. (1850), pp. 140—159.

CAYLEY, A Third Memoir on Quantics: *Phil. Trans.*, t. CXLVI. (1856), pp. 627—647; [144].

ARONHOLD, Theorie der homogenen Functionen dritten Grades von drei Variabeln: *Crelle*, t. LV. (1858), pp. 97—191.

SALMON, *Lessons Introductory to the Modern Higher Algebra:* 8°, Dublin, 1859.

CAYLEY, A Seventh Memoir on Quantics: *Phil. Trans.*, t. CLI. (1861), pp. 277—292; [269].

BRIOSCHI, Sur la théorie des formes cubiques à trois indéterminées: *Comptes Rendus*, t. LVI. (1863), pp. 304—307.

HERMITE, Extrait d'une lettre à M. Brioschi: *Crelle*, t. LXIII. (1864), pp. 30—32, followed by a note by Brioschi, pp. 32—33.

The skew covariant of the ninth order, which is $y^3 - z^3 . z^3 - x^3 . x^3 - y^3$ for the canonical form $x^3 + y^3 + z^3 + 6lxyz$, and the corresponding contravariant $\eta^3 - \zeta^3 . \zeta^3 - \xi^3 . \xi^3 - \eta^3$, alluded to p. 116 of Salmon's *Lessons*, were obtained, the covariant by Brioschi and the contravariant by Hermite, in the last-mentioned papers.

CLEBSCH and GORDAN, Ueber die Theorie der ternären cubischen Formen: *Math. Annalen*, t. I. (1869), pp. 56—89.

The establishment of the complete system of the 34 covariants, contravariants and *Zwischenformen*, or, as I have here called them, the 34 concomitants, was first effected by Gordan in the next following memoir:

GORDAN, Ueber die ternären Formen dritten Grades: *Math. Annalen*, t. I. (1869), pp. 90—128.

And the theory is further considered:

GUNDELFINGER, Zur Theorie der ternären cubischen Formen: *Math. Annalen*, t. VI. (1871), pp. 144—163. The author speaks of the 34 forms as being "theils mit den von Gordan gewählten identisch, theils möglichst einfache Combinationen derselben." They are, in fact, the 34 forms given in the present paper for the canonical form of the cubic, and the meaning of the adopted combinations of Gordan's forms will presently clearly appear.

There is an advantage in using the form $ax^3 + by^3 + cz^3 + 6lxyz$ rather than the Hessian form $x^3 + y^3 + z^3 + 6lxyz$, employed in my Third and Seventh Memoirs on Quantics: for the form $ax^3 + by^3 + cz^3 + 6lxyz$ is what the general cubic

$$(a, b, c, f, g, h, i, j, k, l)(x, y, z)^3$$

becomes by no other change than the reduction to zero of certain of its coefficients; and thus any concomitant of the canonical form consists of terms which are leading terms of the same concomitant of the general form.

The concomitants are functions of the coefficients $(a, b, \ldots, l)$, of (ξ, η, ζ), and of (x, y, z): the dimensions in regard to the three sets respectively may be distinguished as the degree, class, and order; and we have thus to consider the deg-class-order of a concomitant.

Two or more concomitants of the same deg-class-order may be linearly combined together: viz., the linear combination is the sum of the concomitants each multiplied by a mere number. The question thus arises as to the selection of a representative concomitant. As already mentioned, I follow Gundelfinger, viz., my 34 concomitants of the canonical form correspond each to each (with only the difference of a numerical factor of the entire concomitant) to his 34 concomitants of the general form. The principle underlying the selection would, in regard to the general form, have to be explained altogether differently; but this principle exhibits itself in a very remarkable manner in regard to the canonical form $ax^3 + by^3 + cz^3 + 6lxyz$.

Each concomitant of the general form is an indecomposable function, not breaking up into rational factors; but this is not of necessity the case in regard to a canonical form: only a concomitant which *does* break up must be regarded as indecomposable, no factor of such concomitant being rejected, or separated. So far from it, there is, in regard to the canonical form in question, a frequent occurrence of $abc + 8l^3$ or a power thereof, either as a factor of a unique concomitant, or when there are two or more concomitants of the same deg-class-order, then as a factor of a properly selected linear combination of such concomitants: and the principle referred to is, in fact, that of the selection of such combination for the representative concomitant; or (in other words) the representative concomitant is taken so as to contain as a factor the highest power that may be of $abc + 8l^3$. As to the signification of this expression $abc + 8l^3$, I call to mind that the discriminant of the form is $abc(abc + 8l^3)^3$.

As to numerical factors: my principle has been, and is, to throw out any common numerical divisor of all the terms: thus I write $S = -abcl + l^4$, instead of Aronhold's $S = -4abcl + 4l^4$. There is also the question of nomenclature: I retain that of my Seventh Memoir on Quantics, except that I use single letters H, P, &c., instead of the same letters with U, thus HU, PU, &c.; in particular, I use U, H, P, Q instead of Aronhold's f, Δ, S_f, T_f. It is thus at all events necessary to make some change in Gundelfinger's letters; and there is moreover a laxity in his use of accented letters; his B, B', B'', B''', and so in other cases E, E', E'', &c., are used to denote functions derived in a determinate manner each from the preceding one (by the δ-process explained further on); whereas his L, L'; M, M'; N, N' are functions having to each other an altogether different relation; also three of his functions are not denoted by any letters at all. Under the circumstances, I retain only a few of his letters; use the accent where it denotes the δ-process; and introduce barred letters $\bar{J}$, $\bar{K}$, &c., to denote a different correspondence with the unbarred letters J, K, &c. But I attach also to each concomitant a numerical symbol showing its deg-class-order, thus: 541 (degree = 5, class = 4, order = 1) or 1290, (there is no ambiguity in the two-digit numbers 10, 11, 12 which present themselves in the system of the 34 symbols); and it seems to me very desirable that the significations of these deg-class-order symbols should be considered as permanent and unalterable. Thus, in writing $S = 400 = -abcl + l^4$, I wish the 400 to be regarded as denoting its expressed value $-abcl + l^4$: if the same letter S is to be used in Aronhold's sense to denote $-4abcl + 4l^4$, this would be completely expressed by the new definition $S = 4.400$, the meaning of the symbol 400 being explained by reference to the present memoir, or by the actual quotation $400 = -abcl + l^4$.

I proceed at once to the table: for shortness, I omit, in general, terms which can be derived from an expressed term by mere cyclical interchanges of the letters (a, b, c), (ξ, η, ζ), (x, y, z).

Table of the 34 Covariants of the Canonical Cubic $ax^3+by^3+cz^3+6lxyz$ and the linear form $\xi x+\eta y+\zeta z$.

First Part, 10 Forms. Class = Order.

Current No.

