CHAPTER I.

NATURE OF THE PROBLEM. PRELIMINARY CONSIDERATIONS.

1. Integration is a reversal of the operation of Differentiation, the finding of a function of x when the differential coefficient is known. Thus the differential coefficient of x^2e^x, say, is $(2x+x^2)e^x$. We require a method of retracing our steps, and having given the expression $(2x+x^2)e^x$, we aim at the formulation of a method of arriving at the original function x^2e^x. The result of integrating a function of x is called the integral of the function.

2. In the language of the early writers on the subject, a differential coefficient was called a "fluxion." The original expression regarded as derived from the differential coefficient was called the "fluent."

Thus, in Kinetics, if s be the space described by a particle moving with a uniform acceleration f in time t, and with initial velocity u, $s=ut+\frac{1}{2}ft^2$, and the velocity at any time is given by $v=u+ft$. We obtain, by differentiating these expressions,

$$\frac{dv}{dt}=f, \quad \frac{ds}{dt}=u+ft.$$

So f is the differential coefficient (or "fluxion") of v with regard to t,

$u+ft$ is the differential coefficient (or "fluxion") of s with regard to t.

Regarding $u+ft$ and f as the original quantities, their integrals with regard to t (*i.e.* their "fluents") are respectively $ut+\frac{1}{2}ft^2$, *i.e.* s, and $ft+u$, *i.e.* v.

3. It will be noted that, as a constant quantity has no "rate of variation," all unattached constants, *i.e.* constants which do not multiply variables, as for instance u in the formula $v=u+ft$, disappear on differentiation. We may therefore expect constants to reappear upon integration.

Thus it appears that the differential coefficient with regard to time (or "fluxion") of a length, or distance, is a velocity or rate of change of the length. The integral with regard to time (or "fluent") of a velocity is a length. In other words, the problem of the Differential Calculus is, given any quantity which is changing its value continuously, to find the rate of that change; whilst the problem to be attacked in the Integral Calculus is the converse, viz., given the rate of change, to find what the nature of the varying quantity must be.

4. The general character of integration is necessarily tentative. Newton remarked in his *Method of Fluxions*, "It may not be amiss to take notice, that in the Science of Computation all the Operations are of two kinds, either Compositive or Resolutative. The Compositive or Synthetic Operations proceed necessarily and directly, in computing their several *quaesita*, and not tentatively or by way of tryal. Such are Addition, Multiplication, Raising of Powers, and taking of Fluxions. But the Resolutative or Analytical Operations, as Subtraction, Division, Extraction of Roots, and finding of Fluents, are forced to proceed indirectly and tentatively, by long deduction, to arrive at their several *quaesita*; and suppose or require the contrary Synthetic Operations, to prove and compare every step of the process. The Compositive Operations, always when the *data* are finite and terminated, and often when they are interminate or infinite, will produce finite conclusions; whereas, very often in the Resolutative Operations, tho' the *data* are in finite Terms, yet the *quaesita* cannot be obtain'd without an infinite Series of Terms."

5. We have illustrated the object of integration from the fundamental equations of motion of a particle moving with a constant acceleration and with a given initial velocity. This is sufficient for the present. But it will be seen later that the reversal of the operation of differentiation will also enable us to calculate with precision the areas bounded by curved lines, the lengths of such curved lines, the volumes contained by curved surfaces, the areas of such surfaces and many other quantities which it is necessary to find in both Pure and Applied Mathematics.

6. Before embarking upon the general problem of the reversal of a differential operation, it will be instructive to the student to consider how such a reversal could be used in such a problem as the discovery of the area of a space bounded by curved lines.

The plan adopted for this purpose is to imagine the area divided into a very large number of very small elements according to some fixed principle of division. We have then to devise some method of obtaining the limit of the sum of all these elements when each is ultimately infinitesimally small, and at the same time their number is indefinitely increased. And when once such a method of summation is discovered it will be found to be applicable also to many other problems, such as those already mentioned of finding the lengths of specified portions of curves, volumes bounded by specific surfaces, the positions of centroids, etc.

7. In some elementary cases it will be found that the requisite summation can be performed by ordinary algebraical or trigonometrical means. But such processes will be generally tedious and almost always inadequate to the treatment of any but the simplest examples.

A fundamental theorem will, however, be established showing how this summation depends upon the *reversal of a differentiation*. We shall therefore, after a few illustrations, confine our attention for several chapters mainly to the purely analytical problem of reversing the fundamental operation of the Differential Calculus, with the end explained in view. And when the student is well equipped with this powerful weapon we shall proceed to discuss more fully the uses to which the process may be applied.

8. To avoid constant repetition, we may state that throughout the book all coordinate axes will be supposed rectangular, all angles will be supposed measured in circular measure, all logarithms will be supposed Napierian except where otherwise expressly stated, and for the present all variables will be supposed real and all functions will be considered continuous functions of a real variable.

9. NEWTON'S SECOND LEMMA.

In the First Section of the *Principia* (Lemma II.), Newton enunciates and proves the following Theorem:*

If in any figure Aab ... kL bounded by the straight lines Aa, AL and the curve abc ... kL any number of parallelograms Ab, Bc, Cd, etc., be inscribed upon equal bases AB, BC, CD, etc., and having sides Bb, Cc, Dd, etc., parallel to the side Aa of the figure, and the parallelograms aPbp, bQcq, cRdr, etc., be completed; then, if the breadth of these parallelograms be diminished and the number increased indefinitely, the ultimate ratios which the inscribed figure APbQcRdS ... kK, the circumscribed figure Aapbqcrd ... ykzL and the curvilinear figure Aabcde ... kL have to one another are ratios of equality.

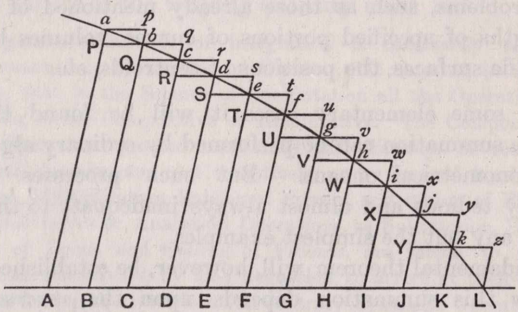

Fig. 1.

To prove this statement it may be observed that the difference of the sums of the inscribed and circumscribed rectilineal figures is the sum of the parallelograms Pp, Qq, Rr, ..., Kz; and as the bases Pb, Qc, ..., KL of these parallelograms are all equal and their aggregate altitude is the sum of their individual altitudes, the sum of these parallelograms is equal to the parallelogram Ap. And in the limit, when the bases AB BC, ..., are diminished indefinitely, the area of this parallelogram which has a finite altitude and indefinitely small breadth becomes less than anything conceivable, however small. Hence the inscribed and circumscribed figures, and therefore also the curvilinear figure whose area is intermediate between the areas of these figures, in the limit become ultimately equal

* See Frost's *Newton's Principia*, pages 17, 18.

10. Newton devotes the next Lemma (III.) to proving that "the same ultimate ratios are also ratios of equality when the breadths of the parallelograms, AB, BC, CD, ... are *unequal*, and are all diminished indefinitely."

This is proved in like manner, and may be established by the student.

It follows that the limit of the sum of either the inscribed parallelograms or of the parallelograms which make up the circumscribed figure ultimately coincides in area with that of the curvilinear figure itself.

11. Analytical expression of the above result.

We shall now obtain an analytical expression for the sum of such a system of inscribed parallelograms.

Suppose it be required to find the area of the portion of space bounded by a given curve AB, whose Cartesian Equation is $y=\phi(x)$, the ordinates AL and BM, and the axis of x, the axes being rectangular, and all ordinates from A to B being finite, and for the purposes of this article, increasing or decreasing from A to B.

Following the method of Newton's Second Lemma, let LM be divided into n equal small parts LQ_1, Q_1Q_2, Q_2Q_3, ..., each of length h; and let a and b be the abscissae of A and B, *i.e.* $OL=a$, $OM=b$. Then $b-a=nh$.

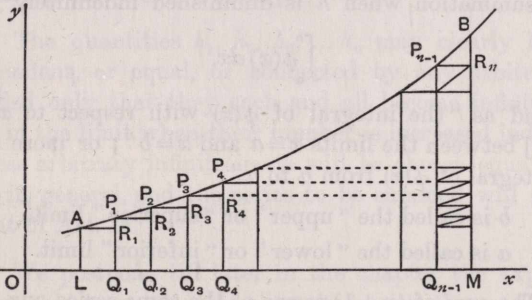

Fig. 2.

The ordinates LA, Q_1P_1, Q_2P_2, etc., $Q_{n-1}P_{n-1}$, MB at the points L, Q_1, Q_2, ..., Q_{n-1}, M are respectively

$\phi(a)$, $\phi(a+h)$, $\phi(a+2h)$, $\phi(a+3h)$, ..., $\phi\{a+(n-1)h\}$, $\phi(b)$.

Complete the rectangles AQ_1, P_1Q_2, P_2Q_3, ...

Now the sum of these n rectangles falls short of the area sought by the sum of the n small figures AR_1P_1, $P_1R_2P_2$, etc. Let each of these be supposed to slide parallel to the x-axis into a corresponding position upon the longest strip, say $P_{n-1}Q_{n-1}MB$. Their sum is then less than the area of this strip, *i.e.* in the limit less than an infinitesimal of the first order, for the breadth $Q_{n-1}M$ is h and is ultimately an infinitesimal of the first order, and the length MB is supposed finite.

Hence the area required is the limit when h is zero (and therefore n infinite) of the sum of the n infinitesimal terms of the first order,

$$h\phi(a)+h\phi(a+h)+h\phi(a+2h)+\ldots+h\phi[a+(n-1)h].$$

This sum may be denoted by

$$\mathop{S}_{a+rh=a}^{a+rh=b-h} \phi(a+rh)\,h \quad \text{or} \quad \sum_a^{b-h}\phi(a+rh)\,h,$$

where S or Σ denotes the "sum" between the limits indicated.

Regarding $a+rh$ as a variable x, the infinitesimal increment h may be written as δx or dx. It is customary also upon taking the limit to replace the symbol S by the more convenient sign $\int$, which is, as a matter of fact, merely only another way of writing the same letter, and the limit of the above summation when h is diminished indefinitely is then written

$$\int_a^b \phi(x)\,dx,$$

and read as "the integral of $\phi(x)$ with respect to x [or of $\phi(x)\,dx$] between the limits $x=a$ and $x=b$"; or more shortly "the integral of $\phi(x)$ from a to b."

b is called the "upper" or "superior" limit,

a is called the "lower" or "inferior" limit.

