CHAPTER II.

STANDARD FORMS.

27. Reversal of Differentiation.

We now proceed to consider Integration as the purely analytical problem of reversal of the operation of Differentiation.

In the *Differential Calculus* the student has learnt how to differentiate a function of any assigned character with regard to the independent variable contained. In other words, having given $y=\psi(x)$, methods have been there explained of obtaining the form of the function $\psi'(x)$ in the equation

$$\frac{dy}{dx}=\psi'(x)\equiv\phi(x), \text{ say.}$$

If we can reverse this operation and obtain the value of $\psi(x)$ when $\psi'(x)$ is the given function of x, we shall be able to perform the operation which has been indicated by the symbol

$$\int_a^b \phi(x)\,dx, \quad \textit{i.e.}\ \int_a^b \psi'(x)\,dx,$$

by merely (1) taking the function $\psi(x)$, (2) substituting b and a alternately for x in this function, and (3) subtracting the latter result from the former; thus obtaining

$$\psi(b)-\psi(a).$$

28. We shall therefore confine our attention for the next few chapters to the problem of this reversal of the operation of the Differential Calculus.

The quantity b has been assumed to have any real value whatever, provided it be finite; we may therefore replace it by x and write the result as

$$\int_a^x \phi(x)\,dx=\psi(x)-\psi(a).$$

When the lower limit is not specified and we are merely enquiring the *form* of the function $\psi(x)$, at present unknown, whose differential coefficient is the known function $\phi(x)$, the notation is

$$\int \phi(x)\,dx = \psi(x),$$

the limits being omitted.

29. Nomenclature.

The nomenclature of these expressions is as follows:

The function $\phi(x)$ whose integral is sought is termed the "**integrand**," and the result $\psi(x)$ is termed the "**integral.**"

$$\int_a^b \phi(x)\,dx \quad \text{or} \quad \psi(b)-\psi(a)$$

is called the "**definite**" integral of $\phi(x)$ between the assigned limits a and b.

$$\int_a^x \phi(x)\,dx \quad \text{or} \quad \psi(x)-\psi(a),$$

where the lower limit is assigned and the upper limit is left undetermined, is called a "**corrected**" integral.

$$\int \phi(x)\,dx \quad \text{or} \quad \psi(x),$$

without any specified limits and regarded merely as the reversal of an operation of the differential calculus, is called an "**indefinite**" or "**uncorrected**" integral.

It is customary to read the expression $\int \phi(x)\,dx$ as "**the integral of $\phi(x)$ with respect to x,**" or as "**the integral of** $\phi(x)\,dx$." And the process of obtaining $\psi(x)$ is called **Integration.**

30. Addition of a Constant.

It will be observed that if $\phi(x)$ be the differential coefficient of $\psi(x)$, it is also the differential coefficient of $\psi(x)+C$, where C is any constant whatever, that is to say, a quantity which does not depend upon the variable x; for the differential coefficient of such a quantity with regard to x is zero. (See Art. 3.)

Accordingly, we might write

$$\int \phi(x)\,dx = \psi(x)+C.$$

This arbitrary constant is, however, not usually expressly written down, but will be understood to be existent in all cases where the lower limit of the integral is not expressed.

31. Different processes of indefinite integration will frequently give results of different form; for instance,

$$\int \frac{1}{\sqrt{1-x^2}}\,dx \quad \text{is} \quad \sin^{-1}x \quad \text{or} \quad -\cos^{-1}x.$$

for the expression $\dfrac{1}{\sqrt{1-x^2}}$ is the differential coefficient of either of these expressions. We cannot infer that $\sin^{-1}x$ and $-\cos^{-1}x$ are equal. What is really true is that $\sin^{-1}x$ and $-\cos^{-1}x$ differ by a constant, for

$$\sin^{-1}x=\frac{\pi}{2}-\cos^{-1}x.$$

So that $$\int \frac{1}{\sqrt{1-x^2}}\,dx=\sin^{-1}x+C,$$

or $$\int \frac{1}{\sqrt{1-x^2}}\,dx=-\cos^{-1}x+C',$$

the arbitrary constants C and C' being necessarily different.

32. **Inverse Differential Notation.**

In agreement with the accepted notation for the Inverse Trigonometrical and Inverse Hyperbolic functions, we might express the equation

$$\int \phi(x)\,dx=\psi(x)$$

as $$\left(\frac{d}{dx}\right)^{-1}\phi(x)=\psi(x),$$

or $$D^{-1}\phi(x)=\psi(x)$$

or $$\frac{1}{D}\phi(x)=\psi(x),$$

and it is not infrequently useful to employ this notation, which very well expresses the interrogative character of the operation we are conducting.

33. General Laws satisfied by the Integrating Symbol

$$\int dx \quad \text{or} \quad \frac{1}{D}.$$

I. It is plain from the meaning of the symbols that

$$\frac{d}{dx}\int \phi(x)\,dx \quad \text{is} \quad \phi(x) \qquad \text{or} \qquad D\left[\frac{1}{D}\phi(x)\right]=\phi(x).$$

But $\displaystyle\int\left[\frac{d}{dx}\phi(x)\right]dx=\phi(x)+C$ or $\displaystyle\frac{1}{D}[D\phi(x)]=\phi(x)+C,$

C being any arbitrary constant.