1 $S = 400 = -abcl + l^4.$

2 $T = 600 = a^2b^2c^2 - 20abcl^3 - 8l^6.$

3 $\Lambda = 011 = \xi x + \eta y + \zeta z.$

4 $\Theta = 222 = x^2[-l^2\xi^2 - 2al\eta\zeta]\ldots$
$+ yz[bc\xi^2 + 2l^2\eta\zeta]\ldots.$

5 $\Theta' = 422 = x^2[l(abc+2l^3)\xi^2 + a(abc-4l^3)\eta\zeta]\ldots$
$+ yz[6bcl^2\xi^2 - 2l(abc+2l^3)\eta\zeta]\ldots.$

6 $\Theta'' = 622 = x^2[-(abc+2l^3)^2\xi^2 + 12al^2(abc+2l^3)\eta\zeta]\ldots$
$+ yz[36bcl^4\xi^2 + 2(abc+2l^3)^2\eta\zeta]\ldots.$

7 $B = 333 = x^3[a^2(c\eta^3 - b\zeta^3)]\ldots$
$+ y^2z[(abc+8l^3)\eta^2\zeta + 12bl^2\zeta^2\xi + 6bcl\xi^2\eta]\ldots$
$+ yz^2[-(abc+8l^3)\eta\zeta^2 - 6bcl\zeta\xi^2 - 12cl^2\xi\eta^2]\ldots.$

8 $B' = 533 = x^3[3a^2l^2(c\eta^3 - b\zeta^3)]\ldots$
$+ y^2z[-l^2(abc+8l^3)\eta^2\zeta + 4bl(-abc+l^3)\zeta^2\xi - bc(abc-10l^3)\xi^2\eta]\ldots$
$+ yz^2[l^2(abc+8l^3)\eta\zeta^2 + bc(abc-10l^3)\zeta\xi^2 - 4cl(-abc+l^3)\xi\eta^2]\ldots.$

9 $B'' = 733 = x^3[9a^2l^4(c\eta^3 - b\zeta^3)]\ldots$
$+ y^2z[l(abc+8l^3)(2abc+l^3)\eta^2\zeta$
$+ b(abc+2l^3)(abc-10l^3)\zeta^2\xi + 6bcl^2(-abc+l^3)\xi^2\eta]\ldots$
$+ yz^2[-l(abc+8l^3)(2abc+l^3)\eta\zeta^2$
$- 6bcl^2(-abc+l^3)\zeta\xi^2 - c(abc+2l^3)(abc-10l^3)\xi\eta^2]\ldots.$

10 $B''' = 933 = x^3[27a^2l^6(c\eta^3 - b\zeta^3)]\ldots$
$+ y^2z[-(abc+8l^3)(abc-l^3)^2\eta^2\zeta + 9bl^2(abc+2l^3)^2\zeta^2\xi$
$- 27bcl^4(abc+2l^3)\xi^2\eta]\ldots$
$+ yz^2[(abc+8l^3)(abc-l^3)^2\eta\zeta^2 + 27bcl^4(abc+2l^3)\zeta\xi^2$
$- 9cl^2(abc+2l^3)^2\xi\eta^2]\ldots.$

Second Part, (4 + 4 =) 8 forms. Class = 0, and Order = 0.

Class = 0.

11 $U = 103 = ax^3 + by^3 + cz^3 + 6lxyz.$

12 $H = 303 = l^2(ax^3 + by^3 + cz^3) - (abc+2l^3)xyz.$

13 $\Psi = 806 = (abc+8l^3)^2\{a^2x^6 + b^2y^6 + c^2z^6 - 10(bcy^3z^3 + caz^3x^3 + abx^3y^3)\}.$

14 $\Omega = 1209 = (abc+8l^3)^3\{by^3 - cz^3 \,.\, cz^3 - ax^3 \,.\, ax^3 - by^3\}.$

Current No. Order = 0.

15 $P = 330 = -l(bc\xi^3 + ca\eta^3 + ab\zeta^3) + (-abc + 4l^3)\xi\eta\zeta.$

16 $Q = 530 = (abc - 10l^3)(bc\xi^3 + ca\eta^3 + ab\zeta^3) - 6l^2(5abc + 4l^3)\xi\eta\zeta.$

17 $F = 460 = b^2c^2\xi^6 + c^2a^2\eta^6 + a^2b^2\zeta^6 - 2(abc + 16l^3)(a\eta^3\zeta^3 + b\zeta^3\xi^3 + c\xi^3\eta^3)$
$- 24l^2(bc\xi^3 + ca\eta^3 + ab\zeta^3)\xi\eta\zeta - 24l(abc + 2l^3)\xi^2\eta^2\zeta^2.$

18 $\Pi = 1290 = (abc + 8l^3)^3\{c\eta^3 - b\zeta^3 . a\zeta^3 - c\xi^3 . b\xi^3 - a\eta^3\}.$

Third Part, (8 + 8 =) 16 forms. Class less or greater than Order.

Class less than Order.

19 $J = 414 = (abc + 8l^3)\{\xi x(by^3 - cz^3) + \eta y(cz^3 - ax^3) + \zeta z(ax^3 - by^3)\}.$

20 $K = 514 = (abc + 8l^3)\{\xi[alx^4 - 2blxy^3 - 2clxz^3 + 3bcy^2z^2]\ldots\}.$

21 $K' = 714 = (abc + 8l^3)\{\xi[(abc + 2l^3)(ax^4 - 2bxy^3 - 2cxz^3) - 18bcl^2y^2z^2]\ldots\}.$

22 $E = 625 = (abc + 8l^3)\{\xi^2(by^3 - cz^3)[2l^2x^2 + bcyz]\ldots$
$+ \eta\zeta(by^3 - cz^3)[4alx^2 + 2l^2yz]\ldots\}.$

23 $E' = 825 = (abc + 8l^3)\{\xi^2(by^3 - cz^3)[l(abc + 2l^3)x^2 - 3bcl^2yz]\ldots$
$+ \eta\zeta(by^3 - cz^3)[a(abc - 4l^3)x^2 + l(abc + 2l^3)yz]\ldots\}.$

24 $E'' = 1025 = (abc + 8l^3)\{\xi^2(by^3 - cz^3)[(abc + 2l^3)^2x^2 + 18bcl^4yz]\ldots$
$+ \eta\zeta(by^3 - cz^3)[-12al^2(abc + 2l^3)x^2 + (abc + 2l^3)^2yz]\ldots\}.$

25 $M = 917 = (abc + 8l^3)^2\{\xi(by^3 - cz^3)[5alx^4 - blxy^3 - clxz^3 - 3bcy^2z^2]\ldots\}.$

26 $M' = 1117 = (abc + 8l^3)^2\{\xi(by^3 - cz^3)[(abc + 2l^3)(5ax^4 - bxy^3 - cxz^3)$
$+ 18bcl^2y^2z^2]\ldots\}.$

Order less than Class.