12. The sum of $(n+1)$ terms of the same series, viz.,

$$h\phi(a)+h\phi(a+h)+h\phi(a+2h)+\ldots \\ +h\phi[a+(n-1)h]+h\phi(a+nh),$$

differs from the above series merely in the addition of the term $h\phi(a+nh)$, *i.e.* $h\phi(b)$, which being an infinitesimal of the first order vanishes when the limit is taken. Hence the limit of this series may also be written

$$\int_a^b \phi(x)\,dx.$$

13. In the same way, if in fig. 2, Art. 11, LQ_1, Q_1Q_2, Q_2Q_3, ..., $Q_{n-1}M$ are not necessarily equal, but are respectively h_1, h_2, h_3, ..., h_n, the ordinates at the several points L, Q_1, Q_2, ..., Q_{n-1} are respectively,

$$\phi(a), \quad \phi(a+h_1), \quad \phi(a+h_1+h_2), \ldots, \quad \phi(b-h_n)$$

and the sum of the inscribed rectangles is

$$h_1\phi(a)+h_2\phi(a+h_1)+h_3\phi(a+h_1+h_2)+\ldots+h_n\phi(b-h_n),$$

and the sum of the residuary areas AR_1P_1, $P_1R_2P_2$, $P_2R_3P_3$, etc., is less than the area of a rectangle whose breadth is the greatest of the quantities h_1, h_2, h_3 ... h_n, and whose height is the greatest ordinate of the given curve; and as in the last article, this sum therefore vanishes in the limit when h_1, h_2, h_3, ... h_n are each made infinitesimally small, provided that the curve has no infinite ordinate either at A, B or between A and B.

Hence the limit of

$$h_1\phi(a)+h_2\phi(a+h_1)+h_3\phi(a+h_1+h_2)+\ldots+h_n\phi(b-h_n),$$

is also the area of the portion $LABM$ described in Art. 11.

[See also Art. 1875, Vol. II.]

14. The quantities h_1, h_2, h_3, ... h_n may clearly be either independent, or equal, or connected by any arbitrary law, provided only that they each and all become infinitesimally small in the limit when their number is increased indefinitely.

These arbitrary infinitesimals will be chosen equal to each other in general, and the series to be summed will therefore be that of Art. 11.

15. We postpone till later in the chapter the explanation of how this summation is connected with the reversal of a differentiation, and illustrate what has been stated as to the finding of areas by a few elementary cases in which the limit of the summation may be found by elementary processes without undue difficulty.

16. Illustrative Examples.

Ex. 1. To calculate $\int_a^b ce^{mx}\,dx$, that is to find the area of the space bounded by the x-axis, the logarithmic curve $y=ce^{mx}$ and two ordinates $x=a$ and $x=b$.

Here we have to evaluate

$$Lt_{h=0}\,ch\left[e^{ma}+e^{m(a+h)}+e^{m(a+2h)}+\dots+e^{m(a+\overline{n-1}h)}\right]$$

where $b=a+nh$.

This expression $=Lt_{h=0}\,che^{ma}\dfrac{e^{nmh}-1}{e^{mh}-1}$

$$=Lt_{h=0}\,c\frac{e^{ma}}{m}\cdot\frac{mh}{e^{mh}-1}\cdot\left[e^{m(b-a)}-1\right]$$

$$=\quad c\frac{e^{ma}}{m}\,.\,1\,.\left[e^{m(b-a)}-1\right],\quad \text{by } \textit{Diff. Cal.} \text{ (Art. 21)},$$

$$=\quad c\,\frac{e^{mb}-e^{ma}}{m}.$$

$\therefore$ the area sought is equal to the rectangle contained by $\dfrac{1}{m}$ (which is of the dimension of a line) and the difference of the initial and final ordinates.

E.g. if now $\dfrac{1}{m}=1$ inch and $a=0$, $b=1$, $c=2$,

the area in question $=2(e-1)=2\times 1{\cdot}71828\ldots$ square inches
$=3{\cdot}43656\ldots$ square inches,
i.e. a little less than $3\frac{1}{2}$ square inches.

Ex. 2. Shew that in the last result, *i.e.* $y=ce^{mx}$, if $A_1, A_2, A_3, \ldots$ be the areas between

$x=0$ and $x=1$, $x=1$ and $x=2$, $x=2$ and $x=3$, etc.,

then $A_1, A_2, A_3, \ldots$ form a G.P. whose common ratio is e^m.

Ex. 3. Calculate the area bounded by the *curve of sines* $y=c\sin mx$, the x-axis and two ordinates $x=a$ and $x=b$ $\left(0<a<b<\dfrac{\pi}{m}\right)$.

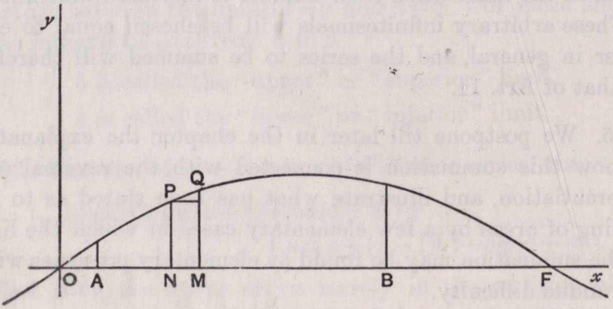

Fig. 3.

Here we are to evaluate $\int_a^b c \sin mx\, dx$,

that is $Lt_{h=0} ch[\sin ma + \sin m(a+h) + \sin m(a+2h) \dots \text{to } n \text{ terms}]$
where $nh = b - a$.

$$\text{This expression} = Lt_{h=0} ch \frac{\sin\left\{ma + (n-1)\frac{mh}{2}\right\} \sin n\frac{mh}{2}}{\sin\frac{mh}{2}}$$

$$= Lt_{h=0} c\left[\cos m\left(a - \frac{h}{2}\right) - \cos m\left\{a + (2n-1)\frac{h}{2}\right\}\right]\frac{\frac{mh}{2}}{\sin\frac{mh}{2}}\cdot\frac{1}{m}$$

$$= c\frac{\cos ma - \cos mb}{m}.$$

Thus, if the limits are such as to take in one half wave length, *i.e.* the portion above the x-axis from $x=0$ to $mx=\pi$, and if $c=1$ inch, the area sought is

$$\frac{\cos 0 - \cos\pi}{m} = \frac{2}{m},$$

or if, say, $m=\frac{1}{10}$, the area is 20 square inches.

Ex. 4. Find the value of $\int_a^b \frac{x^3}{c^2}dx$; that is the area bounded by the cubical parabola $c^2y=x^3$, the x-axis and two ordinates $x=a$ and $x=b$.

Here we have to evaluate

$$Lt_{h=0}\frac{1}{c^2}\sum_{r=0}^{r=n-1}(a+rh)^3h,$$

where $nh=b-a$.

Now $\frac{h}{c^2}[a^3+(a+h)^3+(a+2h)^3+\dots+(a+\overline{n-1}h)^3]$

$$= \frac{h}{c^2}\left[na^3 + 3a^2h\frac{(n-1)n}{2} + 3ah^2\frac{(n-1)n(2n-1)}{6} + h^3\frac{(n-1)^2n^2}{4}\right]$$

$$= \frac{(b-a)}{c^2}\left[a^3 + 3a^2\frac{(b-a)}{2}\left(1-\frac{1}{n}\right) + \frac{1}{2}a(b-a)^2\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right) + \frac{1}{4}(b-a)^3\left(1-\frac{1}{n}\right)^2\right],$$

and when n becomes infinite this becomes

$$= \frac{(b-a)}{c^2}[a^3 + \tfrac{3}{2}a^2(b-a) + a(b-a)^2 + \tfrac{1}{4}(b-a)^3]$$

$$= \frac{(b-a)}{4c^2}(b^3 + b^2a + ba^2 + a^3)$$

$$= \frac{b^4 - a^4}{4c^2}.$$

Ex. 5. Find $\int_a^b \frac{1}{x^2}dx.$

We have to evaluate

$$Lt_{h=0} h\left[\frac{1}{a^2} + \frac{1}{(a+h)^2} + \frac{1}{(a+2h)^2} + \dots + \frac{1}{b^2}\right].$$

This is $\quad > Lt\left[\dfrac{1}{a(a+h)}+\dfrac{1}{(a+h)(a+2h)}+\ldots+\dfrac{1}{b(b+h)}\right]h,$

$$\text{i.e.} > Lt\left[\left(\frac{1}{a}-\frac{1}{a+h}\right)+\left(\frac{1}{a+h}-\frac{1}{a+2h}\right)+\ldots+\left(\frac{1}{b}-\frac{1}{b+h}\right)\right],$$

$$\text{i.e.} > Lt\left(\frac{1}{a}-\frac{1}{b+h}\right),$$

and $\quad < Lt\left[\dfrac{1}{(a-h)a}+\dfrac{1}{a(a+h)}+\ldots+\dfrac{1}{(b-h)b}\right]h$

$$\text{i.e.} < Lt\left[\left(\frac{1}{a-h}-\frac{1}{a}\right)+\left(\frac{1}{a}-\frac{1}{a+h}\right)+\ldots+\left(\frac{1}{b-h}-\frac{1}{b}\right]\right),$$

$$\text{i.e.} < Lt\left(\frac{1}{a-h}-\frac{1}{b}\right),$$

and when h diminishes without limit, each of these expressions becomes $\dfrac{1}{a}-\dfrac{1}{b}$. Thus the value is entrapped between two ultimately equal expressions, and $\int_a^b \dfrac{1}{x^2}dx=\dfrac{1}{a}-\dfrac{1}{b}$.

Ex. 6. **Integration of x^m**, from the definition, between limits a and b $(m \neq -1)$.

Here we have to consider

$$Lt_{h=0}h[a^m+(a+h)^m+(a+2h)^m+\ldots+(a+\overline{n-1}h)^m],$$

where $\dfrac{b-a}{n}=h$ and n is indefinitely large, $m+1$ not being zero.

In the *Differential Calculus for Beginners* (Art. 13) it is proved without the aid of the Binomial Theorem [which was purposely avoided, as it was then proposed later to apply Taylor's Theorem to the expansion of $(x+h)^n$] that

$$Lt_{z=1}\frac{z^{m+1}-1}{z-1}=m+1.$$

Writing $\qquad z=1+\dfrac{h}{y},$

we have
$$Lt_{h=0}\frac{\left(1+\frac{h}{y}\right)^{m+1}-1}{\frac{h}{y}}=m+1,$$

or
$$Lt_{h=0}\frac{(y+h)^{m+1}-y^{m+1}}{hy^m}=m+1.$$

In this result put y successively $a, a+h, a+2h, \ldots, a+(n-1)h$, and we get

$$Lt_{h=0}\frac{(a+h)^{m+1}-a^{m+1}}{ha^m}=Lt_{h=0}\frac{(a+2h)^{m+1}-(a+h)^{m+1}}{h(a+h)^m}=\ldots$$
$$=Lt_{h=0}\frac{(a+nh)^{m+1}-(a+\overline{n-1}h)^{m+1}}{h(a+n-1h)^m}=m+1,$$

or, adding numerators for a new numerator and denominators for a new denominator,

$$Lt\,\frac{(a+nh)^{m+1}-a^{m+1}}{h[a^m+(a+h)^m+(a+2h)^m+\ldots+(a+\overline{n-1}h)^m]}=m+1,$$

i.e. $$Lt_{h=0}\,h[a^m+(a+h)^m+(a+2h)^m+\ldots+(a+\overline{n-1}h)^m]=\frac{b^{m+1}-a^{m+1}}{m+1},$$

i.e. in accordance with the notation of Art. 11,

$$\int_a^b x^m\,dx=\frac{b^{m+1}-a^{m+1}}{m+1}.$$

The letters a and b may represent any finite quantities whatever, provided x^m does not become ∞ between $x=a$ and $x=b$.