II. The operation of integration is **distributive** for a finite number of terms.

For if u_1, u_2, u_3 be any functions of x,

$$\frac{d}{dx}\left\{\int u_1\,dx+\int u_2\,dx+\int u_3\,dx\right\}$$

$$=\frac{d}{dx}\left[\int u_1\,dx\right]+\frac{d}{dx}\left[\int u_2\,dx\right]+\frac{d}{dx}\left[\int u_3\,dx\right]$$

$$=u_1+u_2+u_3,$$

and therefore, omitting additive constants, *i.e.* supposing the lower limit to have been assigned and to be the same in each case,

$$\int u_1\,dx+\int u_2\,dx+\int u_3\,dx=\int (u_1+u_2+u_3)\,dx.$$

Similarly,

$$\int u_1\,dx+\int u_2\,dx-\int u_3\,dx=\int (u_1+u_2-u_3)\,dx.$$

If the lower limits in these several integrations are not the same, the left-hand member of the equation may differ from the right-hand by a constant. It is in this sense that the equality sign is used.

III. The operation of integration is **commutative** with regard to constants.

For if $\dfrac{du}{dx}=v$, and a be any constant,

$$\frac{d}{dx}(au)=a\frac{du}{dx}=av.$$

So that, omitting additive constants of integration,

$$au=\int av\,dx,$$

or $$a\int v\,dx=\int av\,dx,$$

which establishes the theorem.

34. Case of an Infinite Series.

In the case of an infinite series of real quantities,

$$U\equiv u_1+u_2+u_3+\ldots+u_n+\ldots \text{ to } \infty,$$

of which the terms are connected by a definite law, we shall still have

$$\int_{x_1}^{x_2} U\,dx=\int_{x_1}^{x_2} u_1\,dx+\int_{x_1}^{x_2} u_2\,dx+\int_{x_1}^{x_2} u_3\,dx+\ldots \text{ to } \infty \equiv V, \text{ say,}$$

provided the series U itself, and the series V formed by the integrations of the separate terms, are both *uniformly and unconditionally** *convergent* within a range of values of x, viz. $x=b$ to $x=a$, say, where $a>b$, between which quantities both limits of integration x_1 and x_2 lie, that is

$$a>x_2>x_1>b.$$

For let R and S be the remainders after n terms of the series U and V, *i.e.*

$$U=u_1+u_2+\ldots+u_n+R,$$
$$V=\int_{x_1}^{x_2} u_1\,dx+\int_{x_1}^{x_2} u_2\,dx+\ldots+\int_{x_1}^{x_2} u_n\,dx+S.$$

Then, by supposition, both R and S vanish when n is indefinitely increased for all values of x between a and b, and therefore so also does $\int_{x_1}^{x_2} R\,dx$, for it lies between $R'(x_2-x_1)$ and $R''(x_2-x_1)$, where R' and R'' are the greatest and least values of R as x changes continuously from a to b, and which are quantities vanishing in the limit.

Hence, $V-S=\int_{x_1}^{x_2}(U-R)\,dx=\int_{x_1}^{x_2}U\,dx-\int_{x_1}^{x_2}R\,dx$ (Art. 33, II.), and when n is indefinitely increased,

$$\int_{x_1}^{x_2} U\,dx=V.$$

* See Art. 1900, Vol. II.

If then a function $\phi(x)$ can be expanded in a power-series as $\phi(x)=\sum_{r=0}^{r=\infty} A_r x^r$, the series being uniformly and unconditionally convergent from $x=b$ to $x=a$, we can write

$$\int_{x_1}^{x_2}\phi(x)dx=\sum_{r=0}^{r=\infty}A_r\int_{x_1}^{x_2}x^r\,dx=\sum_{r=0}^{r=\infty}A_r\frac{x_2^{\,r+1}-x_1^{\,r+1}}{r+1}\text{ [Art. 16, Ex. 6]},$$

where $$a>x_2>x_1>b;$$

for if $\Sigma A_r x^r$ be uniformly and unconditionally convergent, so also will be

$$\Sigma A_r\frac{x^{r+1}}{r+1}\quad\text{and}\quad\Sigma A_r\frac{x_2^{\,r+1}-x_1^{\,r+1}}{r+1}.$$

Under such circumstances, therefore, we may expand before integrating.

35. **Geometrical Illustrations.**

We may illustrate these facts geometrically.

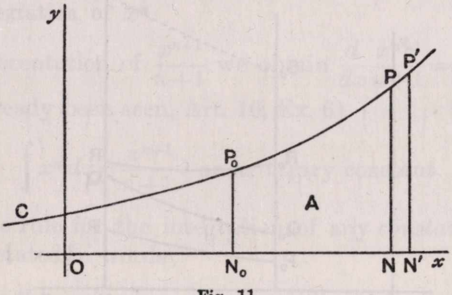

Fig. 11.

Let the graph of $y=\phi(x)$ be represented by the curve CP_0P. Let the coordinates of a fixed point P_0 on the curve be x_0, y_0, let x, y be the coordinates of a current point P on the curve, and let A be the area of the figure P_0N_0NP. Let x increase to $x+\delta x$, and in consequence let y become $y+\delta y$ and A become $A+\delta A$. Then δA is the area of the strip $PNN'P'$ between two contiguous ordinates NP and $N'P'$, and lies in magnitude between $y\,\delta x$ and $(y+\delta y)\delta x$, and therefore $\dfrac{\delta A}{\delta x}$ lies between y and $y+\delta y$.