27 $\bar{J} = 841 = (abc + 8l^3)^2\{x\xi a(c\eta^3 - b\zeta^3) + y\eta b(a\zeta^3 - c\xi^3) + z\zeta c(b\xi^3 - a\eta^3)\}.$

28 $\bar{K} = 541 = (abc + 8l^3)\{x[bc\xi^4 - 2ca\xi\eta^3 - 2ab\xi\zeta^3 - 6al\eta^2\zeta^2]\ldots\}.$

29 $\bar{K}' = 741 = (abc + 8l^3)\{x[l^2(bc\xi^4 - 2ca\xi\eta^3 - 2ab\xi\zeta^3) + a(abc + 2l^3)\eta^2\zeta^2]\ldots\}.$

30 $\bar{E} = 652 = (abc + 8l^3)\{x^2(c\eta^3 - b\zeta^3)[2al\xi^2 + a^2\eta\zeta]\ldots$
$+ yz(c\eta^3 - b\zeta^3)[4l^2\xi^2 + 2al\eta\zeta]\ldots\}.$

31 $\bar{E}' = 852 = (abc + 8l^3)\{x^2(c\eta^3 - b\zeta^3)[a(abc - 4l^3)\xi^2 - 6a^2l^2\eta\zeta]\ldots$
$+ yz(c\eta^3 - b\zeta^3)[4l(abc + 2l^3)\xi^2 + a(abc - 4l^3)\eta\zeta]\ldots\}.$

32 $\bar{E}'' = 1052 = (abc + 8l^3)\{x^2(c\eta^3 - b\zeta^3)[-3al^2(abc + 2l^3)\xi^2 + 9a^2l^4\eta\zeta]\ldots$
$+ yz(c\eta^3 - b\zeta^3)[(abc + 2l^3)^2\xi^2 - 3al^2(abc - 4l^3)\eta\zeta]\ldots\}.$

33 $\bar{M} = 771 = (abc + 8l^3)\{x(c\eta^3 - b\zeta^3)[(abc - 8l^3)\xi^4 - a^2c\xi\eta^3 - a^2b\xi\zeta^3$
$- 12al^2\xi^2\eta\zeta - 6a^2l\eta^2\zeta^2]\ldots\}.$

34 $\bar{M}' = 971 = (abc + 8l^3)\{x(c\eta^3 - b\zeta^3)[l^2(7abc + 8l^3)\xi^4 - 3a^2cl^2\xi\eta^3 - 3a^2bl^2\xi\zeta^3$
$+ 4al(abc - l^3)\xi^2\eta\zeta + a^2(abc - 10l^3)\eta^2\zeta^2]\ldots\}.$

To this may be joined the following Supplemental Table of certain Derived Forms:

Current No.

35 $R = 1200 = \quad 64S^3 - T^2 = - abc\,(abc + 8l^3)^3.$

36 $C = \;\; 703 = - TU + 24SH = (abc + 8l^3)\,\{(-abc + 4l^3)\,(ax^3 + by^3 + cz^3) + 18abclxyz\}.$

37 $D = \;\; 903 = \quad 8S^2U - 3TH = (abc + 8l^3)\,\{l^2\,(5abc + 4l^3)\,(ax^3 + by^3 + cz^3) + 3abc\,(abc - 10l^3)\,xyz\}.$

38 $Y = \;\; 930 = \quad 3TP - 4SQ = (abc + 8l^3)^2\,\{l\,(bc\xi^3 + ca\eta^3 + ab\zeta^3) - 3abc\xi\eta\zeta\}.$

39 $Z = 1130 = - 48S^2P + TQ = (abc + 8l^3)^2\,\{(abc + 2l^3)\,(bc\xi^3 + ca\eta^3 + ab\zeta^3) + 18abcl^2\xi\eta\zeta\}.$

40 $\Phi = 1660 = \quad 12\,(abc + 8l^3)^3 F - 288STP^2 + 768S^2PQ - 8TQ^2$
$= (abc + 8l^3)^4\,\{b^2c^2\xi^6 + c^2a^2\eta^6 + a^2b^2\zeta^6 - 10abc\,(a\eta^3\zeta^3 + b\zeta^3\xi^3 + c\xi^3\eta^3)\},$

viz. these are derived forms characterized by having a power of $abc + 8l^3$ as a factor: R is the discriminant; C, D, Y, Z occur in Aronhold, and in my Seventh memoir on Quantics [269]: Φ in Clebsch and Gordan's memoir of 1869.

I regard as known forms Λ, U, H, P, Q, S, T, F, that is, the eight forms 3, 11, 12, 15, 16, 1, 2, 17; the remaining 26 forms are expressed in terms of these by formulæ involving notations which will be explained, viz. we have

13 $\Psi = \quad 3\,(\mathrm{bc}' + \mathrm{b}'\mathrm{c} - 2\mathrm{ff}', \ldots, \mathrm{gh}' + \mathrm{g}'\mathrm{h} - \mathrm{af}' - \mathrm{a}'\mathrm{f}, \ldots)(X,\ Y,\ Z)(X',\ Y',\ Z') + TU^2.$

14 $\Omega = \quad \tfrac{1}{18}\,\mathrm{Jac}\,(U,\ H,\ \Psi).$

18 $\Pi = - \tfrac{1}{36}\,[\mathrm{Jac}]\,(P,\ Q,\ F).$

4 $\Theta = \quad (\mathrm{bc} - \mathrm{f}^2, \ldots, \mathrm{gh} - \mathrm{af}, \ldots)(\xi,\ \eta,\ \zeta)^2.$

5 $\Theta' = \quad \tfrac{1}{2}\,\delta\Theta.$

6 $\Theta'' = \quad \tfrac{1}{2}\,\delta^2\Theta.$

7 $B = - \tfrac{1}{3}\,\mathrm{Jac}\,(U,\ \Theta,\ \Lambda).$

8 $B' = \quad \tfrac{1}{6}\,\delta B.$

9 $B'' = \quad \tfrac{1}{24}\,\delta^2 B.$

10 $B''' = \quad \tfrac{1}{48}\,\delta^3 B.$

19 $J = - \tfrac{1}{3}\,\mathrm{Jac}\,(U,\ H,\ \Lambda).$

27 $\bar{J} = \quad \tfrac{1}{3}\,[\mathrm{Jac}]\,(P,\ Q,\ \Lambda).$

20 $K = - \tfrac{3}{2}\,\{\partial_\xi\Theta\partial_x H + \partial_\eta\Theta\partial_y H + \partial_\zeta\Theta\partial_z H\} - SU\Lambda.$

21 $K' = - (\delta)\,K.$

28 $\bar{K} = \quad 3\,\{\partial_x\Theta\partial_\xi P + \partial_y\Theta\partial_\eta P + \partial_z\Theta\partial_\zeta P\} + Q\Lambda.$

29 $\bar{K}' = \quad \tfrac{1}{6}\,(\delta)\,\bar{K}.$

22 $E = - \tfrac{1}{18}\,\mathrm{Jac}\,(K,\ U,\ \Lambda).$

23 $E' = - \tfrac{1}{4}\,(\delta)\,E.$

24 $E'' = \ \frac{1}{4}(\delta^2)E.$

30 $\bar{E} = -\frac{1}{9}\,\text{Jac}(\bar{K},\ U,\ \Lambda).$

31 $\bar{E}' = -\frac{1}{2}(\delta)\bar{E}.$

32 $\bar{E}'' = -\frac{1}{8}(\delta^2)\bar{E}.$

25 $M = \ \frac{1}{36}\,\text{Jac}(U,\ \Psi,\ \Lambda).$

26 $M' = -(\delta)M.$

33 $\bar{M} = -\frac{1}{6}[\text{Jac}](P,\ F,\ \Lambda).$

34 $\bar{M}' = \ \frac{1}{6}(\delta)\bar{M}.$

In explanation of the notations, observe that

$$U = \ ax^3 + by^3 + cz^3 + 6lxyz,$$
$$H = l^2(ax^3 + by^3 + cz^3) - (abc + 2l^3)\,xyz.$$

Hence, writing

$$6H = \ a'x^3 + b'y^3 + c'z^3 + 6l'xyz,$$

we have

$$a',\ b',\ c',\ l' = 6al^2,\ 6bl^2,\ 6cl^2,\ -(abc+2l^3).$$

And this being so, we write

$$X,\ Y,\ Z \qquad = ax^2 + 2lyz,\ by^2 + 2lzx,\ cz^2 + 2lxy,$$
$$\text{a, b, c, f, g, h} = ax,\ by,\ cz,\ lx,\ ly,\ lz,$$

for $\frac{1}{3}$ of the first differential coefficients, and $\frac{1}{6}$ of the second differential coefficients of U; and in like manner

$$X',\ Y',\ Z' \qquad = a'x^2 + 2l'yz,\ b'y^2 + 2l'zx,\ c'z^2 + 2l'xy,$$
$$\text{a}',\ \text{b}',\ \text{c}',\ \text{f}',\ \text{g}',\ \text{h}' = a'x,\ b'y,\ c'z,\ l'x,\ l'y,\ l'z,$$

for $\frac{1}{3}$ of the first differential coefficients, and $\frac{1}{6}$ of the second differential coefficients of $6H$.