When a is taken exceedingly small and ultimately zero it is necessary in the proof to suppose h an infinitesimal of higher order, for it has been assumed that in the limit $\frac{h}{y}$ is zero for all the values given to y.

When $b=1$ and $a=0$, the theorem ultimately becomes

$$\int_0^1 x^m dx=\frac{1}{m+1}\quad\text{if } (m+1) \text{ be positive,}$$
$$\text{or } =\infty\quad\text{if } (m+1) \text{ be negative.}$$

This result may be written also

$$Lt_{n=\infty}\frac{1}{n}\left[\left(\frac{1}{n}\right)^m+\left(\frac{2}{n}\right)^m+\left(\frac{3}{n}\right)^m+\ldots+\left(\frac{n-1}{n}\right)^m\right]=\frac{1}{m+1},\ \text{or } \infty$$

according as $m+1$ is positive or negative.

The Limit $$Lt_{n=\infty}\frac{1}{n}\left[\left(\frac{1}{n}\right)^m+\left(\frac{2}{n}\right)^m+\ldots+\left(\frac{n}{n}\right)^m\right],$$

or, which is the same thing,

$$Lt_{n=\infty}\,\frac{1^m+2^m+3^m+\ldots+n^m}{n^{m+1}}$$

differs from the former by $\frac{1}{n}$, *i.e.* by 0 in the limit, and is therefore also $\frac{1}{m+1}$, or ∞, according as $m+1$ is positive or negative.

The case when $m+1=0$ needs special consideration. It is at once derivable from the result

$$\int_a^b x^m\,dx=\frac{b^{m+1}-a^{m+1}}{m+1}$$

as a limiting form.

$$\begin{aligned}Lt_{m+1=0}\int_a^b x^m\,dx&=Lt_{m+1=0}\,\frac{b^{m+1}-a^{m+1}}{m+1}\\&=Lt\,\frac{b^{m+1}-1}{m+1}-Lt\,\frac{a^{m+1}-1}{m+1}\\&=\log b-\log a\quad (\textit{Diff. Cal.}\ \text{Art. } 21)\\&=\log\frac{b}{a}.\end{aligned}$$

EXAMPLES.

1. Find the values of $\int_a^b x\,dx$ and $\int_a^b x^2\,dx$, and interpret the results geometrically.

2. Find the area of the portion of the parabola $x^2=4ay$ cut off by the latus rectum.

3. Prove by summation that

$$(\alpha)\int_a^b \sinh x\,dx=\cosh b-\cosh a;$$

$$(\beta)\int_a^b \frac{1}{\sqrt{x}}\,dx \quad =2(\sqrt{b}-\sqrt{a});$$

$$(\gamma)\int_a^b \cos mx\,dx=\frac{1}{m}(\sin mb-\sin ma).$$

4. In a right circular cone of height h and semivertical angle a, the axis is divided into a large number, n, of equal portions, and planes are drawn through the points of division perpendicular to the axis, the cone being thus divided into a large number of circular laminae. If x be the distance from the vertex of any of these laminae, show that to the first order of small quantities its volume may be written

$$\pi x^2 \tan^2 a\,\delta x,\ \delta x \text{ being the thickness of the lamina.}$$

Find, by taking the limit of the summation of such quantities, the volume of the cone.

Show also that the volume of a frustum of thickness T is

$$\frac{T}{3}(A+\sqrt{AB}+B),$$

where A and B are the areas of the two ends.

5. A quantity y is an unknown function of another quantity x. When x has the values

5 8 10 12 14 16

y is found by observation to be

2·0 2·6 3·2 3·8 5·0 6·5

respectively, and the errors of observation cannot be more than 5 per cent.; draw the simplest continuous curve which can represent y, and estimate its slope when $x=15$.

Find also the value of x for which the slope of the curve is equal to $\frac{y}{x}$. Estimate the value of the definite integral $\int_{11}^{15} y\,dx$.

17. THE FUNDAMENTAL PROPOSITION.

Let $\phi(x)$ be any function of a real variable x, finite, continuous and single valued, for all values of x from $x=a$ to $x=b$ inclusive. Let a be less than b, each being finite, and suppose the difference $b-a$ to be divided into n portions each equal to h, so that $b-a=nh$. It is required to find the *limit of the sum* of the series

$$h[\phi(a)+\phi(a+h)+\phi(a+2h)+\ldots+\phi(b-h)+\phi(b)]$$

when h is diminished indefinitely, and therefore n increased without limit, keeping the product $nh=b-a$.

That this limit is finite may at once be made clear.

For if $h\phi(a+rh)$, say, be the greatest term, the sum is

$$<(n+1)\,h\phi(a+rh),$$

$$i.e. <(b-a)\,\phi(a+rh)+h\phi(a+rh),$$

which is finite, since by hypothesis $\phi(x)$ is finite for all values of x intermediate between b and a.

Let $\psi(x)$ be another function of x such that $\phi(x)$ is its differential coefficient, *i.e.* such that

$$\phi(x)=\frac{d}{dx}\psi(x)=\psi'(x).$$

We shall then prove that

$$Lt_{h=0}h[\phi(a)+\phi(a+h)+\phi(a+2h)+\ldots+\phi(b)]=\psi(b)-\psi(a).$$

By definition,

$$\phi(a)=Lt_{h=0}\frac{\psi(a+h)-\psi(a)}{h},$$

and therefore $\quad \phi(a)=\dfrac{\psi(a+h)-\psi(a)}{h}+a_1,$

where a_1 is a quantity whose limit is zero when h diminishes indefinitely; thus

$$h\phi(a) \qquad =\psi(a+h) \quad -\psi(a) \qquad\qquad +ha_1.$$

Similarly,

$$h\phi(a+h) \qquad =\psi(a+2h)-\psi(a+h) \qquad +ha_2,$$

$$h\phi(a+2h) \qquad =\psi(a+3h)-\psi(a+2h) \qquad +ha_3,$$

etc.,

$$h\phi\{a+(n-1)h\}=\psi(a+nh)-\psi\{a+(n-1)h\}+ha_n,$$

where the quantities $a_2, a_3, \ldots, a_n$ are all, like a_1, quantities whose limits are zero when h diminishes indefinitely.

By addition,

$$h[\phi(a)+\phi(a+h)+\phi(a+2h)+\ldots+\phi(b-h)]$$
$$=\psi(a+nh)-\psi(a)+h[a_1+a_2+\ldots+a_n].$$

Let α be the greatest of the quantities $\alpha_1, \alpha_2, \dots, \alpha_n$.

Then $h[\alpha_1+\alpha_2+\dots+\alpha_n]$ is $< nh\alpha$,

that is $<(b-a)\alpha$,

and therefore vanishes in the limit.

Thus

$$Lt_{h=0}h[\phi(a)+\phi(a+h)+\phi(a+2h)+\dots+\phi(b-h)]$$
$$=\psi(b)-\psi(a).$$

The term $h\phi(b)$ is itself also in the limit zero; hence, if we desire, it may be added to the left-hand member of this result, without affecting it; and it may then be stated that

$$Lt_{h=0}h[\phi(a)+\phi(a+h)+\phi(a+2h)+\dots+\phi(b-h)+\phi(b)]$$
$$=\psi(b)-\psi(a),$$

$$\textit{i.e.}\ \int_a^b \phi(x)\,dx=\psi(b)-\psi(a),$$

where $$\frac{d\psi}{dx}=\phi(x).$$

The result $\psi(b)-\psi(a)$ is frequently denoted by

$$\Big[\psi(x)\Big]_a^b.$$

From this result it appears that when the form of the function $\psi(x)$, of which $\phi(x)$ is the differential coefficient, is obtained, the *process of algebraic or trigonometric summation to obtain* $\int_a^b \phi(x)dx$ *may be avoided.*

18. The letters b and a are supposed in the above work to denote *finite* quantities. We shall now *extend the notation* so as to let $\int_a^\infty \phi(x)dx$ express the limit when b becomes infinitely large of $\psi(b)-\psi(a)$, *i.e.*

$$\int_a^\infty \phi(x)dx=Lt_{b=\infty}\int_a^b \phi(x)dx.$$

Similarly, by $\int_\infty^b \phi(x)dx$ we shall be understood to mean

$$Lt_{a=\infty}[\psi(b)-\psi(a)] \quad \text{or} \quad Lt_{a=\infty}\int_a^b \phi(x)dx.$$

ILLUSTRATIVE EXAMPLES.

Taking the same examples as have been already considered otherwise in Art. 16,

1. ce^{mx} is the differential coefficient of $\frac{c}{m}e^{mx}$.

Therefore $$\int_a^b ce^{mx}\,dx = \frac{c}{m}(e^{mb} - e^{ma}),$$
the result obtained in Ex. 1, p. 8.

2. $c\sin mx$ is the differential coefficient of $-\frac{c}{m}\cos mx$.

Therefore $$\int_a^b c\sin mx\,dx = \left[-\frac{c}{m}\cos mx\right]_a^b = \frac{c}{m}(\cos ma - \cos mb),$$
the result of Ex. 3, p. 9.

3. $\frac{x^3}{c^2}$ is the differential coefficient of $\frac{x^4}{4c^2}$.

Therefore $$\int_a^b \frac{x^3}{c^2}dx = \frac{b^4 - a^4}{4c^2},$$
the result of Ex. 4 of p. 9.

4. $\frac{1}{x^2}$ is the differential coefficient of $-\frac{1}{x}$.

Therefore $$\int_a^b \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_a^b = \frac{1}{a} - \frac{1}{b},$$
the result of Ex. 5 of p. 10.

Comparing these solutions with those of the same problems of Art. 16, the student will at once see the advantage derived from a use of the fundamental proposition of Art. 17.

5. $\frac{1}{x}$ is the differential coefficient of $\log x$.

Therefore $$\int_a^b \frac{1}{x}\,dx = \Big[\log x\Big]_a^b = \log b - \log a = \log\frac{b}{a}.$$

6. $+e^{-x}$ is the differential coefficient of $-e^{-x}$.