Hence, in the limit, when δx is made indefinitely small we have

$$\frac{dA}{dx}=y.$$

Hence $$A=\int y\,dx.$$

So long as the lower limit is unassigned the reckoning of the area may start from any arbitrary position of the ordinate N_0P_0, and the case is that of the "**indefinite**" integral.

When the lower limit is assigned, say $x=ON_0$, the area is reckoned from the ordinate N_0P_0 to any arbitrary ordinate NP, and is $\int_{ON_0}^{x}\phi(x)\,dx$, and is then "**corrected.**"

When both limits ON_0 and ON are numerically assigned the integral $\int_{ON_0}^{ON}\phi(x)\,dx$ is "**definite.**"

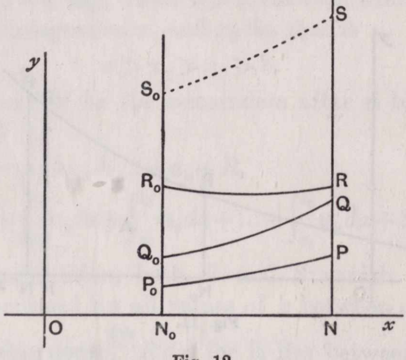

Fig. 12.

If there be several curves (a finite number of them, and all continuous, and none of the ordinates infinite within the limits of integration),

$y=F_1(x)$, $y=F_2(x)$, $y=F_3(x)$

$\equiv u_1$, $\equiv u_2$, $\equiv u_3$, viz. the curves P_0P, Q_0Q, R_0R,

and a curve be derived from them by the algebraic addition of ordinates so that

$$Y=F_1(x)+F_2(x)+F_3(x),\text{ viz. the curve } S_0S,$$

then the distributive property II. of the integration symbol asserts that

Area P_0N_0NP + area Q_0N_0NQ + area R_0N_0NR = area S_0N_0NS.

Again, if a curve be given by the equation

$$y=F(x),\ i.e.\ \text{curve } P_0P,$$

and a new one be derived by increasing all the ordinates in the ratio $a:1$ so as to have an equation

$$y=aF(x),\ i.e.\ \text{curve } S_0S, \text{ say},$$

the commutative rule III. asserts that

$$\text{Area } S_0N_0NS = a \times \text{area } F_0N_0NP.$$

If the lower limit be not the same in each case, as assumed in the figure, the stated results would, instead of being equal, differ by constants which depend upon the positions of the initial ordinates in the several cases.

36. **Integration of x^n.**

By Differentation of $\dfrac{x^{n+1}}{n+1}$ we obtain $\dfrac{d}{dx}\dfrac{x^{n+1}}{n+1}=x^n$. Hence (as has already been seen, Art. 16, Ex. 6).

$$\int x^n\,dx=\frac{x^{n+1}}{n+1}+\text{an arbitrary constant.}$$

Thus the rule for the integration of any *constant* power of x may be stated in words;

Increase the index by unity, and divide by the new index.

E.g.
$$\int x^7\,dx=\frac{x^8}{8}; \qquad \int x^{\frac{3}{5}}\,dx=\frac{x^{\frac{8}{5}}}{\frac{8}{5}}=\tfrac{5}{8}x^{\frac{8}{5}};$$

$$\int x^{-\frac{11}{3}}\,dx=\frac{x^{-\frac{8}{3}}}{-\frac{8}{3}}=-\tfrac{3}{8}x^{-\frac{8}{3}}; \qquad \int x^{-0\cdot45}\,dx=\frac{x^{\cdot55}}{\cdot55};$$

$$\int \sqrt[p]{x^q}\,dx=\frac{x^{\frac{q}{p}+1}}{\frac{q}{p}+1}=\frac{p}{p+q}x^{\frac{p+q}{p}}.$$

$$\int dx,\quad i.e.\ \int 1\,dx \text{ or } \int x^0\,dx=x.$$

37. The case of x^{-1}.

It will be remembered that x^{-1} or $\frac{1}{x}$ is the differential coefficient of $\log_e x$. Thus,

$$\int x^{-1}dx \quad \text{or} \quad \int \frac{1}{x}\,dx = \log_e x.$$

This therefore forms an apparent exception to the general rule,

$$\int x^n\,dx = \frac{x^{n+1}}{n+1}.$$

It is, however, only *apparent*. For we may deduce the logarithmic form as a limiting case. Supplying the arbitrary constant C, we have

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C = \frac{x^{n+1}-1}{n+1} + A,$$

where $A = C + \frac{1}{n+1}$ and still is an arbitrary constant, *i.e.* does not contain x. Taking the limit when $n+1=0$, $\frac{x^{n+1}-1}{n+1}$ takes the form $\log_e x$ (*Diff. Cal.*, Art. 21). And as C is an *arbitrary* constant, we may suppose that it contains a negatively infinite portion $-\frac{1}{n+1}$, together with another *arbitrary* portion A.

Then $$Lt_{n=-1}\int x^n\,dx = \log x + A.$$

This has also been seen in Art. 16, Ex. 6.

38. In the same way as in the integrations of x^n and x^{-1} we have

$$\frac{d}{dx}(ax+b)^{n+1} = (n+1)a(ax+b)^n$$

and $$\frac{d}{dx}\log(ax+b) = \frac{a}{ax+b},$$

and therefore $$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{(n+1)a}$$

and $$\int \frac{dx}{ax+b} = \frac{1}{a}\log(ax+b).$$

[Although $\int dx$ is really one symbol indicating integration with regard to x, we shall often find $\int \frac{1}{ax+b}\,dx$ printed for convenience as $\int \frac{dx}{ax+b}$, $\int \frac{1}{\sqrt{x^2+a^2}}\,dx$ printed as $\int \frac{dx}{\sqrt{x^2+a^2}}$, etc]

39. We are now in a position to integrate any expression of the form
$$\frac{\phi(x)}{ax+b},$$
where $\phi(x)$ indicates any rational integral algebraic function of x.