Jac is written to denote the Jacobian, viz.:

$$\text{Jac}(U,\ H,\ \Psi) = \begin{vmatrix} \partial_x U, & \partial_y U, & \partial_z U \\ \partial_x H, & \partial_y H, & \partial_z H \\ \partial_x \Psi, & \partial_y \Psi, & \partial_z \Psi \end{vmatrix},$$

and in like manner [Jac] to denote the Jacobian, when the differentiations are in regard to $(\xi,\ \eta,\ \zeta)$ instead of $(x,\ y,\ z)$: δ is the symbol of the δ-process, or substitution of the coefficients $(a',\ b',\ c',\ l')$ in place of $(a,\ b,\ c,\ l)$; in fact,

$$\delta = a'\partial_a + b'\partial_b + c'\partial_c + l'\partial_l:$$

δ, δ^2, &c., each operate directly on a function of $(a,\ b,\ c,\ l)$, the $(a',\ b',\ c',\ l')$ of the symbol δ being in the first instance regarded as constants, and being replaced ultimately by their values; for instance,

$$\delta abc = a'bc + ab'c + abc',\ \ \delta^2 abc = 2(ab'c' + a'bc' + a'b'c),\ \ \delta^3 abc = 6a'b'c'.$$

In several of the formulæ, instead of δ or δ^2, the symbol used is (δ) or (δ^2); in these cases, the function operated upon contains the factor $(abc+8l^3)$ or $(abc+8l^3)^2$, and is of the form $(abc+8l^3)(aU+bV+cW)$ or $(abc+8l^3)^2\ (a^2U+abV+\&c.)$: the meaning is, that the δ or δ^2 is supposed to operate through the $(abc+8l^3)\,a$, or $(abc+8l^3)^2\,a^2$, &c., as if this were a constant, upon the U, V, &c., only; thus: $(\delta).(abc+8l^3)(aU+bV+cW)$ is used to denote $(abc+8l^3)(a\delta U+b\delta V+c\delta W)$. As to this, observe that, operating with δ instead of (δ), there would be the additional terms $U\delta(abc+8l^3)\,a+\&c.$; we have in this case

$$\delta(abc+8l^3)\,a,\ =a(2a'bc+ab'c+abc'+24l^2l')+8l^3a',$$
$$=24a^2bcl^2-24al^2(abc+2l^3)+48al^5,\ =0;$$

or the rejected terms in fact vanish. For $(\delta^2).(abc+8l^3)(aU+bV+cW)$, operating with δ^2, we should have, in like manner, terms $U\delta^2(abc+8l^3)\,a$, &c.; here

$$\delta^2(abc+8l^3)\,a=a'^2bc+2aba'c'+2aca'b'+a^2b'c'+24l^2a'l'+24all'^2,$$

which is found to be $=-24a(abc+8l^3)(-abcl+l^4)$, that is, $=-24S(abc+8l^3)\,a$; and the terms in question are thus $=-24S(abc+8l^3)(aU+bV+cW)$, viz.

$$(abc+8l^3)(aU+bV+cW)$$

being a covariant, this is also a covariant; that is, in using (δ^2) instead of δ^2, we in fact reject certain covariant terms; or say, for instance, δ^2E being a covariant, then $(\delta^2)\,E$ is also a covariant, but a different covariant. The calculation with (δ) or (δ^2) is more simple than it would have been with δ or δ^2. See *post*, the calculations of K, $\overline{K}'$, &c.

I give for each of the 26 covariants a calculation showing how at least a single term of the final result is arrived at, and, in the several cases for which there is a power of $abc+8l^3$ as a factor, showing how this factor presents itself.

Calculations for the 26 *Covariants.*

13. $\Psi=3(\mathrm{bc}'+\mathrm{b}'\mathrm{c}-2\mathrm{ff}',\ \ldots,\ \mathrm{gh}'+\mathrm{g}'\mathrm{h}-\mathrm{af}'-\mathrm{a}'\mathrm{f},\ \ldots\between X,\ Y,\ Z\between X',\ Y',\ Z')+TU^2,$
$=3((bc'+b'c)\,yz-2ll'x^2,\ \ldots,\ 2ll'yz-(al'+a'l)\,x^2,\ \ldots\between ax^2+2lyz,\ \ldots\between a'x^2+2l'yz,\ \ldots)$
$+T(a^2x^6+\ldots).$

The whole coefficient of x^6 is

$$-6ll'aa'+Ta^2,\ =36a^2l^3(abc+2l^3)+Ta^2,$$

viz. the coefficient of a^2x^6 is

$$=36l^3(abc+2l^3)+a^2b^2c^2-20abcl^3-8l^6$$
$$=a^2b^2c^2+16abcl^3+64l^6$$
$$=(abc+8l^3)^2.$$

14. $$\Omega=\tfrac{1}{18}\operatorname{Jac}(U,\ H,\ \Psi),\ =\tfrac{1}{2}\begin{vmatrix} X, & X', & \tfrac{1}{6}\partial_x\Psi \\ Y, & Y', & \tfrac{1}{6}\partial_y\Psi \\ Z, & Z', & \tfrac{1}{6}\partial_z\Psi \end{vmatrix}.$$

Here

$$YZ' - Y'Z = (by^2 + 2lzx)(c'z^2 + 2l'xy) - (cz^2 + 2lxy)(b'z^2 + 2l'xy)$$

$$= (bc' - b'c)\,y^2z^2 + (2bl' - b'l)\,xy^3 - 2(cl' - c'l)\,xz^3$$

$$= -2(abc + 8l^3)\,x\,(by^3 - cz^3);$$

$$\tfrac{1}{2}\,.\,\tfrac{1}{6}\,\partial_x\Psi = \tfrac{1}{2}(a^2x^5 - 5abx^2y^3 - 5acx^2z^3).$$

