782.

ON MONGE'S "MÉMOIRE SUR LA THÉORIE DES DÉBLAIS ET DES REMBLAIS."

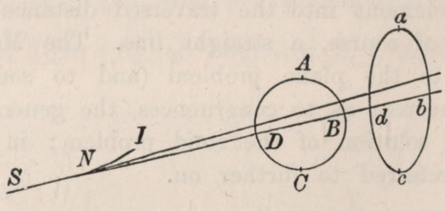
[From the *Proceedings of the London Mathematical Society*, vol. XIV. (1883), pp. 139—142. Read March 8, 1883.]

THE Memoir referred to, published in the *Mémoires de l'Académie*, 1781, pp. 666—704, is a very remarkable one, as well for the problem of earthwork there considered as because the author was led by it to his capital discovery of the curves of curvature of a surface. The problem is, from a given area, called technically the Déblai, to transport the earth to a given equal area, called the Remblai, with the least amount of carriage. Taking the earth to be of uniform infinitesimal thickness over the whole of each area (and therefore of the same thickness for both areas), the problem is a plane one; viz. stating it in a purely geometrical form, the problem is: Given two equal areas, to transfer the elements of the first area to the second area in such wise that the sum of the products of each element into the traversed distance may be a minimum; the route of each element is, of course, a straight line. And we have the corresponding solid problem: Given two equal volumes, to transfer the elements of the first volume to the second volume in such wise that the sum of the products of each element into the traversed distance may be a minimum; the route of each element is, of course, a straight line. The Memoir is divided into two parts: the first relating to the plane problem (and to some variations of it): the second part contains a theorem as to congruences, the general theory of the curvature of surfaces, and finally a solution of the solid problem; in regard to this, I find a difficulty which will be referred to further on.

I have said that Monge gives a theorem as to congruences. This is not stated quite in the best form,—viz. instead of speaking of a singly infinite system of lines, or even of the lines drawn according to a given law from the several points of a *surface*, he speaks of the lines drawn according to a given law from the several points of a *plane* (but, of course, any congruence whatever of lines can be so represented); and he establishes the theorem that each line of the system is intersected by each of two consecutive lines,—viz. taking (x', y') as the coordinates of the point of intersection of any line with the plane of xy, he obtains, as the condition of intersection with the consecutive line a quadric equation in (dx', dy'). He then considers the normals of a surface, (which, as lines drawn according to a given law from any point of a *surface*, require a slightly different analytical investigation), establishes for them the like theorem, and shows moreover that the two directions of passage on the surface to a consecutive point are at right angles to each other; or, what is the same thing, that in the two sets of developable surfaces formed by the intersecting normals, each surface of the one set intersects each surface of the other set in a straight line, and at right angles. He speaks expressly of the lines of greatest and least curvature, and generally establishes the whole theory of the curvature of surfaces in a very complete and satisfactory manner; the particular case of surfaces of the second order is not considered. It may be remarked that, although not explicitly stating it, he must have seen that a congruence of lines is not, *in general*, a system of normals of a surface (that is, the lines of a congruence cannot be, in general, cut at right angles by any surface); he, in fact, assumes (quite correctly, but a proof should have been given) that a congruence of lines for which the two sets of developable surfaces intersect at right angles is a system of normals of a surface.

Reverting to the before-mentioned problem (plane or solid), I remark that this is a problem of minimum *sui generis*. Considering the first area or volume as divided in any manner into infinitesimal elements, we have to divide the second area or volume into corresponding equal elements, in such wise that the sum of the products of each element of the first area or volume into its distance from the corresponding element of the second area or volume may be a minimum; but, for doing this, we have no means of forming the analytical expression of any function which is to be, by the formulæ of the differential calculus or the calculus of variations, made a minimum.