This can be done in two ways:

(1) By ordinary division of $\phi(x)$ by $ax+b$ we can express
$$\frac{\phi(x)}{ax+b} \text{ in the form } Q+\frac{R}{ax+b},$$
where Q consists of a series of descending powers of x and R is independent of x.

Every term is then integrable by the foregoing rules, and the result will be partly algebraic and partly logarithmic, the last term being $\frac{R}{a}\log(ax+b)$. The condition that it should be entirely algebraic is obviously that R should vanish, *i.e.* $\phi\left(-\frac{b}{a}\right)=0$, or that $\phi(x)$ should contain $ax+b$ as a factor.

E.g.
$$\int\frac{x^4+x^3}{x+2}\,dx=\int\left(x^3-x^2+2x-4+\frac{8}{x+2}\right)dx$$
$$=\frac{x^4}{4}-\frac{x^3}{3}+x^2-4x+8\log(x+2).$$

(2) A second process would be to put $ax+b=ay$, *i.e.* $x=y-\frac{b}{a}$ and then
$$\frac{\phi(x)}{ax+b}=\frac{\phi\left(y-\frac{b}{a}\right)}{ay}.$$

Then expand $\phi\left(y-\frac{b}{a}\right)$ in descending powers of y, thus expressing the fraction ultimately in the form $Q'+\frac{R'}{y}$, where Q is a series of powers of y and R' is independent of y.

Thus $\frac{\phi(x)}{ax+b}$ is expressed in a series of powers of $\left(x+\frac{b}{a}\right)$, together with a term $\frac{R'}{x+\frac{b}{a}}$, R' being independent of x and each term is again integrable.

Thus, in the foregoing case, $\int \frac{x^4+x^3}{x+2}\,dx$, putting $x+2=y$,

$$\frac{x^4+x^3}{x+2}=\frac{(y-2)^4+(y-2)^3}{y}=y^3-7y^2+18y-20+\frac{8}{y}$$
$$=(x+2)^3-7(x+2)^2+18(x+2)-20+\frac{8}{x+2}.$$

Hence

$$\int \frac{x^4+x^3}{x+2}\,dx=\frac{(x+2)^4}{4}-\frac{7}{3}(x+2)^3+9(x+2)^2-20(x+2)+8\log(x+2).$$

The results are of different form, but of course equivalent, except that they differ by a constant.

40. It is also to be observed that since the differential coefficients of $[\phi(x)]^{n+1}$ and $\log\phi(x)$ are respectively

$$(n+1)[\phi(x)]^n\phi'(x) \quad \text{and} \quad \frac{\phi'(x)}{\phi(x)},$$

we have
$$\int [\phi(x)]^n\phi'(x)\,dx=\frac{[\phi(x)]^{n+1}}{n+1}$$

and
$$\int \frac{\phi'(x)}{\phi(x)}\,dx=\log\phi(x).$$

The second of these results especially is of great use. It may be put into words thus:

The integral of any fraction of which the numerator is the differential coefficient of the denominator is **log (denominator).**

41. For example:

$$\int (ax^2+bx+c)^n(2ax+b)\,dx=\frac{(ax^2+bx+c)^{n+1}}{n+1}.$$

$$\int \frac{2ax+b}{ax^2+bx+c}\,dx=\log(ax^2+bx+c).$$

$$\int \cot x\,dx=\int\frac{\cos x}{\sin x}\,dx=\log\sin x.$$

$$\int \tan x\,dx=-\int\frac{-\sin x}{\cos x}\,dx=-\log\cos x=\log\sec x.$$

$$\int \frac{e^x-e^{-x}}{e^x+e^{-x}}\,dx=\log(e^x+e^{-x}).$$

42. More generally, since the differential coefficient of

$$F[\phi(x)] \quad \text{is} \quad F'[\phi(x)]\phi'(x),$$

we clearly have

$$\int F'[\phi(x)]\phi'(x)\,dx=F[\phi(x)].$$

Thus, for example, $\int \frac{1}{1+\sin^2 x}\cos x\,dx=\tan^{-1}\sin x.$

Examples.

Write down the indefinite integrals of :

1. x^{10}, x^{-10}, 1, 0, $x^{\frac{7}{5}}$, $x^{-\frac{5}{7}}$, $\sqrt[3]{x^{-2}}$, $\dfrac{1}{\sqrt{x}}$, $x^7 \times x^{-\frac{3}{2}}$,

2. $a\sqrt{x}+\dfrac{b}{\sqrt{x}}$, $a\sqrt[p]{x}+\dfrac{b}{\sqrt[p]{x}}$, $(ax^{\frac{1}{p}}+b)(cx^{\frac{1}{q}}+d)$, $\dfrac{ax^2+bx+c}{x^2}$.

3. $\dfrac{(ax^2+bx+c)(ax^{-2}+bx^{-1}+c)}{x}$, $\dfrac{1}{a-x}$, $\dfrac{1}{(a-x)^2}$, $\dfrac{1}{(a-x)^p}$.