Therefore $$\int_0^\infty e^{-x}\,dx = Lt_{\substack{b=\infty\\a=0}}\Big[-e^{-x}\Big]_a^b = (-e^{-\infty}) - (-e^0) = 1$$

EXAMPLES.

1. Write down the values of

(1) $\int_0^1 x\,dx,\quad \int_0^1 x^2\,dx,\quad \int_0^1 x^5\,dx,\quad \int_0^1 x^n\,dx;$

(2) $\int_0^{\frac{\pi}{2}} \sin x\,dx,\quad \int_0^{\frac{\pi}{2}} \cos x\,dx,\quad \int_0^{\frac{\pi}{4}} \sec^2 x\,dx,\quad \int_0^{\frac{\pi}{4}} \sec x\tan x\,dx;$

(3) $\int_0^1 \frac{1}{1+x^2}\,dx,\quad \int_0^1 \frac{1}{\sqrt{1-x^2}}\,dx,\quad \int_0^1 \frac{1}{1+x}\,dx,\quad \int_0^1 e^x\,dx;$

and interpret each result geometrically as the evaluation of an area.

19. **Geometrical Illustration of Proof.**

The proof of the above theorem of Art. 17 may be interpreted geometrically thus:

Let AB be a portion of a curve, of which the ordinate is finite and continuous at all points between A and B, as also the tangent of the angle which the tangent to the curve makes with the x-axis.

Let the abscissae of A and B be a and b respectively. Draw the ordinates AN, BM. Let the portion NM be divided into n equal parts, each of length h. Erect ordinates at each of these points of division, cutting the curve in $P, Q, R, \ldots$, etc. Draw the successive tangents AP_1, PQ_1, QR_1, etc., and the lines AP_2, PQ_2, QR_2, etc., *parallel* to the x-axis, and let the equation of the curve be $y=\psi(x)$, where $\psi'(x)=\phi(x)$.

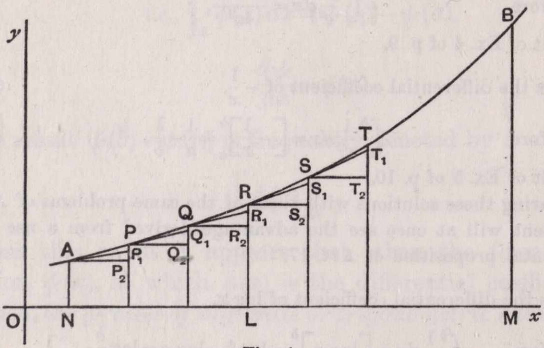

Fig. 4.

Then $\phi(a)$, $\phi(a+h)$, $\phi(a+2h)$, etc., are respectively

$$\psi'(a), \quad \psi'(a+h), \quad \psi'(a+2h), \text{ etc.,}$$

i.e. $\tan P_2AP_1$, $\tan Q_2PQ_1$, $\tan R_2QR_1$, etc.,

and $h\phi(a)$, $h\phi(a+h)$, etc., are respectively

the lengths P_2P_1, Q_2Q_1, R_2R_1, etc.

Now, it is clear that the algebraical sum of

$$P_2P, \quad Q_2Q, \quad R_2R, \ldots,$$

is $\qquad MB-NA, \quad i.e. \ \psi(b)-\psi(a).$

Hence

$$(P_2P_1+Q_2Q_1+R_2R_1+\ldots)+[P_1P+Q_1Q+\ldots]=\psi(b)-\psi(a).$$

Now, the portion between square brackets may be shown to diminish indefinitely with h. For if R_1R, for instance, be the greatest of the several quantities P_1P, Q_1Q, etc., the sum

$$[P_1P+Q_1Q+\ldots] \text{ is } < nR_1R, \quad i.e. < (b-a)\frac{R_1R}{h}.$$

But if the abscissa of Q be called x, then

$$LR_2=\psi(x), \quad R_2R_1=h\psi'(x),$$

and $$LR=\psi(x+h)=\psi(x)+h\psi'(x)+\frac{h^2}{\lfloor 2}\psi''(x+\theta h)$$

(*Diff. Cal.* Art. 130),

so that $$R_1R=\frac{h^2}{\lfloor 2}\psi''(x+\theta h)=\frac{h^2}{2}\phi'(x+\theta h),$$

and $$(b-a)\frac{R_1R}{h}=\frac{b-a}{2}h\phi'(x+\theta h),$$

which is an infinitesimal in general of the first order.

Thus $\quad Lt_{h=0}(P_2P_1+Q_2Q_1+R_2R_1+\ldots)=\psi(b)-\psi(a),$

or

$$Lt_{h=0}h[\phi(a)+\phi(a+h)+\phi(a+2h)+\ldots+\phi(b-h)]=\psi(b)-\psi(a).$$

Also, since $Lt_{h=0}h\phi(b)=0$, we have, by addition,

$$Lt_{h=0}h[\phi(a)+\phi(a+h)+\phi(a+2h)+\ldots+\phi(b)]=\psi(b)-\psi(a).$$

20. Case of an Unknown Curve passing through a given system of Points.

In a certain graph, such, for instance, as the graph on a temperature chart, the temperature being noted at stated intervals, the following table gives the corresponding abscissae and ordinates of eleven points on the curve:

x	1	1·1	1·2	1·3	1·4	1·5	1·6	1·7	1·8	1·9	2
y	·900	·879	·856	·831	·804	·775	·744	·711	·676	639	·600

On the assumption that the graph is that of a continuous function of x, and the ordinate continually decreasing in the intervals between the several stated values, it is required to calculate $\int_1^2 y\,dx$, *i.e.* to find the area bounded by the curve, the x-axis and the extreme ordinates.

Constructing the inscribed and circumscribed parallelograms as explained in Art. 9,

The sum of the circumscribed figures is

$$\cdot1\times[\cdot9+\cdot879+\cdot856+\ldots+639]=\cdot7815;$$

The sum of the inscribed figures is

$$\cdot 1 \times [\cdot 879 + \cdot 856 + \ldots + \cdot 639 + \cdot 600] = \cdot 7515.$$

The first is clearly too large by the sum of the ten small triangular-shaped elements outside the area to be found.

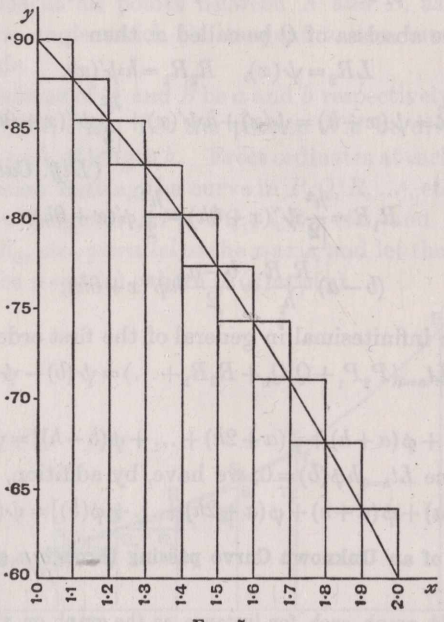

FIG. 5.

The second is too small by the sum of the ten triangular-shaped elements which are omitted.

The mean of these results, viz. $\dfrac{\cdot 7815 + \cdot 7515}{2} = \cdot 7665$, will be a much closer approximation, but will be a little too small, because it omits the very small areas which lie between the *chords* which join successive points on the graph and the corresponding arcs.

Hence, as a closer approximation, we may take

$$\int_1^2 y\,dx = \cdot 7665 \text{ square units.}$$

[From a finite number of ordinates it is impossible to assign the equation to the curve, but it is customary to take the *simplest algebraic* curve which satisfies the prescribed conditions. In the present case the simplest curve to fit the data will be found to be $y = 1 - \dfrac{x^2}{10}$.

Any other curve of the form

$$y=1-\frac{x^2}{10}+(x-1)(x-1\cdot1)(x-1\cdot2)\dots(x-2)\phi(x)$$

where $\phi(x)$ is any integral algebraic expression, would go through the same points, but is much more complicated.

The true area on the supposition of the curve being $y=1-\frac{x^2}{10}$ will be found by the result of Art. 16, Ex. 6, to be $\left[x-\frac{x^3}{30}\right]_1^2$, *i.e.* $\frac{23}{30}$, or $\cdot7666\dots$, which shows errors as follows:

In the first estimate, - ·0148 in excess, *i.e.* a 1·9 % error in excess,
" second " - ·0152 in defect, *i.e.* a 2·0 % error in defect,
" mean " - ·0002 in defect, *i.e.* a 0·03 % error in defect.]

21. Simpson's Rule.

If a curve be partially defined as passing through an odd number of points whose abscissae are in arithmetical progression, *e.g.* the points

$$(a, y_1),\quad (a+h, y_2),\quad (a+2h, y_3)\dots(a+\overline{n-1}h, y_n),$$

and if the same assumptions be made as in the last article as to continuity, etc., it is possible to find a very close approximation to the area of the curve, which is useful in many practical cases, as follows:

Consider first the case of the parabola whose equation is

$$y=a+bx+cx^2,$$

and let a, b, c be chosen so as to make this curve go through

$$(-h, y_1),\quad (0, y_2),\quad (h, y_3).$$

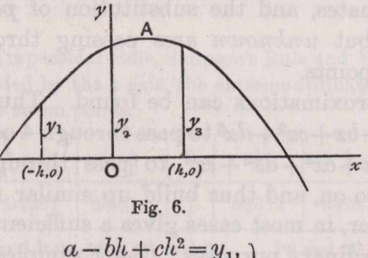

Fig. 6.

Then
$$\left.\begin{aligned} a-bh+ch^2&=y_1,\\ a\quad\quad&=y_2,\\ a+bh+ch^2&=y_3.\end{aligned}\right\}$$

So that $a=y_2,\quad b=\frac{y_3-y_1}{2h},\quad c=\left(\frac{y_1+y_3}{2}-y_2\right)\Big/h^2.$

Now the area bounded by the x-axis, the parabola and the ordinates y_1 and y_2 is, by Art. 16, Ex. 6,

$$\int_{-h}^{h}(a+bx+cx^2)\,dx=\left[ax+\frac{bx^2}{2}+\frac{cx^3}{3}\right]_{-h}^{h}$$
$$=2ah+\frac{2c}{3}h^3$$
$$=h\{2y_2+\tfrac{1}{3}(y_1-2y_2+y_3)\}$$
$$=\frac{h}{3}(y_1+4y_2+y_3).$$

If we apply this rule to the case in question, passing parabolic arcs through the (1st, 2nd, 3rd points), (3rd, 4th, 5th), (5th, 6th, 7th), etc., we have the following approximative rule, viz.

$$\begin{aligned}A=\frac{h}{3}[y_1+4y_2+y_3 & \\ &+y_3+4y_4+y_5\\ &\quad+y_5+4y_6+y_7\\ &\qquad+\ldots+y_{n-2}+4y_{n-1}+y_n]\end{aligned}$$
$$=\frac{h}{3}[y_1+4y_2+2y_3+4y_4+2y_5+4y_6+\ldots+4y_{n-1}+y_n]$$
$$=\frac{h}{3}[y_1+y_n+2(y_3+y_5+y_7+\ldots)+4(y_2+y_4+y_6+\ldots)],$$

i.e. $\frac{h}{3}$(sum of first and last + twice sum of all other odd ordinates + four times the sum of the even ordinates).