For the plane problem, Monge obtains the solution by means of the very simple consideration that the routes of two elements must not cross each other; in fact, imagine an element A transferred to a, and an equal element B transferred to b: the lines Aa, Bb must not cross each other, for if they did, drawing the two lines



Ab and Ba, the sum $Aa + Bb$ would be greater than the sum $Ab + Ba$, contrary to the condition of the minimum. Imagine the areas intersected by two consecutive lines as shown in the figure: the filament between these two lines may be regarded as a right line; and, assuming that some one element of the filament BD is transferred to a point of bd (that is, so as to coincide with an element of the filament bd), it follows that every other element of BD must be transferred so as to coincide with some other element of bd; and this obviously implies that the filaments BD and bd must be equal. Observe that, this being so, it is immaterial which element of BD is transferred to which element of bd; in whatever way this is done, the sum of the products will be the same*. The two lines may be regarded as the normals of a curve; and the problem thus is, to find a curve such that, drawing the normals thereof to intersect the two areas, then that the filaments BD and bd, cut off by consecutive normals on the two areas respectively, shall be equal. This leads to a differential equation of the second order for the normal curve; one of the constants of integration remains arbitrary, for the normal curve is any one of a system of parallel curves. It is to be observed that the filaments are the increments of the areas BCD and bcd; these increments are equal; a position of the line must be the common tangent Cc of the two areas (this, in fact, constitutes the condition for the determination of one of the arbitrary constants), and for this position the areas are each $=0$. Hence, in general, the areas must be equal; or the problem is, to find a curve such that any normal thereof cuts off equal areas BCD and bcd.

If, instead of the normal curve, we consider the curve which is the envelope of the several lines, or, what is the same thing, the locus of the point N, then we could, in like manner, obtain for this curve a differential equation of the first order: the constant of integration would be determined by the condition that Cc is a tangent. The curve in question is, of course, the evolute of the normal curve.

The several lines which intersect the two areas give rise to a finite arc IS of this evolute, and, as remarked by Monge, it is only when this arc IS lies (as in the figure) *outside* the two areas, that we have a true minimum.

Passing now to the solid problem, we may imagine a congruence of lines intersecting the two volumes; each line of the congruence is intersected by two consecutive lines, and the lines of the congruence thus form two sets of developable surfaces, each surface of the one set intersecting each surface of the other set. And, considering two consecutive surfaces of the one set, and two consecutive surfaces of the other set, these include between them a filament; and, treating the filament as a right line, it seems to follow (although it is more difficult to present the reasoning in a rigorous form) that, if any one element of the filament BD be transferred to any one element of the filament bd, then that every other element of the filament BD must be

* The most simple case is, take in the same straight line two equal segments AB, ab; it is immaterial how the elements of AB are transferred to ab, the sum of the products of each element into the traversed distance will be in every case the same. Analytically, if $dx=dx'$, then

$$\int (x'-x)\,dx=\int x'dx'-\int x\,dx,$$

the equation $dx'=dx$ meaning $x'=x+$ a discontinuous constant. In the actual case of the filament, the formula is, if $rdr=r'dr'$, then

$$\int (r'-r)\,r\,dr=\int r'^2\,dr'-\int r^2\,dr.$$

transferred to some other element of the filament bd; and, this being so, the two filaments must be equal. But Monge goes on to argue that the condition of the minimum further requires that the developable surfaces shall cut at right angles, *and I cannot say that I see this.* He says (pp. 700, 701), "We know already that the routes must be the intersections of two sets of developable surfaces such that each surface of the first set intersects those of the second set in right lines; it remains to be found under what angles these surfaces must cut each other to satisfy the minimum. But it is evident that these angles must be right angles, for with these angles the elementary spaces comprised between four developable surfaces will be greater, and for equal distances the transported mass will be greater; therefore, in the case of a minimum, the routes must be the intersections of two sets of developable surfaces such that each surface of the one set cuts those of the second set in straight lines and at right angles." And, this being so, he infers, and it in fact follows, that the routes are the normals of a surface.

Admitting the conclusion, the problem becomes as follows:—Given two volumes, it is required to find a surface such that, drawing the normals thereof to intersect the two volumes, and considering the filament bounded by the developable surfaces which belong to two consecutive curves of curvature of the one set and those belonging to two consecutive curves of curvature of the other set, the portions cut off on the two volumes respectively may be equal. And we are thus led to a partial differential equation of the second order for determining the equation $z = f(x, y)$ of the required surface. As in the plane problem, it is immaterial how the elements of the one filament are transferred to the other filament.