4. $\dfrac{1}{a-x}+\dfrac{1}{a+x}$, $\dfrac{x}{a+x}$, $\dfrac{1}{(a+x)^2}+\dfrac{1}{(a-x)^2}$, $\dfrac{(x-a)(x+a)(x^2+a^2)}{x^4}$.

5. Calculate $\int_0^2 x^{0\cdot 3}\,dx$; $\int_3^5 x^{-\frac{1}{3}}\,dx$; $\int_1^2 \dfrac{1}{2x+3}dx$.

6. Calculate $\int_2^3 (a+bx^3)^2\, 3bx^2\,dx$ for the values $a=2$, $b=5$.

7. $\displaystyle\int_a^{2a}\left(\sqrt{\frac{a}{x}}+\sqrt{\frac{x}{a}}\right)^2 dx.$

8. If the retardation of a particle be 2 foot-seconds per second, and its initial velocity 10 f.s., when and where will it come to a stop?

9. Given $pv=$constant, and that $p=40$ when $v=10$, calculate $\int_{10}^{20} p\,dv$. What does this integration mean?

10. Calculate $\int (x-1)(x-2)(x-3)(x-4)\,dx$

between limits (*a*) 0 and 1, (*b*) 1 and 2, (*c*) 2 and 3, (*d*) 3 and 4, (*e*) 4 and 5.

Explain the signs which occur in the results. Illustrate by a graph.

11. Write down the indefinite integrals of :

(i) $(ae^x+b)^n e^x$, (ii) $\dfrac{ce^x}{ae^x+b}$, (iii) $\left(ax+\dfrac{b}{x}+c\right)^n\left(\dfrac{ax^2-b}{x^2}\right)$,

(iv) $(ax^p+be^x)^n(pax^{p-1}+be^x)$.

12. Integrate (i) $\displaystyle\int\frac{ae^{ax}+be^{bx}}{e^{ax}+e^{bx}}dx$, (ii) $\int \cot 2x\,dx$, (iii) $\int \tanh x\,dx$,

(iv) $\displaystyle\int\frac{2ax^n+b}{(ax^{2n}+bx^n+c)^p}x^{n-1}\,dx$.

13. Integrate

(i) $\displaystyle\int\frac{dx}{(1+x^2)\tan^{-1}x}$, (ii) $\displaystyle\int\frac{dx}{(1+x^2)(\tan^{-1}x)^n}$, (iii) $\displaystyle\int\frac{(\sin^{-1}x)^n}{\sqrt{1-x^2}}dx$,

(iv) $\displaystyle\int\frac{1}{\sin^{-1}x}\frac{dx}{\sqrt{1-x^2}}$, (v) $\displaystyle\int\frac{\text{vers}^{-1}x}{\sqrt{2x-x^2}}dx$.

14. Integrate (i) $\int \frac{dx}{x \log x}$, (ii) $\int \frac{1}{x \log x} \frac{1}{\log \log x} dx$,

(iii) $\int \frac{1}{x \log x} \frac{1}{\log \log x} \frac{1}{(\log \log \log x)^n} dx$,

(iv) $\int \frac{dx}{x\, l(x) l^2(x) l^3(x) \ldots l^r(x) [l^{r+1}(x)]^n}$,

where $l^r x$ represent $\log \log \log \ldots \log x$, the log being repeated r times.

43. It will now be perceived that, the operations of the Integral Calculus being of a tentative nature, success in Integration will depend in the first place on a knowledge of the results of differentiating the ordinary simple functions which occur in Algebra and Trigonometry. It is therefore necessary to learn the table of Standard Forms which is now appended. It is practically the same list as that already learnt for Differentiation, and the proofs of the facts stated lie in differentiating the right-hand members of the several results. The list was printed on page 46 of the Author's *Differential Calculus*. There are a few additions, as we are now specifically considering Integration. The list will be gradually extended, and a supplementary list will be given when the results have been established.

44. Preliminary Table of Results to be committed to Memory.

(1) $\int x^n\, dx = \frac{x^{n+1}}{n+1}$. (2) $\int \frac{1}{x}\, dx = \log_e x$.

(3) $\int e^x\, dx = e^x$. (4) $\int a^x\, dx = \frac{a^x}{\log_e a}$.

(5) $\int \cos x\, dx = \sin x$. (6) $\int \sin x\, dx = -\cos x$.

(7) $\int \sec^2 x\, dx = \tan x$. (8) $\int \operatorname{cosec}^2 x\, dx = -\cot x$.

(9) $\int \sec x \tan x\, dx = \int \frac{\sin x}{\cos^2 x}\, dx = \sec x$.

(10) $\int \operatorname{cosec} x \cot x\, dx = \int \frac{\cos x}{\sin^2 x}\, dx = -\operatorname{cosec} x$.

$$\left.\begin{array}{l}(11)\ \int \tan x\, dx = \log_e \sec x. \\ (12)\ \int \cot x\, dx = \log_e \sin x.\end{array}\right\}\ \text{(Art. 41.)}$$

(13) $\displaystyle\int\frac{dx}{\sqrt{a^2-x^2}}=\sin^{-1}\frac{x}{a}$ or $\displaystyle-\cos^{-1}\frac{x}{a}$.*

(14) $\displaystyle\int\frac{dx}{a^2+x^2}=\frac{1}{a}\tan^{-1}\frac{x}{a}$ or $\displaystyle-\frac{1}{a}\cot^{-1}\frac{x}{a}$.