This is known as Simpson's Rule. It will be noticed that it consists in the division of the area by an odd number of equidistant ordinates, and the substitution of *parabolic* arcs for the *actual* but *unknown* arcs passing through consecutive groups of 3 points.

Other approximations can be found. Thus we may take a curve $y=a+bx+cx^2+dx^3$ to pass through 4 consecutive points, or $y=a+bx+cx^2+dx^3+ex^4$ to pass through 5 consecutive points, and so on, and thus build up similar rules. Simpson's Rule, however, in most cases gives a sufficiently close approximation for ordinary purposes. (See Examples 27, 28, page 33.)

22. The Trapezoidal Rule and Weddle's Rule.

The approximation previously adopted in Art. 20 of the mean of the inscribed and circumscribed rectangles may be expressed in similar manner, as

$$h\left(\frac{y_1+y_2}{2}+\frac{y_2+y_3}{2}+\frac{y_3+y_4}{2}+\ldots+\frac{y_{n-1}+y_n}{2}\right)$$

$$=\frac{h}{2}(y_1+2y_2+2y_3+2y_4+\ldots+2y_{n-1}+y_n)$$

$$=\frac{h}{2}(\text{sum of first and last ordinates}+\text{twice the sum of all the rest}),$$

which is a convenient form, but not usually so accurate as Simpson's Rule.

It consists, as already explained, of substituting chords joining consecutive points for their arcs, and as we are summing a series of Trapezoids this is known as the Trapezoidal Rule.

23. Other Approximative Rules.

Other rules will be found in Examples 27, 28 at the end of this chapter, and in Examples 24, 25, 26, page 61.

A very convenient rule was given by Weddle, *Math. Journal*, vol. ix., for the case where there are *seven equidistant ordinates*, $y_1, y_2, y_3, \ldots, y_7$ at mutual distances h, viz.

$$\tfrac{3}{10}h[y_1+y_3+y_5+y_7+5(y_2+y_4+y_6)+y_4],$$

i.e. $\frac{3}{10}\times$ mutual distance [Σ odds $+\,5\,\Sigma$ evens $+$ middle].

(Weddle's Rule.)

We transcribe this for convenience, but the proof is one most conveniently treated by finite difference methods. It will be found in Boole's *Finite Differences*, pages 47-48.

Boole remarks that in all applications of such approximate formulae "it is desirable to avoid extreme differences among the ordinates."

Ex. Apply the Trapezoidal Rule, Simpson's Rule and Weddle's Rule to find the area bounded by the x-axis, the extreme ordinates and the arc of a circle through the seven points:

$x=$	$-\frac{3}{6}$	$-\frac{2}{6}$	$-\frac{1}{6}$	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$
$y=$	·86602	·94281	·98614	1	·98614	·94281	·86602

First and last.	2nd, 4th and 6th.	3rd and 5th.
·86602	·94281	·98614
·86602	1·00000	·98614
1·73204	·94281	1·97228
	2·88562	2
	4	3·94456
	11·54248	

For Trapezoidal Rule, Area $= \frac{1}{6}(\cdot 86602 + 2\cdot 88562 + 1\cdot 97228)$
$= \frac{1}{6}(5\cdot 72392) = \underline{\cdot 95398.}$

For Simpson's Rule, Area $= \frac{1}{18}(1\cdot 73204 + 3\cdot 94456 + 11\cdot 54248)$
$= \frac{1}{18}(17\cdot 21908)$
$= \underline{\cdot 95661.}$

For Weddle's Rule, Area $= \frac{1}{20}(3\cdot 70432 + 14\cdot 42810 + 1\cdot 00000)$
$= \underline{\cdot 95662}$

This area, being the area of that part of a semicircle whose centre is at the origin and radius unity bounded by two ordinates $x = \cdot 5$, $x = -\cdot 5$, may be seen to have its area correctly $= \frac{\pi}{6} + \frac{\sqrt{3}}{4} = \cdot 956611\ldots$, and therefore Simpson's Rule gives a result accurate to the last figure.

[See Boole, *Finite Differences*, p. 49.]

The approximation by Weddle's Rule does not appreciably differ from that by Simpson's Rule.

The Trapezoidal Rule errs in defect by $\cdot 00263$, *i.e.* by about $\cdot 3\,\%$ of the whole.

24. Determination of a Volume of Revolution.

Let it be required to find the volume formed by the revolution of a given curve AB about an axis in its own plane which it does not cut.

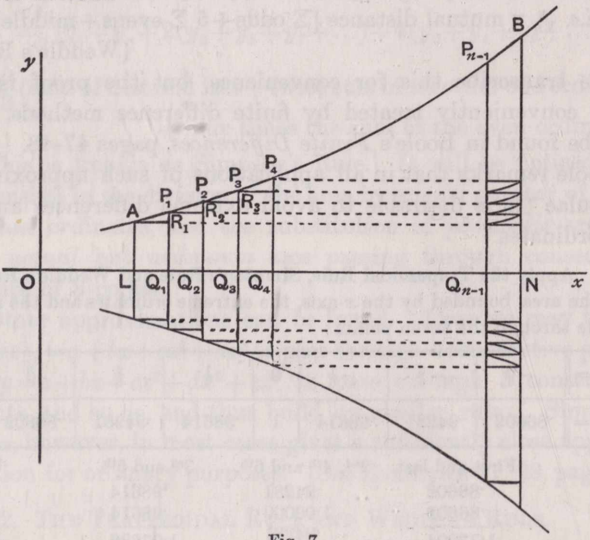

Fig. 7.

Taking the axis of revolution as the x-axis, the figure may be described exactly as in Art. 11. The elementary rectangles

AQ_1, P_1Q_2, P_2Q_3, etc., trace in their revolution circular discs of equal thickness and of volumes $\pi AL^2 . LQ_1$, $\pi P_1Q_1{}^2 . Q_1Q_2$, etc. The several annular portions formed by the revolution of the portions AR_1P_1, $P_1R_2P_2$, $P_2R_3P_3$, etc., may be considered to slide parallel to the x-axis into a corresponding position upon the disc of greatest radius, say that formed by the revolution of the figure $P_{n-1}Q_{n-1}NB$. Their sum is less than this disc, *i.e.* in the limit less than an infinitesimal of the first order, for the breadth $Q_{n-1}N$ is h, according to the notation of Art. 11, and is ultimately an infinitesimal of the first order, and the radius NB is, as in that article, supposed finite, as also all other ordinates of the curve from A to B.

Hence the volume required is the limit when $h=0$ (and therefore $n=\infty$) of the sum of the series

$$\pi[\phi(a)]^2h+\pi[\phi(a+h)]^2h+\pi[\phi(a+2h)]^2h+\dots$$
$$+\pi[\phi(a+\overline{n-1}h)]^2h,$$

or, as it may be written,

$$\pi\int_a^b[\phi(x)]^2dx \quad \text{or} \quad \pi\int_a^b y^2dx,$$

the equation of the curve being $y=\phi(x)$ and the extreme ordinates $x=a$ and $x=b$, as in the article cited.

25. Illustrative Examples.

Ex. 1. The portion of the parabola $y^2=4ax$ bounded by the line $x=c$ revolves about the axis. Find the volume generated.

Let the portion required be that formed by the revolution of the area APM about the axis, being bounded by the curve, the axis and an ordinate MP. (See Fig. 8.)

Dividing as in Art. 24 into elementary circular laminae, we have

$$\text{Vol.}=\int_0^c \pi y^2dx=4a\pi\int_0^c x\,dx=4a\pi\frac{c^2}{2}=2\pi ac^2 \text{ (Art. 16, Ex. 6)}$$
$$=\tfrac{1}{2}\pi PM^2 . AM$$
$$=\tfrac{1}{2}\text{ cylinder of radius } PM \text{ and height } AM$$
$$=\tfrac{1}{2}\text{ vol. of circumscribing cylinder.}$$

[Or, if expressed as a series,

$$4a\pi\int_0^c x\,dx=4a\pi\, Lt\,\frac{1}{n}\left[\left(\frac{1}{n}\right)+\left(\frac{2}{n}\right)+\left(\frac{3}{n}\right)+\dots+\left(\frac{n-1}{n}\right)\right]c^2$$
$$=4\pi a\frac{c^2}{2}=2\pi ac^2 \text{ as before.]}$$

Ex. 2. Find the area of the portion PAP' of the same parabola, PP' being the double ordinate through P.

$$\text{Area } PAM=\int_0^c y\,dx=2\sqrt{a}\int_0^c x^{\frac{1}{2}}dx=2\sqrt{a}\frac{c^{\frac{3}{2}}}{\frac{3}{2}}$$
$$=\tfrac{4}{3}\sqrt{ac^3}=\tfrac{2}{3}c\sqrt{4ac}=\tfrac{2}{3}AM\,.\,MP$$

$\therefore$ Area $PAP'=\frac{2}{3}$ of the circumscribing rectangle $RPP'R'$.

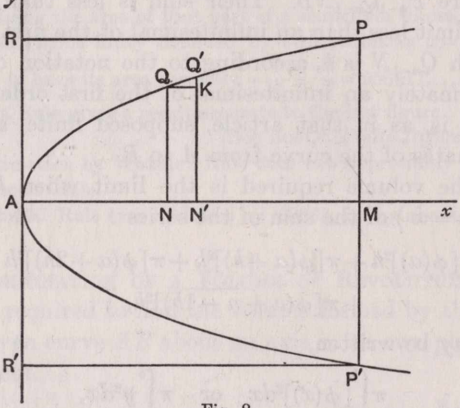

Fig. 8.

[Or we may proceed thus: Divide c into n equal portions, and erect ordinates. Let QN be the ordinate at $x=\frac{r}{n}c$.

Then Area $PAM=Lt\sum_{r=0}^{r=n-1}\sqrt{4arh}\,.\,h$, where $h=\frac{c}{n}$,

$$=2a^{\frac{1}{2}}c^{\frac{3}{2}}\,Lt\frac{1}{n^{\frac{3}{2}}}\left[1^{\frac{1}{2}}+2^{\frac{1}{2}}+3^{\frac{1}{2}}+(n-1)^{\frac{1}{2}}\right]$$
$$=2a^{\frac{1}{2}}c^{\frac{3}{2}}\,.\,\tfrac{2}{3}=\tfrac{2}{3}c\sqrt{4ac}, \text{ as before.]}$$

Ex. 3. The portion of a circle $x^2+y^2=a^2$ between ordinates $x=h_1$, $x=h_2$ rotates about the x-axis. Find the volume of the frustum of the sphere generated.