Hence the whole is

$$= -(abc + 8l^3)\{a^2x^6(by^3 - cz^3) + b^2y^6(cz^3 - ax^3) + c^2z^6(ax^3 - by^3)\},$$

$$= (abc + 8l^3)(by^3 - cz^3)(cz^3 - ax^3)(ax^3 - by^3).$$

18. $$\Pi = -\tfrac{1}{36}[\text{Jac}](P,\ Q,\ F) = -\tfrac{1}{36}\begin{vmatrix} \partial_\xi P, & \partial_\xi Q, & \partial_\xi F \\ \partial_\eta P, & \partial_\eta Q, & \partial_\eta F \\ \partial_\zeta P, & \partial_\zeta Q, & \partial_\zeta F \end{vmatrix},$$

viz. if, in this calculation, we write

$$6P = \mathrm{a}\,\xi^3 + \mathrm{b}\,\eta^3 + \mathrm{c}\,\zeta^3 + 6\mathrm{l}\,\xi\eta\zeta,\ \textit{i.e.}\ \mathrm{a},\ \mathrm{b},\ \mathrm{c},\ \mathrm{l} = -6lbc,\ -6lca,\ -6lab,\ -abc + 4l^3,$$

$$Q = \mathrm{a}'\xi^3 + \mathrm{b}'\eta^3 + \mathrm{c}'\zeta^3 + 6\mathrm{l}'\xi\eta\zeta,\ \text{,,}\ \mathrm{a}',\ \mathrm{b}',\ \mathrm{c}',\ \mathrm{l}' = (abc - 10l^3)(bc,\ ca,\ ab),\ -l^2(5abc + 4l^3),$$

then

$$\Pi = -\tfrac{1}{4}\begin{vmatrix} \mathrm{a}\xi^2 + 2\mathrm{l}\eta\zeta, & \mathrm{a}'\xi^2 + 2\mathrm{l}'\eta\zeta, & \tfrac{1}{6}\partial_\xi F \\ \mathrm{b}\eta^2 + 2\mathrm{l}\zeta\xi, & \mathrm{b}'\eta^2 + 2\mathrm{l}'\zeta\xi, & \tfrac{1}{6}\partial_\eta F \\ \mathrm{c}\zeta^2 + 2\mathrm{l}\xi\eta, & \mathrm{c}'\zeta^2 + 2\mathrm{l}'\xi\eta, & \tfrac{1}{6}\partial_\zeta F \end{vmatrix}.$$

Here

$$(\mathrm{b}\eta^2 + 2\mathrm{l}\zeta\xi)(\mathrm{c}'\zeta^2 + 2\mathrm{l}'\xi\eta) - (\mathrm{b}'\eta^2 + 2\mathrm{l}'\zeta\xi)(\mathrm{c}\zeta^2 + 2\mathrm{l}\xi\eta)$$

$$= (\mathrm{bc}' - \mathrm{b}'\mathrm{c})\,\eta^2\zeta^2 + 2(\mathrm{bl}' - \mathrm{b}'\mathrm{l})\,\xi\eta^3 - 2(\mathrm{cl}' - \mathrm{c}'\mathrm{l})\,\xi\zeta^3,$$

or since

$$\mathrm{bc}' - \mathrm{b}'\mathrm{c} = 0,$$

$$\mathrm{bl}' - \mathrm{b}'\mathrm{l} = -6lca\,.\,-l^2(5abc + 4l^3) - (abc - 10l^3)\,ca\,(-abc + 4l^3)$$

$$= ca\,\{6l^3(5abc + 4l^3) + (abc - 4l^3)(abc - 10l^3)\}$$

$$= ca\,(abc + 8l^3)^2,$$

and the like for $\mathrm{cl}' - \mathrm{c}'\mathrm{l}$, the expression is

$$= 2(abc + 8l^3)^2(ca\eta^3 - ab\zeta^3)\,\xi;$$

and the whole is thus

$$= -\tfrac{1}{2}(abc + 8l^3)^2\{(ca\eta^3 - ab\zeta^3)\,\xi\,.\,\tfrac{1}{6}\partial_\xi F + \dots\}$$

$$= -\tfrac{1}{2}(abc + 8l^3)^2\{(ca\eta^3 - ab\zeta^3)[b^2c^2\xi^6 - (abc + 16l^3)(b\zeta^3\xi^3 + c\xi^3\eta^3) + \&\text{c.}]$$

$$+ (ab\zeta^3 - bc\xi^3)[c^2a^2\eta^6 - (abc + 16l^3)(c\xi^3\eta^3 + a\eta^3\zeta^3) + \&\text{c.}]$$

$$+ (bc\xi^3 - ca\eta^3)[a^2b^2\zeta^6 - (abc + 16l^3)(a\eta^3\zeta^3 + b\zeta^3\xi^3) + \&\text{c.}]\}.$$

Here the coefficient of $\xi^6\eta^3$, inside the $\{\}$, is

$$ab^2c^3 + bc^2(abc + 16l^3),\ = 2bc^2(abc + 8l^3),$$

and consequently the whole is

$$= -(abc+8l^3)^3 (bc^2\xi^6\eta^3 - \ldots),$$
$$= (abc+8l^3)^3 \{(c\eta^3 - b\zeta^3)(a\zeta^3 - c\xi^3)(b\xi^3 - a\eta^3)\}.$$

4. $$\Theta = (\text{bc} - \text{f}^2, \ldots, \text{gh} - \text{af}, \ldots \between \xi, \eta, \zeta)^2$$
$$= (bcyz - l^2x^2)\xi^2 + \ldots + 2(l^2yz - alx^2)\eta\zeta + \ldots$$

which are the terms of the final result

$$\Theta = x^2[-l^2\xi^2 - 2al\eta\zeta] + yz[bc\xi^2 + 2l^2\eta\zeta].$$

5 and 6. The δ-process applied to the terms of Θ just written down gives

$$\Theta' = \tfrac{1}{2}\delta\Theta = x^2[-ll'\xi^2 - (al' + a'l)\eta\zeta] + yz[\tfrac{1}{2}(bc' + b'c)\xi^2 + 2ll'\eta\zeta],$$
$$\Theta'' = \tfrac{1}{2}\delta^2\Theta = x^2[-l'^2\xi^2 - 2a'l'\eta\zeta] + yz[b'c'\xi^2 + 2l'^2\eta\zeta];$$

substituting for a', b', c', l' their values, we have the corresponding terms of Θ' and Θ'' respectively.

7. $$B = -\tfrac{1}{3}\,\text{Jac}(U, \Theta, \Lambda), = -\begin{vmatrix} X, & \partial_x\Theta, & \xi \\ Y, & \partial_y\Theta, & \eta \\ Z, & \partial_z\Theta, & \zeta \end{vmatrix}.$$

A term is $X(\eta\partial_z\Theta - \zeta\partial_y\Theta)$, and if, in this calculation, we write

$$\Theta = (A, B, C, F, G, H \between x, y, z)^2, \textit{ i.e. } A = -l^2\xi^2 - 2al\eta\zeta, \ldots, F = \tfrac{1}{2}bc\xi^2 + l^2\eta\zeta,$$

then the term is

$$= (ax^2 + 2lyz)\{x \,.\, 2(G\eta - H\zeta) + y \,.\, 2(F\eta - B\zeta) + z \,.\, 2(C\eta - F\zeta)\}.$$

Here

$$2(G\eta - H\zeta) = \eta(ca\eta^2 + l^2\zeta\xi) - \zeta(ab\zeta^2 + l^2\xi\eta), = a(c\eta^3 - b\zeta^3),$$

and hence the whole term in x^3 is $= a^2x^3(c\eta^3 - b\zeta^3)$.