(15) $\displaystyle\int\frac{dx}{x\sqrt{x^2-a^2}}=\frac{1}{a}\sec^{-1}\frac{x}{a}$ or $\displaystyle-\frac{1}{a}\operatorname{cosec}^{-1}\frac{x}{a}$.

(16) $\displaystyle\int\frac{dx}{\sqrt{2ax-x^2}}=\operatorname{vers}^{-1}\frac{x}{a}$ or $\displaystyle-\operatorname{covers}^{-1}\frac{x}{a}$.

45. It is a help to the memory to observe the dimensions of each side. For instance, x and a being supposed linear, $\displaystyle\int\frac{dx}{\sqrt{a^2-x^2}}$ is of zero dimensions. There could, therefore, be no $\dfrac{1}{a}$ prefixed to the integral. On the other hand, $\displaystyle\int\frac{dx}{a^2+x^2}$ is of dimensions -1. Hence the result of integration must be of dimensions -1. Thus the integral could not be $\tan^{-1}\dfrac{x}{a}$, which is of zero dimensions. There should, therefore, be no difficulty in remembering in which cases the factor $\dfrac{1}{a}$ appears, and when it does not.

Also, so long as we are dealing with the trigonometrical functions, whenever the result begins with the letters "co," it must be with a negative sign. The reason is obvious; the cosine, cosecant, coversine and their inverses are all decreasing functions as x increases through the first quadrant, and their differential coefficients are negative.

The rule of the "co" does not apply to the hyperbolic functions.

Examples.

Write down the indefinite integrals of the following functions :

1. $\dfrac{1}{x+1},\ \dfrac{x-a}{x+a},\ \dfrac{x}{x^2+a^2},\ \dfrac{x+a}{x^2+a^2},\ \dfrac{x^2}{x^3+a^3},\ \dfrac{x^{n-1}}{x^n+a^n}$.

2. $2^x,\ 2x,\ \dfrac{2}{x},\ x^2,\ x^3+3^x,\ a+b^x+c^{2x}+d^{3x}$.

3. $\cos^2\dfrac{x}{2},\ \sin^2\dfrac{x}{2},\ \cot x+\tan x,\ \cos x\left(\dfrac{1}{\sin x}+\dfrac{1}{\sin^2 x}\right)$.

4. $\dfrac{1}{\sqrt{9-x^2}},\ \dfrac{1}{\sqrt{9+x^2}},\ \dfrac{1}{\sqrt{x^2-9}},\ \dfrac{1}{9+x^2},\ \dfrac{1}{9-x^2},\ \dfrac{1}{x^2-9}$.

5. $\dfrac{1}{x\sqrt{x^2-4}},\ \dfrac{x+1}{x\sqrt{x^2-4}},\ \dfrac{ax+b}{\sqrt{c^2-x^2}},\ \dfrac{ax+b}{\sqrt{x^2-c^2}},\ \dfrac{ax+b}{\sqrt{x^2+c^2}}$.

6. $\dfrac{1}{\sqrt{x-x^2}},\ \dfrac{1}{x\sqrt{3x^2-27}},\ \dfrac{1}{\sqrt{27-3x^2}},\ \dfrac{x^2-4}{x^2+4},\ \dfrac{x^2+4}{x^2-4}$.

* See also Art. 1890, Vol. II.

7. Write down the indefinite integrals of:

(i) $\int \sin^{-3} x \cos x\, dx$, (ii) $\int \cot x \sec^2 x\, dx$, (iii) $\int (e^x + a)^n e^x\, dx$,

(iv) $\int (x^3 + a^3)^n x^2\, dx$, (v) $\int (ax^3 + bx + c)^n (3ax^2 + b)\, dx$.

8. Write down the indefinite integrals of:

(i) $\int \dfrac{\frac{1}{1+x^2}}{\tan^{-1} x}\, dx$, (ii) $\int \dfrac{dx}{\sqrt{1-x^2}(\sin^{-1} x)^2}$, (iii) $\int \dfrac{dx}{x(\log x)^3}$.

9. Evaluate (i) $\int_1^2 \dfrac{2x+1}{(x^2+x+1)}\, dx$, (ii) $\int_1^2 \dfrac{2x+1}{(x^2+x+1)^2}\, dx$.

10. Draw the graph of $64(x-2)(x-3)(2x-5)$, and show that the area between $x=2$ and $x=2{\cdot}5$, bounded by the curve and the x-axis, is

$$32\left[(x-2)^2(x-3)^2\right]_2^{2.5}, \text{ i.e.} = 2 \text{ square units.}$$

Verify by multiplying out and integrating each term.

11. Write down the values of:

(i) $\int_0^x \dfrac{e^{3x} + e^{5x}}{e^x + e^{-x}}\, dx$, (ii) $\int_0^x \dfrac{e^{(n+1)x} - e^{(n-1)x}}{\sinh x}\, dx$,

(iii) $\int_0^1 \dfrac{e^{2x} + e^{4x}}{e^{3x}}\, dx$, (iv) $\int_a^b \cosh(\log x)\, dx$.

12. Evaluate

(i) $\int_0^{\frac{\pi}{2}} \cos x\, dx$, (ii) $\int_0^{\frac{\pi}{2}} \cos^2 x\, dx$,

(iii) $\int_0^{\frac{\pi}{4}} \cos 2x\, dx$, (iv) $\int_0^x (\cosh x + \cos x)\, dx$.