Let the portion required be that formed by the portion $N_1P_1P_2N_2$ of the circle revolving about N_1N_2 (Fig. 9).

Here we are to evaluate

$$\pi\int_{h_1}^{h_2}y^2dx=\pi\int_{h_1}^{h_2}(a^2-x^2)dx=\pi\left[a^2x-\frac{x^3}{3}\right]_{h_1}^{h_2} \quad \text{(Art. 16, Ex. 6)}$$
$$=\pi a^2(h_2-h_1)-\frac{\pi(h_2{}^3-h_1{}^3)}{3}.$$

It is convenient for mensuration purposes to express this in terms of the radii of the ends of the frustum and its thickness.

Let T be the thickness $= h_2 - h_1$ and $y_1{}^2 = a^2 - h_1{}^2$,

$$y_2{}^2 = a^2 - h_2{}^2.$$

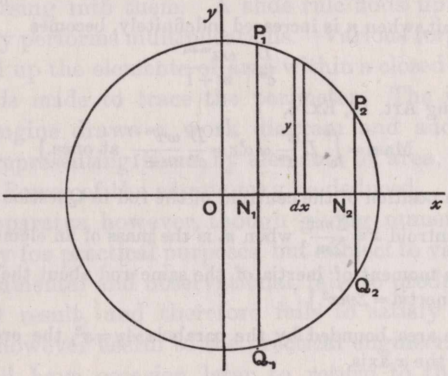

Fig. 9.

Then
$$\begin{aligned}\text{Vol.} &= \tfrac{1}{6}\pi T(6a^2 - 2h_1{}^2 - 2h_2{}^2 - 2h_1h_2)\\ &= \tfrac{1}{6}\pi T\{3(a^2 - h_1{}^2) + 3(a^2 - h_2{}^2) + (h_2 - h_1)^2\}\\ &= \tfrac{1}{6}\pi T(3y_1{}^2 + 3y_2{}^2 + T^2)\\ &= \frac{T}{6}(3\pi y_1{}^2 + 3\pi y_2{}^2 + \pi T^2)\end{aligned}$$

$= \frac{1}{6}$ thickness $\times$ [3 sum of circular faces + circle on thickness as radius].

Cor. For the whole sphere

$$T = 2a, \quad y_1 = y_2 = 0, \quad V = \tfrac{4}{3}\pi a^3.$$

EXAMPLES.

1. Find the volume of the prolate spheroid formed by the revolution of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ about the x-axis.

2. Find the mass of a rod whose density varies as the m^{th} power of the distance from one end. $\left(\rho = D\dfrac{x^m}{c^m}\right.$, say, where D and c are constants.$\Big)$

Let a be the length of the rod,

ω the sectional area.

Divide as before into n equal elementary portions.

The volume of the $(r+1)^{\text{th}}$ element from the end of zero density is $\omega\dfrac{a}{n}$. Its density varies from $\dfrac{D}{c^m}\left(\dfrac{ra}{n}\right)^m$ to $\dfrac{D}{c^m}\left[\dfrac{(r+1)a}{n}\right]^m$. Its mass is therefore intermediate between

$$D\frac{\omega a^{m+1}}{c^m}\frac{r^m}{n^{m+1}} \quad \text{and} \quad D\frac{\omega a^{m+1}}{c^m}\frac{(r+1)^m}{n^{m+1}},$$

and the mass of the rod lies between

$$D\frac{\omega a^{m+1}}{c^m}\frac{1^m+2^m+3^m+\ldots+(n-1)^m}{n^{m+1}} \quad \text{and} \quad D\frac{\omega a^{m+1}}{c^m}\frac{2^m+3^m+\ldots+n^m}{n^{m+1}},$$

and in the limit, when n is increased indefinitely, becomes

$$\frac{D}{c^m}\cdot\frac{\omega a^{m+1}}{m+1}.$$

[Or, assuming Art. 16, Ex. 6,

$$\text{Mass}=\int_0^a D\frac{x^m}{c^m}\omega\,dx=\frac{D}{c^m}\frac{\omega a^{m+1}}{m+1} \text{ at once.]}$$

3. Find the position of the centroid of the rod in Question 2.

[For the centroid $\bar{x}=\frac{\Sigma mx}{\Sigma m}$, when m is the mass of an element.]

4. Find the moment of inertia of the same rod about the lighter end. [Moment of Inertia $=\Sigma mx^2$.]

5. Find the area bounded by the parabola $4y=x^2$, the ordinates $x=2$ and $x=4$ and the x-axis,

(1) by means of inscribed rectangles,

(2) " circumscribed rectangles,

taking ordinates at distances ·1, and compare the results with that obtained by integration.

The sum of the inscribed rectangles is

$$\tfrac{1}{4}\times\tfrac{1}{10}(2^2+2{\cdot}1^2+2{\cdot}2^2+\ldots+3{\cdot}9^2).$$

The sum of the circumscribed rectangles is

$$\tfrac{1}{4}\times\tfrac{1}{10}(2{\cdot}1^2\times 2{\cdot}2^2+\ldots+3{\cdot}9^2+4^2).$$

The values of these expressions are respectively (taking the squares from Bottomley's tables or summing otherwise),

4·5175, which is a little too small,

and 4·8175, which is a little too large.

Their mean is 4·6675.

The true value is

$$\int_2^4\frac{x^2}{4}dx=\left[\frac{x^3}{12}\right]_2^4=\frac{4^3-2^3}{12}=\frac{14}{3}=4{\cdot}666\ldots.$$

6. Plot the graph of $y=\frac{1}{1+x^2}$ and mark on your figure the area represented by the definite integral $\int_0^1\frac{dx}{1+x^2}$.

Evaluate this integral by mensuration, and hence obtain an approximation for π. Note that $\frac{1}{1+x^2}=\frac{d}{dx}\tan^{-1}x$.

26. Mechanical Integration.

In a sense, any mechanical contrivance which performs additions and registers the results is an Integrating machine for the particular class of function to which it may be adapted. Cash registers which record the day's takings, gas meters, water meters, electric-light meters, all record the amount passing into them. A slide rule adds up logarithms, and thereby performs multiplications. Various forms of planimeters add up the elements of area within a closed curve when a pointer is made to trace the perimeter. The indicator of a steam engine draws a work diagram and adds up work elements, representing them by elements of area, from which the Horse-Power of the engine may be deduced.

Such apparatus, however, though giving numerical results satisfactory for practical purposes, but subject to various errors both instrumental and observational, fails to produce an exact algebraical result, and therefore fails to satisfy the mathematician, however useful to the practical engineer.

We shall have occasion later to return to the theory of some apparatus of this kind. For the present it is sufficient to mention its existence.

To sum up then; we have discussed Four Methods of Integration, *i.e.* of finding

$$\int_a^b \phi(x)\,dx:$$

I. By obtaining

$$Lt_{h=0}\, h\left[\phi(a)+\phi(a+h)+\phi(a+2h)+\ldots+\phi(b)\right].$$

II. By finding a function $\psi(x)$ such that $\dfrac{d\psi}{dx}=\phi(x)$, from which we obtain

$$\int_a^b \phi(x)\,dx=\psi(b)-\psi(a).$$

III. By drawing the graph of $y=\phi(x)$ and by some means or other obtaining its area, by the Trapezoidal or Simpson's or some other approximative rule, as, for instance, by drawing on squared paper and counting all the squares within the contour with a "give and take" rule round the perimeter.

IV. By approximating to the area of the contour by mechanical means.

It is obvious that III. and IV. can only give approximate results, though such results may approach a very high degree of accuracy.

For exact results we have to apply Method I. or II. As has been seen, Method I. leads to very difficult algebraic or trigonometric summation, except in the very simplest cases. Hence we are forced upon Method II. for exact general work. This method we therefore shall in future rely upon and begin to develop the explanation of it in the next chapter.

EXAMPLES.

1. If the acceleration of a moving point be $\phi''(t)$, the initial velocity be u and $\phi'(0) = \phi(0) = 0$, show that, t being the time from a given epoch,

$$v = u + \phi'(t), \quad s = ut + \phi(t),$$

where v and s are respectively the velocity at time t and the space described.

If the acceleration be $10 \cos \omega t$ and the initial velocity be zero, show that

$$v = \frac{10}{\omega} \sin \omega t,$$

$$s = C - \frac{10}{\omega^2} \cos \omega t,$$

where C is a constant. To what kind of motion does this refer?

Show that the "periodic time" is $\frac{2\pi}{\omega}$.

2. If A be the area bounded by a curve, the coordinate axes and the ordinate at a given abscissa x, show that $y = \frac{dA}{dx}$, and hence that $A = \int_0^x y\,dx$. What difference would it make if the measurement of A commences from a standard ordinate y_0 whose abscissa is x_0?

If V be the volume of water in a pond, and A the horizontal sectional area at a height x above the bottom of the pond, show that

$$= \int_0^h A\,dx,$$

where h is the depth of the pond.

3. A large number of circular discs of the same thickness $\frac{h}{n}$ and successive radii

$$\frac{a}{n}, \ \frac{2a}{n}, \ \frac{3a}{n}, \ \frac{4a}{n}, \ldots, a,$$

are threaded through their centres upon a straight wire and lie with their plane faces in contact. Show that their total volume differs from that of a cone of height h and with a for the radius of its base by the ultimately vanishing quantity

$$\frac{\pi a^2 h}{6}\frac{3n+1}{n^2}.$$

If $n=1000$, show that the error in taking this sum as the volume of the cone is ·1505 per cent. of the true volume.

4. Consider a sphere of diameter $2a$ to be divided into $2n$ thin laminae of equal thickness by a series of parallel planes; show that the volume of the sphere is

$$2\,Lt_{n=\infty}\sum_{r=0}^{r=n}\pi\left(1-\frac{r^2}{n^2}\right)\frac{a^3}{n},$$

and that this limit is $\frac{4}{3}\pi a^3$.

Obtain by a similar method the volume of the spheroid formed by the revolution of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ round the axis of length $2b$.

5. Show by the method of summation that the volume of a paraboloid of revolution bounded by a plane at right angles to the axis is one half of the circumscribing cylinder.

Verify by consideration of the integral

$$4a\int_0^h x\,dx.$$

6. Draw on squared paper (one inch squares divided into tenths is convenient) a quadrant of a circle of radius 5 inches. Divide one of the bounding radii into 10 half-inch divisions, and erect ordinates at each point. Complete the inscribed and escribed rectangles. Show that the sum of the inscribed rectangles is 18·15 square inches very nearly. Also show that the mean of the inscribed and escribed rectangles falls short of the true area of the quadrant by about $\frac{6}{25}$ of a square inch.