8, 9, 10. The coefficient of $x^3\eta^3$ in B is a^2c, and hence in δB, δ^2B, δ^3B the coefficients of this term are $2a'ac + a^2c'$, $2a'^2c + 4aa'c'$, $6a'^2c'$, whence in

$$B', B'', B''' = \tfrac{1}{6}\delta B, \tfrac{1}{24}\delta^2B, \tfrac{1}{48}\delta^3B \text{ respectively,}$$

the coefficients are

$$\tfrac{1}{6}(a^2c' + 2aa'c),\quad \tfrac{1}{12}(a'^2c + 2aa'c'),\quad \tfrac{1}{8}a'^2c',$$
$$= \quad 3l^2a^2c, \qquad 9l^4a^2c, \qquad 27l^6a^2c \text{ respectively.}$$

19 $$J = -\tfrac{1}{3}\,\text{Jac}(U, H, \Lambda) = -\tfrac{1}{2}\begin{vmatrix} X, & X', & \xi \\ Y, & Y', & \eta \\ Z, & Z', & \zeta \end{vmatrix};$$

a term is $-\tfrac{1}{2}(YZ' - Y'Z)\xi$, where, as in a previous calculation,

$$YZ' - Y'Z = -2(abc + 8l^3)x(by^3 - cz^3).$$

Hence, the whole is

$$= (abc + 8l^3)\{\xi x(by^3 - cz^3) + \eta y(cz^3 - ax^3) + \zeta z(ax^3 - by^3)\}.$$

27. $$\bar{J} = \tfrac{1}{3}[\text{Jac}](P, Q, \Lambda) = \tfrac{1}{2}\begin{vmatrix} \mathrm{a}\xi^2 + 2\mathrm{l}\eta\zeta, & \mathrm{a}'\xi^2 + 2\mathrm{l}'\eta\zeta, & x \\ \mathrm{b}\eta^2 + 2\mathrm{l}\zeta\xi, & \mathrm{b}'\eta^2 + 2\mathrm{l}'\zeta\xi, & y \\ \mathrm{c}\zeta^2 + 2\mathrm{l}\xi\eta, & \mathrm{c}'\zeta^2 + 2\mathrm{l}'\xi\eta, & z \end{vmatrix},$$

if, as in a previous calculation

$$6P = \mathrm{a}\xi^3 + \mathrm{b}\eta^3 + \mathrm{c}\zeta^3 + 6\mathrm{l}\xi\eta\zeta, \quad Q = \mathrm{a}'\xi^3 + \mathrm{b}'\eta^3 + \mathrm{c}'\zeta^3 + 6\mathrm{l}'\xi\eta\zeta.$$

Here, as before,

$$(\mathrm{b}\eta^2 + 2\mathrm{l}\zeta\xi)(\mathrm{c}'\zeta^2 + 2\mathrm{l}'\xi\eta) - (\mathrm{b}'\eta^2 + 2\mathrm{l}'\zeta\xi)(\mathrm{c}\zeta^2 + 2\mathrm{l}\xi\eta) = 2(abc + 8l^3)^2(ca\eta^3 - ab\zeta^3)\,\xi.$$

Hence, the whole is

$$= (abc + 8l^3)^2\{x\xi a(c\eta^3 - b\zeta^3) + y\eta b(a\zeta^3 - c\xi^3) + z\zeta c(b\xi^3 - a\eta^3)\}.$$

20. $$K = -\tfrac{3}{2}(\partial_\xi\Theta\partial_x H + \partial_\eta\Theta\partial_y H + \partial_\zeta\Theta\partial_z H) - SU\Lambda,$$

which, H being

$$= \tfrac{1}{6}(a'x^3 + b'y^3 + c'z^3 + 6l'xyz),$$

and putting

$$\Theta = (A, B, C, F, G, H \between \xi, \eta, \zeta)^2, \quad A = -l^2x^2 + bcyz, \ldots, F = -alx^2 + l^2yz, \ldots,$$

is

$$\begin{aligned} = -\tfrac{3}{2}\{&(a'x^2 + 2l'yz)(A\xi + H\eta + G\zeta) - (-abcl + l^4)\,U(\xi x + \eta y + \zeta z) \\ &+ (b'y^2 + 2l'zx)(H\xi + B\eta + F\zeta) \\ &+ (c'z^2 + 2l'xy)(G\xi + F\eta + C\zeta)\}. \end{aligned}$$

The whole coefficient of ξ is thus

$$\begin{aligned} &= -\tfrac{3}{2}\{(a'x^2 + 2l'yz)A + (b'y^2 + 2l'zx)H + (c'z^2 + 2l'xy)G\} - (-abcl + l^4)\,Ux \\ &= -\tfrac{3}{2}\{(a'x^2 + 2l'yz)(-l^2x^2 + bcyz) + (b'y^2 + 2l'zx)(-clz^2 + l^2xy) \\ &\qquad + (c'z^2 + 2l'xy)(-bly^2 + l^2zx)\} - (-abcl + l^4)\{ax^4 + bxy^3 + cxz^3 + 6lx^2yz\}, \end{aligned}$$

and herein the coefficient of x^4 is

$$= \tfrac{3}{2}a'l^2 - al(-abc + l^3), = 9al^4 - al(-abc + l^3), = (abc + 8l^3)\,al;$$

viz. we have thus the term $(abc + 8l^3)\,\xi \,.\, alx^4$ of the final result.

21. $K' = -(\delta)K$, where K is of the form $(abc + 8l^3)(aU + bV + cW)$; operating with (δ), we obtain $(abc + 8l^3)(a\delta U + b\delta V + c\delta W)$. Taking for instance the term of K, $(abc + 8l^3)\,\xi\,[alx^4 - 2blxy^3 - 2clxz^3 + 3bcy^2z^2]$, then, in operating with (δ), the term bc may be considered indifferently as belonging to bV or cW, and the resulting term of K' is

$$\begin{aligned} K' = -(\delta)K &= -(abc + 8l^3)\,\xi\,[al'x^4 - 2bl'xy^3 - 2cl'xz^3 + 3bc'y^2z^2], \\ &= \quad (abc + 8l^3)\,\xi\,[(abc + 2l^3)(ax^4 - 2bxy^3 - 2cxz^3) - 18bcl^2y^2z^2]. \end{aligned}$$

28. $$\overline{K} = 3\{\partial_x\Theta\partial_\xi P + \partial_y\Theta\partial_\eta P + \partial_z\Theta\partial_\zeta P\} + Q\Lambda\text{; viz. writing}$$

$$\Theta = (A,\ B,\ C,\ F,\ G,\ H \between x,\ y,\ z)^2,\quad A = -l^2\xi^2 - 2al\eta\zeta, \ldots\ F = \tfrac{1}{2}bc\xi^2 + l^2\eta\zeta, \ldots,$$

then this is

$$\begin{aligned}
= 3\,\{&[-3bcl\xi^2 + (-abc + 4l^3)\,\eta\zeta]\,2\,(Ax + Hy + Gz)\\
&+ [-3cal\eta^2 + (-abc + 4l^3)\,\zeta\xi]\,2\,(Hx + By + Fz)\\
&+ [-3abl\zeta^2 + (-abc + 4l^3)\,\xi\eta]\,2\,(Gx + Fy + Cz)\}\\
&+ \{(abc - 10l^3)(bc\xi^3 + ca\eta^3 + ab\zeta^3) - 6l^2(5abc + 4l^3)\,\xi\eta\zeta\}\,(\xi x + \eta y + \zeta z).
\end{aligned}$$