13. Evaluate

(i) $\int_0^{\frac{\pi}{4n}} \sec^2 nx\, dx$, (ii) $\int_0^{\frac{\pi}{4}} \sec x \tan x\, dx$,

(iii) $\int_1^{\sqrt{3}} \dfrac{dx}{1+x^2}$, (iv) $\int_0^1 \dfrac{dx}{\sqrt{1-x^2}}$.

14. Evaluate

(i) $\int \dfrac{x^n - a^n}{x-a}\, dx$, (ii) $\int \dfrac{x^n}{x-a}\, dx$, (iii) $\int \dfrac{x^5 - 2x + 1}{x-1}\, dx$,

(iv) $\int \dfrac{x^5}{x-1}\, dx$, (v) $\int \dfrac{x^3 - 6x^2 + 11x - 6}{x^2 - 3x + 2}\, dx$.

46. The processes of Integration being necessarily of a tentative nature and founded upon a knowledge of the forms obtained by differentiating the known functions—algebraic, logarithmic, exponential, trigonometric or hyperbolic, or the inverse forms, it will be realized that many expressions may be written down which are not the differential coefficients of such known functions or of any combination of them. A little consideration will show that this is necessarily the case.

If the inverse sine had never received the consideration of mathematicians, the expression $\frac{1}{\sqrt{1-x^2}}$ would have been the differential coefficient of something so far uninvented. In the same way, if the invention of a logarithm had not preceded the necessity for the integration of x^{-1}, the integral of $\frac{1}{x}$ would have been lacking and have presented difficulty.

Hence it will be seen that it is only certain classes of algebraic, trigonometrical, exponential, logarithmic, or hyperbolic functions, or the corresponding inverse functions, that admit of integration in finite terms. Some functions there are which admit of integration in terms of an infinite series though such series may not be otherwise expressible as the expansion of any known function. For example,

$\frac{1}{\sqrt{1-x^{100}}}$ is not the differential coefficient of any known function. But supposing $x<1$,

$$\int_0^x \frac{dx}{(1-x^{100})^{\frac{1}{2}}} = \int_0^x \left(1+\tfrac{1}{2}x^{100}+\frac{1.3}{2.4}x^{200}+\ldots\right)dx$$

$$=x+\tfrac{1}{2}\frac{x^{101}}{101}+\frac{1.3}{2.4}\frac{x^{201}}{201}+\ldots,$$

an infinite series, not capable of summation, but nevertheless useful for approximative purposes, supposing x to be a positive proper fraction, if such arithmetical approximation be required.

And to go back to the case of x^{-1}, it is also clear that as by the failure to integrate it, by considering it a case of $\int x^n\,dx = \frac{x^{n+1}}{n+1}$, there would have been a gap in the list of integrals of powers of x, viz.,

$$\int x^2\,dx=\frac{x^3}{3},\quad \int x\,dx=\frac{x^2}{2},\quad \int x^0\,dx=\frac{x^1}{1},\quad \int \mathbf{x^{-1}\,dx=?}\quad \int x^{-2}\,dx=\frac{x^{-1}}{-1},$$

the properties of a function which had x^{-1} for its differential coefficient could not long have remained undiscovered.

For if $F(x)$ stand for $\int \frac{1}{x}\,dx$, we must have

$$F(x)+F(y)=\int\frac{dx}{x}+\int\frac{dy}{y}$$

$$=\int\left(\frac{dx}{x}+\frac{dy}{y}\right)=\int\frac{x\,dy+y\,dx}{xy}$$

$$=\int\frac{d(xy)}{xy}=F(xy),\quad i.e.\ xy=F^{-1}[F(x)+F(y)],$$

which constitutes the fundamental theorem of logarithms and indicates how an addition may be used to perform a multiplication when tables of $F(x)$ have been constructed.

In a similar way, the expression $\int_0^\phi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$, where k is a constant < 1, presents itself in the consideration of many problems geometric and kinetic. Now $\frac{1}{\sqrt{1-k^2\sin^2\theta}}$ is not the differential coefficient of any combination of algebraic, exponential or circular functions. Hence, this is a case in point. This is an integral where necessity for discussion has arisen prior to a knowledge of the expression of which it is the differential coefficient. Calling it u,

$$u=\int_0^\phi \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}.$$

We call the upper limit ϕ the amplitude of u, and write $\phi=\text{am } u$, and inversely $u=\text{am}^{-1}\phi$. Thus u receives a name. It is a function whose leading properties we propose to discuss later.

EXAMPLES.

1. Write down the indefinite integrals of:

(1) $\frac{(x+a)(x+b)(x+c)}{x^4}$, (2) $\frac{x^3+a^3+b^3-3abx}{x+a+b}$,

(3) $\sqrt[p-q]{x^{\frac{q+r}{r-p}}}\ \sqrt[q-r]{x^{\frac{r+p}{p-q}}}\ \sqrt[r-p]{x^{\frac{p+q}{q-r}}}$, (4) $\frac{a\cos x-b\sin x}{a\sin x+b\cos x+c}$,

(5) $e^{a\log x}+e^{x\log a}$, (6) $\frac{\tan^{-1}\frac{x}{3}}{9+x^2}$,

(7) $\frac{1}{\sin x\cos x}$, (8) $\frac{\cos x(1+\sin x)}{\sin^2 x}$,

(9) $\frac{1}{1-\cos x}$, (10) $\sin\left(x+\frac{\pi}{4}\right)$,

(11) $\frac{x^2+\sin^2 x}{x^2+1}\sec^2 x$, (12) $(1+\sin x\cos x)\sec^2 x$,

(13) $\tan x(1+\sec x)$, (14) $\frac{a\sin^3 x+b\cos^3 x}{\sin^2 x\cos^2 x}$,

(15) $(\cos x-\sin x)(2+\sin 2x)\sec^2 x\operatorname{cosec}^2 x$,

(16) $(a+\tan x)(b+\tan x)\sec^2 x$,

(17) $\dfrac{1}{x[1+(\log x)^2]}$, (18) $\dfrac{1}{x}\cos(\log x)$,

(19) $\dfrac{x^5+1}{x-1}$, (20) $\dfrac{e^x}{a^2e^{2x}+1}$.