7. Rectangles of the same breadth and of areas

$$\left(\frac{c}{n}\right)^t,\ \frac{1}{2}\left(\frac{2c}{n}\right)^t,\ \frac{1}{3}\left(\frac{3c}{n}\right)^t,\ \ldots,\ \frac{1}{r}\left(\frac{rc}{n}\right)^t,\ \ldots,\ \frac{1}{n}c^t,$$

are set up side by side on bases in a straight line.

Shew that when n is very great, the sum of their areas differs little from that enclosed by $y=x^{t-1}$, $y=0$, $x=c$.

Assume t to be positive.

Evaluate $$Lt_{n=\infty}\sum_{r=1}^{r=n}\frac{1}{r}\left(\frac{rc}{n}\right)^t.$$ [I. C. S. Exam. 1902.]

8. In the curve in which the abscissa varies as the logarithm of the ordinate, prove that the area bounded by the curve, the x-axis and any two ordinates varies as the difference of the ordinates.

9. Approximate to the integral $\int_2^3 \frac{10}{x}dx$, regarding it as a summation (1) of inscribed parallelograms as in Art. 9,

(2) of circumscribed parallelograms,

and compare with the result of integration.

[The results are 3·9724 and 4·1391, the reciprocals being taken from Bottomley's tables. Their mean is 4·0557. The result to three places of decimals as computed from $10\log_e \frac{3}{2}$ is 4·055.]

10. Draw a sketch showing the curvilinear area which is represented by the definite integral

$$\int_1^{10} \frac{10}{x}dx,$$

and evaluate the area approximately from the figure.

Without plotting, indicate roughly by dotted lines on your sketch the relative positions of the curvilinear areas represented by the definite integrals

$$\int_1^{10} 10x^{-0\cdot9}dx \quad \text{and} \quad \int_1^{10} 10x^{-1\cdot1}dx,$$

and calculate the values of these integrals.

Calculate also

$$\int_1^{10} 10x^{-0\cdot99}dx \quad \text{and} \quad \int_1^{10} 10x^{-1\cdot01}dx.$$ [I. C. S., 1908.]

11. In any curve in which the ordinate $PN \propto$ the n^{th} power of the abscissa, show that if any two ordinates be taken, viz. P_1N_1 and P_2N_2, and two others, P_3N_3 and P_4N_4, which are twice as far from the y-axis as P_1N_1 and P_2N_2 respectively, then

$$\text{Area } P_3P_4N_4N_3 : \text{Area } P_1P_2N_2N_1 :: 2^{n+1} : 1.$$

12. Prove that the area of the diagram formed by

$$\begin{aligned} x &= 0 \quad \text{from } (0, 0) \text{ to } (0, 4), \\ y &= 4 \quad \text{from } (0, 4) \text{ to } (1, 4), \\ x^2 - 10x + 25 &= 4y \quad \text{from } (1, 4) \text{ to } (5, 0), \\ y &= 0 \quad \text{from } (5, 0) \text{ to } (0, 0), \end{aligned}$$

is $9\frac{1}{3}$ square units.

13. In the construction of reservoir walls of great height, Rankine adopted the following plan:

Taking a vertical x-axis on which depths and ordinates are measured in feet, the ordinates to the outer and inner faces are shown in the following scheme:

Depth in feet.	0	10	20	30	40	50	60
Ordinate to outer face in feet.	17·40	19·72	22·35	25·29	28·69	32·53	36·86
Ordinate to inner face in feet.	1·34	1·52	1·72	1·94	2·21	2·50	2·83
Depth in feet.	70	80	90	100	110	120	130
Ordinate to outer face in feet.	41·75	47·31	53·61	60·75	68·84	78·00	88·39
Ordinate to inner face in feet.	3·21	3·64	4·12	4·67	5·29	6·00	6·80
Depth in feet.	140	150	160	170	180		
Ordinate to outer face in feet.	100·15	113·49	128·60	146·72	165·14		
Ordinate to inner face in feet.	7·70	8·73	9·90	11·21	12·70		

(The two sets of ordinates are measured in opposite directions from the vertical.)

[RANKINE, *Applied Mechanics*, p. 638, and *Engineer*, Jan. 5, 1872.]

Construct a diagram showing the wall in elevation, and estimate in cubic yards the volume of material necessary to construct 100 yards length of the wall.

14. Find the centre of gravity of a rod whose density varies

(1) as the distance from one end;

(2) as the square of the distance from one end.

Find also the moment of inertia of the rod about the light end in each case.

15. Find the mass of a circular disc in which the density varies as the n^{th} power of the distance from the centre. $(n > -2.)$

Also find the moment of inertia of this disc about an axis through the centre at right angles to the plane of the disc.

16. If the graphs of $a \sin \dfrac{\pi x}{2b}$ and $a \sin \dfrac{\pi x}{b}$ be drawn, show that the areas bounded by the x-axis, the curves and the ordinate $x = b$ are equal.

17. A single wave on the sea is in the form defined by the curve of sines, $y = a \sin \frac{\pi x}{b}$. Show that the quantity of water raised above the mean sea level contained in a length c of the wave measured on the surface at right angles to the direction of progression, is $2abc/\pi$, the raised portion extending from $x = 0$ to $x = b$.

If c be 100 yards, $b = 20$ feet, $a = 2$ feet, and a cubic foot of water weighs $62\frac{1}{2}$ lbs. weight, find the number of tons weight in the portion of the wave higher than the mean sea level.

18. Show that when n becomes infinitely large,

$$Lt \frac{1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n(n+1)}{n^3}$$

is the same as $\quad Lt \dfrac{1^2 + 2^2 + 3^2 + \ldots + n^2}{n^3}$.

Illustrate geometrically.

19. Show that the limit when $n = \infty$ of the ratio of the sum of all possible products, two and two together, of the first n natural numbers, to n^4, is $\frac{1}{8}$; and that the limit of the ratio of the sum of all products, three and three together, to n^6, is $\frac{1}{48}$.

20. If there be gas of volume v and pressure p below a piston in a cylinder of sectional area A and occupying a length x of the cylinder, show that in its expansion, so as to occupy a length $x + dx$ of the cylinder, the work done by the gas upon the piston is

$$pA\,dx \text{ or } p\,dv,$$

and that if the expansion continues so that the piston moves through a finite distance—say from $x = x_1$ to $x = x_2$, the work done on expansion is

$$\int_{x_1}^{x_2} p\,dv.$$

Remembering that $\quad \dfrac{d}{dx} \log x = \dfrac{1}{x}$,

find the value of this integral in the two cases:

(1) Isothermal expansion, $pv = c$;

(2) Adiabatic expansion, $pv^\gamma = c'$.

Find in foot-lbs. the work done in the expansion of 10 cubic feet of gas, initially at a pressure of 1000 lbs. per square foot, to 40 cubic feet;

(1) According to the law, $pv = c$;

(2) According to the law, $pv^{1 \cdot 41} = c'$.

21. If the graph of $e^{\frac{x}{a}}$ be drawn, prove that the areas bounded by the curve, the x-axis and a set of equidistant ordinates are in geometrical progression, whose common ratio is the same as the common ratio of the tangents of the angles which the tangents at the ends of the successive ordinates make with the x-axis.

22. Show that the area bounded by a parabola, the axis and an ordinate is two-thirds of the circumscribing rectangle.

23. The circle $x^2 + y^2 = 5a^2$ and the parabola $y^2 = 4ax$ revolve about their common axis. Show that the smaller lens-shaped solid formed has for its volume

$$\tfrac{2}{3}\pi a^3(5\sqrt{5} - 4).$$

24. If $x_1, x_2, x_3, \ldots, x_{n-1}$ be a series of quantities taken between $a(=x_0)$ and $b(=x_n)$, prove that when n is made infinite, and the difference between any two consecutive terms of the series becomes indefinitely small, the limit of

$$\sum_{r=1}^{r=n}(x_r - x_{r-1})f(x_{r-1})$$

is $\phi(b) - \phi(a)$, where $\dfrac{d}{dx}\phi(x) = f(x)$.

Verify this in the case where $f(x) = \log x$, and the series $a, x_1, x_2, \ldots,$ is geometrical. [Oxford, *2nd Public Examination*, 1900.]

25. Plot the value of $\cos^2 x$ for 10° intervals from 0° to 90°, and thus find as close an approximation as you can to

$$\int_0^{\frac{\pi}{2}} \cos^2 x \, dx$$

without integration. $\left[\text{The true value is } \dfrac{\pi}{4}.\right]$

26. If a cylindrical hole be drilled through a solid sphere, the axis of the cylinder passing through the centre of the sphere, show that the volume of the portion of the sphere left is equal to the volume of a sphere whose diameter is the length of the hole

27. If the curve $y = a + bx + cx^2 + dx^3$ pass through the extremities of four equidistant ordinates y_1, y_2, y_3, y_4, the distance apart being h, show that the area bounded by the extreme ordinates, the curve and the x-axis is $\dfrac{3h}{8}(y_1 + 3y_2 + 3y_3 + y_4)$.

[Simpson's "Three-eighths' Rule."]

28. If the curve $y = a + bx + cx^2 + dx^3 + ex^4$ pass through the extremities of 5 equidistant ordinates y_1, y_2, y_3, y_4, y_5, at mutual

distances h, show that the area bounded by the extreme ordinates, the curve and the x-axis is

$$h\frac{14(y_1+y_5)+64(y_2+y_4)+24y_3}{45}.$$

[BOOLE, *Finite Differences.*]

29. If a parabola whose axis is parallel to the y-axis pass through the points (a, y_1), (b, y_2), (c, y_3), show that its equation is

$$y=y_1\frac{(x-b)(x-c)}{(a-b)(a-c)}+y_2\frac{(x-c)(x-a)}{(b-c)(b-a)}+y_3\frac{(x-a)(x-b)}{(c-a)(c-b)},$$

and find the area bounded by the curve, the x-axis and the extreme ordinates y_1 and y_3.

30. In the cycloid $\left.\begin{aligned} x&=10(\theta+\sin\theta)\\ y&=10(1-\cos\theta)\end{aligned}\right\}$

tabulate the values of x and y for intervals of $\frac{\pi}{36}$ from $\theta=0$ to $\theta=\frac{\pi}{2}$. Hence obtain approximate results for

$$(1)\ \int y\,dx, \qquad (2)\ \int x\,dy$$

corresponding to the above limits for θ.

31. If $f(x)>\phi(x)$ where $a<x<b$, and if both functions are finite for this range of the variable, including both limits, prove that

$$\int_a^b f(x)\,dx>\int_a^b \phi(x)\,dx.$$

Explain why these conditions must be postulated. Must the functions also be continuous? [I. C. S, 1905.]

32. Prove that the integral

$$\int_0^{0\cdot5}\frac{dx}{\sqrt{1-x^n}}$$

is for all values of n greater than 2, nearly equal to 0·5.

[I. C. S., 1905.]