The whole coefficient of x is thus

$$\begin{aligned}
= 3\,\{&[-3bcl\xi^2 + (-abc + 4l^3)\,\eta\zeta]\,(-2l^2\xi^2 - 4al\eta\zeta)\\
&+ [-3cal\eta^2 + (-abc + 4l^3)\,\zeta\xi]\,(ab\zeta^2 + l^2\xi\eta)\\
&+ [-3abl\zeta^2 + (-abc + 4l^3)\,\xi\eta]\,(ac\eta^2 + l^2\zeta\xi)\}\\
&+ \{(abc - 10l^3)(bc\xi^4 + ca\xi\eta^3 + ab\xi\zeta^3) - 6l^2(5abc + 4l^3)\,\xi^2\eta\zeta\};
\end{aligned}$$

herein the coefficient of ξ^4 is $18bcl^3 + (abc - 10l^3)\,bc$, $= (abc + 8l^3)\,bc$, giving, in the final result, the term $(abc + 8l^3)\,\xi\,.\,bcx^4$.

29. $$\overline{K}' = \tfrac{1}{6}(\delta)\,\overline{K}.$$

Here $\overline{K}$ is of the form $(abc + 8l^3)(aU + bV + cW)$, and we have

$$\overline{K}' = \tfrac{1}{6}(abc + 8l^3)(a\delta U + b\delta V + c\delta W).$$

A term of $aU + bV + cW$ is $x\,[bc\xi^4 - 2ca\xi\eta^3 - 2ab\xi\zeta^3 - 6al\eta^2\zeta^2]$, where $bc\xi^4$ may be considered as belonging indifferently to bV or cW; and so for the other terms. The resulting term in $\tfrac{1}{6}(a\delta U + b\delta V + c\delta W)$ is thus

$$\tfrac{1}{6}x\,[bc'\xi^4 - 2ca'\xi\eta^3 - 2ab'\xi\zeta^3 - 6al'\eta^2\zeta^2],$$

which is

$$= x\,[l^2(bc\xi^4 - 2ca\xi\eta^3 - 2ab\xi\zeta^3) + a\,(abc + 2l^3)\,\eta^2\zeta^2],$$

and we have thus a term of $\overline{K}'$.

22. $$E = -\tfrac{1}{18}\,\mathrm{Jac}\,(K,\ U,\ \Lambda):$$

K contains the factor $abc + 8l^3$, and if, omitting this factor, the value of K is called $A\xi + B\eta + C\zeta$, then we have

$$\begin{aligned}
E = -\tfrac{1}{6}\{&(\xi\partial_x A + \eta\partial_x B + \zeta\partial_x C)(Y\zeta - Z\eta) + (\xi\partial_y A + \eta\partial_y B + \zeta\partial_y C)(Z\xi - X\zeta)\\
&+ (\xi\partial_z A + \eta\partial_z B + \zeta\partial_z C)(X\eta - Y\xi)\},
\end{aligned}$$

and the term herein in ξ^2 is $-\tfrac{1}{6}\xi^2(Z\partial_y A - Y\partial_z A)$, where A is

$$= alx^4 - 2blxy^3 - 2clxz^3 + 3bcy^2z^2;$$

viz. the coefficient of ξ^2 is

$$= -\tfrac{1}{6}\{(cz^2 + 2lxy)(-6blxy^2 + 6bcyz^2) - (by^2 + 2lzx)(-6clxz^2 + 6bcy^2z)\}$$
$$= b^2cy^4z - bc^2yz^4 + 2bl^2x^2y^3 - 2cl^2x^2z^3$$
$$= (2l^2x^2 + bcyz)(by^3 - cz^3).$$

Hence, restoring the omitted factor $(abc + 8l^3)$, we have in E the term

$$(abc + 8l^3)\,\xi^2(by^3 - cz^3)\,[2l^2x^2 + bcyz].$$

23, 24. $$E' = -\tfrac{1}{4}(\delta)E,\ E'' = \tfrac{1}{4}(\delta^2)E:$$

E is of the form $(abc + 8l^3)(aU + bV + cW)$, and, as before, in a term such as

$$(abc + 8l^3)\,\xi^2(by^3 - cz^3)(2l^2x^2 + bcyz),$$

we operate with δ or δ^2 only on the factor $2l^2x^2 + bcyz$; and in E' and E'' respectively, operating upon this factor, we obtain

$$-\tfrac{1}{4}\{4ll'x^2 + (bc' + b'c)\,yz\},\ \text{and}\ \tfrac{1}{4}\{4l'^2x^2 + 2b'c'yz\},$$

viz. we thus obtain in E' the term

$$(abc + 8l^3)\,\xi^2(by^3 - cz^3)\,[l(abc + 2l^3)\,x^2 - 3bcl^2yz],$$

and in E'' the term

$$(abc + 8l^3)\,\xi^2(by^3 - cz^3)\,[(abc + 2l^3)^2x^2 + 18bcl^4yz].$$

30. $$\bar{E} = -\tfrac{1}{9}\,\text{Jac}(\bar{K}, U, \Lambda),\ = -\tfrac{1}{3}\begin{vmatrix} \partial_x\bar{K}, & X, & \xi \\ \partial_y\bar{K}, & Y, & \eta \\ \partial_z\bar{K}, & Z, & \zeta \end{vmatrix},$$

and, if omitting in $\bar{K}$ the factor $abc + 8l^3$, we write $\bar{K} = Ax + By + Cz$, where

$$A = bc\xi^4 - 2ca\xi\eta^3 - 2ab\xi\zeta^3 - 6al\eta^2\zeta^2,\ \text{this is} = -\tfrac{1}{3}\begin{vmatrix} A, & X, & \xi \\ B, & Y, & \eta \\ C, & Z, & \zeta \end{vmatrix},$$

which contains the term

$$\tfrac{1}{3}X(B\zeta - C\eta),\ = \tfrac{1}{3}(ax^2 + 2lyz)\{\zeta(ca\eta^4 - 2ab\eta\zeta^3 - 2bc\eta\xi^3 - 6bl\zeta^2\xi^2)$$
$$- \eta(ab\zeta^4 - 2bc\zeta\xi^3 - 2ca\zeta\eta^3 - 6cl\xi^2\eta^2)\},$$
$$= (ax^2 + 2lyz)(c\eta^3 - b\zeta^3)(2l\xi^2 + a\eta\zeta).$$

Hence, restoring the factor $abc + 8l^3$, we have the terms

$$\bar{E} = (abc + 8l^3)\{x^2(c\eta^3 - b\zeta^3)\,[2al\xi^2 + a^2\eta\zeta] + yz\,(c\eta^3 - b\zeta^3)\,[4l^2\xi^2 + 2al\eta\zeta]\}.$$

31 and 32. $$\bar{E}' = -\tfrac{1}{2}(\delta)\,\bar{E},\quad \bar{E}'' = -\tfrac{1}{8}(\delta^2)\,\bar{E}:$$