33. A claret glass is 6 cm. deep and its rim is 5 cm. in diameter. Its vertical section is nearly parabolic. Calculate its capacity in c.c. to the nearest integer. [I. C. S., 1905.]

34. Trace the curve $y=x^m(1-x)^{2m}$ from $x=0$ to $x=1$ for the values $m=0{\cdot}5$ and 2. Show the two curves on one diagram. Show that the area enclosed by the curve and the x-axis diminishes as m increases. [I. C. S, 1902.]

35. A cask has a head diameter of a inches, a bung diameter of b inches, and length c. Find an expression for its volume, supposing

that a section along a stave is an arc of a curve of sines, the curvature vanishing at the ends of the stave.

Evaluate the result when $a=13$, $b=17$, $c=18$. [I. C. S., 1902.]

36. Find the value of $\int_0^{0\cdot3} x^{0\cdot4}(1-x)^{0\cdot6}\,dx$ to two significant figures,

(1) graphically,

(2) by calculation. [I. C. S., 1903.]

37. Show, without integration, that

$$I \equiv \int_0^{0\cdot644} \frac{64\,d\theta}{(5+3\cos\theta)^2}$$

lies between ·644 and ·753.

[PETERHOUSE AND SIDNEY SUSSEX SCHOLARSHIP EXAM., 1917.]

Differentiate $\quad 5\tan^{-1}\left(\frac{1}{2}\tan\frac{\theta}{2}\right)-\frac{6\sin\theta}{5+3\cos\theta}$,

and hence prove that the true value of I is about ·68.

(Take $\tan^{-1}\frac{1}{3}=\cdot322$ and $\tan^{-1}\frac{1}{6}=\cdot165$.)

38. In a diagram of the work done by the expansion of steam in a cylinder, given by Watt in 1782, there are 20 ordinates at equal (unit) distances. The respective lengths of the ordinates, of which the first is one unit distance from the beginning of the diagram, are 1, 1, 1, 1, 1, ·830, ·711, ·625, ·555, ·500, ·454, ·417, ·385, ·357, ·333, ·312, ·294, ·277, ·262, ·250, representing the steam pressure in pounds weight per square inch as the piston arrives at a position corresponding to the several ordinates. The initial ordinate is also of unit length. The steam pressure is supposed to be constant (14 lbs. weight per square inch), whilst the piston travels over the first five divisions, and then the steam being cut off suddenly, the pressure is assumed to fall according to Boyle's Law ($pv=$ constant). Show that the area of this diagram is very little more than 11·562 square units, and that the mean pressure is ·578 lb. weight per square inch.

Justify Watt's statement "whereby it appears that only $\frac{1}{4}$ of the steam necessary to fill the whole cylinder is employed, and that the effect is more than half of the effect which would have been produced by one whole cylinder full of steam, if it had been allowed to enter freely above the piston during the whole length of its descent." [GOODEVE, *On the Steam Engine.*]

39. If steam at pressure p lbs. weight per square inch be admitted into a cylinder of length a feet, and be cut off when the piston has completed $\frac{1}{n}$ of its stroke, and the steam pressure then fall according to Boyle's Law for the rest of the stroke, prove by the Integral Calculus that if the piston area be A square inches, and there be no back pressure, the work done in one stroke is

$$\frac{Aap}{n}\log_e en \text{ foot-pounds.}$$

Show also that the approximate result found by the method of dividing the Indicator diagram as in the preceding question, and assuming the cut-off to be at half-stroke, differs from the true result by about 1·5 per cent. of the estimated work.

$$\left[\text{Assume } \int_a^b \frac{dx}{x} = \log\frac{b}{a},\ \log_e 2 = \cdot 69314718.\right]$$

40. Steam is admitted into a cylinder at double the atmospheric pressure (atmosph. pres. = 15 lbs. wt. per sq. inch), and on the opposite side of the piston the pressure is atmospheric continually. The steam is cut off at half stroke. Divide the stroke into 20 equal parts. Suppose the pressure at the beginning of each of these portions to remain uniform until the piston reaches the next in order, and assume the fall of pressure after cut-off to be that of Boyle's Law. Show that with these assumptions the work done in one stroke is nearly 8466 foot-lbs.; the area of the piston being 200 square inches and the length of the stroke 40 inches. [Draw the work diagram as accurately as possible on squared paper.]

41. An ellipse, whose major axis is 10 cm. and eccentricity 0·4, has a perimeter $20\int_0^{\frac{\pi}{2}}\sqrt{1-0\cdot 16\sin^2\phi}\,d\phi$ cm. in length. Draw on a large scale the graph of $\sqrt{1-0\cdot 16\sin^2\phi}$ as a function of ϕ from the following values, the angle ϕ being in radians:

ϕ	0·0	0·175	0·350	0·524	0·785	1·047	1·257	1 571
$\sin\phi$	0·0	0·030	0·117	0·250	0·500	0·750	0·904	1·000

Hence find graphically the value of the integral and the perimeter of the ellipse.

Check your result by drawing the ellipse, and stepping along the perimeter with your dividers opened to 1 cm. [I. C. S., 1907.]

42. Draw in one figure the graphs of $\frac{1}{x}$, $-\frac{1}{x}$, $\frac{\sin x}{x}$, showing how they are related; the angle x being taken in radians.

From the graph of $\frac{\sin x}{x}$ deduce the general shape of the graph of $\int_0^x \frac{\sin x}{x}\,dx$, finding its proportions roughly. What is the approximate value of the integral when x is large? [Use a large scale of representation and draw the graph, say, from $x=0$ to $x=15$. It is sufficient to describe the shape when x is negative.] [I. C. S., 1907.]

43. In an electric circuit of resistance R ohms, L is the self-inductance. The voltage suddenly rises to a value V. The current in ampères is I. The law of growth of the current is

$$I=\frac{V}{R}\left(1-e^{-\frac{R}{L}t}\right) \quad \text{and} \quad I=\frac{dQ}{dt}.$$

Taking Q to vanish initially, prove that $Q=\frac{V}{R}t-\frac{VL}{R^2}\left(1-e^{-\frac{R}{L}t}\right)$, and illustrate the growth of I graphically.

44. A current is changing according to the law

$$\frac{dQ}{dt}=I=a+bt-ct^2,$$

where t is measured in seconds and Q vanishes with t.

Also the voltage is given by $V=RI+L\frac{dI}{dt}$ where $R\equiv$ resistance and $L\equiv$ self-inductance. Express Q and V in terms of t.

45. The figure shows the indicator-diagram of a gas engine

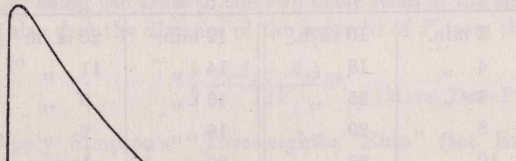

Fig. 10.

which works on the Otto cycle. Estimate the horse-power of the engine from the diagram and from the following data:

Diameter of cylinder $9\frac{1}{2}''$,
Length of stroke $16''$,
Revolutions per minute 180. [MECH. SC. TRIP.]

46. Apply Weddle's Rule for the approximate evaluation of a definite integral, viz.

$$\int_0^{6h} u_x\,dx = \frac{3h}{10}\left[u_0 + u_{2h} + u_{4h} + u_{6h} + 5(u_h + u_{5h}) + 6u_{3h}\right],$$

to evaluate $\int_0^{\frac{\pi}{2}} \log\sin\theta\,d\theta$ to four places of decimals, and compare your result with the known value $\frac{\pi}{2}\log\frac{1}{2}$. [BOOLE, *Fin. Diff.*, p. 49.]

47. Prove from first principles that if

$$x_1,\quad x_2,\quad x_3,\ \ldots,\ x_n$$

be finite real quantities such that, as n tends to infinity and x_n to x, the limit of

$$(x_2 - x_1)^2 + (x_3 - x_2)^2 + \ldots + (x_n - x_{n-1})^2$$

is zero, then the limit of the sum

$$x_1{}^4(x_2 - x_1) + x_2{}^4(x_3 - x_2) + \ldots + x^4{}_{n-1}(x_n - x_{n-1})$$

is

$$\frac{x^5 - x_1{}^5}{5}.$$

[OXF. FIRST P., 1913.]

48. The velocity of a train which starts from rest is given by the following table, the time being reckoned in minutes from the start and the speed in miles per hour:

2 min.	10 m./h.	12 min.	20 m./h.
4 ,,	18 ,,	14 ,,	11 ,,
6 ,,	25 ,,	16 ,,	5 ,,
8 ,,	29 ,,	18 ,,	2 ,,
10 ,,	32 ,,	20 ,,	At rest.

Estimate approximately the total distance run in the 20 minutes. [MATH. TRIP. PT. I., 1913.]

49. Show that if $\phi(x)$ be any polynomial of the fifth degree,

$$\int_0^1 \phi(x)\,dx = \tfrac{1}{18}\{5\phi(\alpha) + 8\phi(\tfrac{1}{2}) + 5\phi(\beta)\},$$

where α and β are the roots of $x^2 - x + \frac{1}{10} = 0$.

[MATH. TRIP. PT. I., 1909.]

50. The specific heat of a substance at temperature $t°$ is $\dfrac{dQ}{dt}$, where Q is the quantity of heat required to raise 1 gram of the substance from some fixed temperature to $t°$.

The specific heat of water (s, in joules) at a temperature of $t°$ being given by the following table:

$t° =$	0°	10°	20°	30°	40°	50°
$s =$	4·219	4·195	4·181	4·174	4·173	4·174
$t° =$	60°	70°	80°	90°	100°	
$s =$	4·178	4·184	4·190	4·197	4·205	

show that to raise 1 gram of water from 0° to 100° requires 418·5 joules approximately. [MATH. TRIP. PT. I. 1910.]

51. Prove that $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x}{x} dx > \int_{\frac{\pi}{2}}^{\pi} \frac{\sin x}{x} dx.$

[MATH. TRIP. PT. I., 1912.]

52. A uniform solid is bounded by the surface obtained by revolving the area

$$y^2 = ax^2 + 2bx + c$$

about the axis of x. A slice is cut from the solid by two plane sections perpendicular to the axis at a distance h apart; prove that the volume of the slice is V, where

$$V = \tfrac{1}{2}(A_1 + A_2)h - \tfrac{1}{6}\pi a h^3,$$

A_1 and A_2 being the areas of the two plane faces of the slice.

Show also that the distance of the centroid of V from the face A_1 is equal to

$$\frac{h}{2} + \frac{(A_2 - A_1)}{12V} h^2.$$ [MATH. TRIP. PT. I., 1914.]

53. Apply Simpson's "Three-eighths' Rule" (see Ex. 27) to approximate to the value of the integral

$$\int_0^{\frac{\pi}{6}} (1 + 6 \sin\theta)^{\frac{1}{2}} d\theta.$$

[MATH. TRIP. PT. I., 1917.]