$\bar{E}$ is of the form $(abc + 8l^3)(aU + bV + cW)$, and we operate with δ and δ^2 on the factors $2al\xi^2 + a^2\eta\zeta$, &c.; viz.

$$\delta(2al\xi^2 + a^2\eta\zeta) = 2(al' + a'l)\,\xi^2 + 2aa'\eta\zeta,\quad \delta^2(2al\xi^2 + a^2\eta\zeta) = 4a'l'\xi^2 + 2a'^2\eta\zeta,$$

and we thus obtain in $\bar{E}'$ the term

$$(abc+8l^3)\, x^2 (c\eta^3 - b\zeta^3)\, [a\,(abc-4l^3)\,\xi^2 - 6a^2l^2\eta\zeta],$$

and in $\bar{E}''$ the term

$$(abc+8l^3)\, x^2 (c\eta^3 - b\zeta^3)\, [-3al^2 (abc+2l^3)\,\xi^2 + 9a^2l^4\eta\zeta].$$

25. $M = \frac{1}{36}\,\mathrm{Jac}\,(U,\ \Psi,\ \Lambda)$: this, omitting the factor $(abc+8l^3)^2$ of Ψ, is

$$= \tfrac{1}{2} \begin{vmatrix} ax^2 + 2lyz, & ax^2 (ax^3 - 5by^3 - 5cz^3), & \xi \\ by^2 + 2lzx, & by^2 (by^3 - 5cz^3 - 5ax^3), & \eta \\ cz^2 + 2lxy, & cz^2 (cz^3 - 5ax^3 - 5by^3), & \zeta \end{vmatrix};$$

the coefficient of ξ herein is

$$\begin{aligned} &= \tfrac{1}{2}\,\{(bcy^2z^2 + 2clxz^3)(cz^3 - 5ax^3 - 5by^3) - (bcy^2z^2 + 2blxy^3)(by^3 - 5cz^3 - 5ax^3)\},\\ &= \tfrac{1}{2}\,\{bcy^2z^2 (-6by^3 + 6cz^3) + 2lx\,[-b^2y^6 + c^2z^6 + 5ax^3 (by^3 - cz^3)]\},\\ &= \quad (by^3 - cz^3)\,[5alx^4 - blxy^3 - clxz^3 - 3bcy^2z^2]. \end{aligned}$$

Hence, restoring the factor $(abc+8l^3)^2$, we have the term

$$(abc+8l^3)^2 . \,\xi\,(by^3 - cz^3)\,[5alx^4 - blxy^3 - clxz^3 - 3bcy^2z^2].$$

26. $M' = -(\delta)\,M$. Here M is of the form $(abc+8l^3)^2 (a^2U + \&c.)$; and the δ operates through the $(abc+8l^3)^2 a^2$, &c.; we, in fact, have in M' the term

$$-(abc+8l^3)^2 . \,\xi\,(by^3 - cz^3)\,[5al'x^4 - bl'xy^3 - cl'xz^3 - 3bc'y^2z^2],$$

which is

$$= (abc+8l^3)^2 . \,\xi\,(by^3 - cz^3)\,[(abc+2l^3)(5ax^4 - bxy^3 - cxz^3) + 18bcl^2y^2z^2].$$

33. $$\bar{M} = -\tfrac{1}{6}\,[\mathrm{Jac}]\,(P,\ F,\ \Lambda),\ = -\tfrac{1}{6} \begin{vmatrix} -3lbc\xi^2 + (-abc+4l^3)\,\eta\zeta, & \partial_\xi F, & x \\ -3lca\eta^2 + (-abc+4l^3)\,\zeta\xi, & \partial_\eta F, & y \\ -3lab\zeta^2 + (-abc+4l^3)\,\xi\eta, & \partial_\zeta F, & z \end{vmatrix},$$

and the whole coefficient of x is thus

$$= \tfrac{1}{6}\,\{[3lca\eta^2 + (abc-4l^3)\,\zeta\xi]\,\partial_\zeta F - [3lab\zeta^2 + (abc-4l^3)\,\xi\eta]\,\partial_\eta F\},$$

or substituting for $\frac{1}{6}\partial_\zeta F$, $\frac{1}{6}\partial_\eta F$ their values, this is

$$\begin{aligned} = \{3lca\eta^2 + (abc-4l^3)\,\zeta\xi\}\,&[a^2b^2\zeta^5 - (abc+16l^3)(b\zeta^2\xi^3 + a\zeta^2\eta^3) \\ &\quad - 4l^2 (bc\xi^4\eta + ca\xi\eta^4 + 4ab\xi\eta\zeta^3) - 8l\,(abc+2l^3)\,\xi^2\eta^2\zeta] \\ - \{3lab\zeta^2 + (abc-4l^3)\,\xi\eta\}\,&[a^2c^2\eta^5 - (abc+16l^3)(a\eta^2\zeta^3 + c\eta^2\xi^3) \\ &\quad - 4l^2 (bc\zeta\xi^4 + 4ca\xi\eta^3\zeta + ab\xi\zeta^4) - 8l\,(abc+2l^3)\,\xi^2\eta\zeta^2]. \end{aligned}$$

Collecting, first, the terms independent of $abc - 4l^3$, and, next, those which contain $abc - 4l^3$, each set contains the factor $c\eta^3 - b\zeta^3$, and the whole is $= c\eta^3 - b\zeta^3$ multiplied by

$$- 3la^3bc\eta^2\zeta^2 - 3a^2l(abc + 8l^3)\eta^2\zeta^2 - 12l^3(abc\xi^4 + a^2c\xi\eta^3 + a^2b\xi\zeta^3) - 24al^2(abc + 2l^3)\xi^2\eta\zeta$$
$$+ (abc - 4l^3)\{a^2c\xi\eta^3 + a^2b\xi\zeta^3 - (abc + 16l^3)\xi^4 + 12al^2\xi^2\eta\zeta\};$$

and here collecting the terms in ξ^4, $\xi(c\eta^3 + b\zeta^3)$, $\xi^2\eta\zeta$, and $\eta^2\zeta^2$, each of these contains the factor $abc + 8l^3$, and, finally, the term of $\overline{M}$ is

$$= (abc + 8l^3)(c\eta^3 - b\zeta^3)[(abc - 8l^3)\xi^4 - a^2c\xi\eta^3 - a^2b\xi\zeta^3 - 12al^2\xi^2\eta\zeta - 6a^2l\eta^2\zeta^2]\,x.$$

34. $$\overline{M}' = \tfrac{1}{6}(\delta)\,\overline{M}.$$

Here M is of the form $(abc + 8l^3)(aU + bV + cW)$; and, operating with δ through the $(abc + 8l^3)\,a$, &c., we obtain in $\overline{M}'$ the term

$$\tfrac{1}{6}(abc + 8l^3)\,x\,(c\eta^3 - b\zeta^3)[(a'bc + ab'c + abc' - 24l^2l')\,\xi^4 + \&c.],$$

where

$$a'bc + ab'c + abc' - 24l^2l' = 18abcl^2 + 24l^2(abc + 2l^3),\ = 6l^2(7abc + 8l^3),$$

and the term thus is

$$= (abc + 8l^3)\,x\,(c\eta^3 - b\zeta^3)[(7abc + 8l^3)\,l^2\xi^4 + \ldots].$$

This concludes the series of calculations.

Cambridge, England, 17 *May*, 1881.