2. (*a*) If $f(x)=\dfrac{1}{1-x}$, prove that $\displaystyle\int fff(x)dx=\frac{x^2}{2}$.

(*b*) If $f(x)=a+bx$, prove that $\displaystyle\int f^r(x)dx=a\frac{b^r-1}{b-1}x+b^r\frac{x^2}{2}$; where $f^r(x)$ means $fff\ldots f(x)$, the functionality symbol f occurring r times.

3. Show by expansion that

(*a*) $\displaystyle\int\log(1+x)dx=\frac{x^2}{1.2}-\frac{x^3}{2.3}+\frac{x^4}{3.4}-\frac{x^5}{4.5}+\ldots=(1+x)\log(1+x)-x$,

(*b*) $\displaystyle\int\log\frac{1+x+x^2}{1-x+x^2}dx$

$$=2\left\{\frac{x^2}{1.2}-\frac{2x^4}{3.4}+\frac{x^6}{5.6}+\frac{x^8}{7.8}-\frac{2x^{10}}{9.10}+\frac{x^{12}}{11.12}+\frac{x^{14}}{13.14}-\frac{2x^{16}}{15.16}+\ldots\right\},$$

(*c*) $\displaystyle\int(1+x)^n dx=x+\binom{n}{1}\frac{x^2}{2}+\binom{n}{2}\frac{x^3}{3}+\binom{n}{3}\frac{x^4}{4}+\binom{n}{4}\frac{x^5}{5}+\ldots$,

where $\displaystyle\binom{n}{r}\equiv\frac{n(n-1)\ldots(n-r+1)}{1.2\ldots r}$.

4. Prove by Differentiation or Integration from the Binomial Expansion of $(1+x)^n$, where n is a positive integer,

(*a*) $1C_1+2C_2+3C_3+\ldots+nC_n=n2^{n-1}$,

(*b*) $1.2C_2+2.3C_3+3.4C_4+\ldots+(n-1)nC_n=n(n-1)2^{n-2}$,

(*c*) $1C_1+3C_3+5C_5+\ldots=n2^{n-2}$,

(*d*) $1^2C_1+2^2C_2+3^2C_3+\ldots+n^2C_n=n(n+1)2^{n-2}$,

(*e*) $1^3C_1+2^3C_2x+3^3C_3x^2+\ldots+n^3C_nx^{n-1}$

$$=x\{n^2x^2+(3n-1)x+1\}(1+x)^{n-3},$$

(*f*) $1C_0+2C_1+3C_2+\ldots+(n+1)C_n=2^{n-1}(n+2)$,

(*g*) $\dfrac{C_0}{1}+\dfrac{C_1}{2}+\dfrac{C_2}{3}+\ldots+\dfrac{C_n}{n+1}=\dfrac{2^{n+1}-1}{n+1}$,

(*h*) $\dfrac{C_0}{1.2}+\dfrac{C_1}{2.3}+\dfrac{C_2}{3.4}+\ldots+\dfrac{C_n}{(n+1)(n+2)}=\dfrac{2^{n+2}-n-3}{(n+1)(n+2)}$,

(*i*) $\dfrac{C_0}{1.2.3}-\dfrac{C_1}{2.3.4}+\dfrac{C_2}{3.4.5}-\ldots+\dfrac{(-1)^nC_n}{(n+1)(n+2)(n+3)}=\dfrac{1}{2(n+3)}$,

(*j*) $\dfrac{3.4}{1.2}C_0-\dfrac{4.5}{2.3}C_1+\dfrac{5.6}{3.4}C_2-\ldots+(-1)^n\dfrac{(n+3)(n+4)}{(n+1)(n+2)}C_n$

$$=2\frac{3n+5}{(n+1)(n+2)}.$$

5. Prove from the expansions of $\sin x$ and $\cos x$ in powers of x that $\int_0^x \sin x\,dx = 1 - \cos x$ and that $\int_0^x \cos x\,dx = \sin x$.

6. Prove from the expansion of $\exp x$ that

$$\int_{-\infty}^{x} \exp x\,dx = \exp x \quad [\exp x \equiv e^x].$$

7. Prove that

$$(a)\ \int_1^x \frac{(1+x)^n}{x}\,dx = \log x + \binom{n}{1}(x-1) + \binom{n}{2}\frac{x^2-1}{2} + \binom{n}{3}\frac{x^3-1}{3} + \dots,$$

$$(b)\ \int_2^x \frac{(x+1)^n}{x-1}\,dx = 2^n \log(x-1) + \binom{n}{1} 2^{n-1}[(x-1)-1]$$
$$+ \binom{n}{2} 2^{n-2}\frac{(x-1)^2-1}{2} + \binom{n}{3} 2^{n-3}\frac{(x-1)^3-1}{3} + \dots.$$

8. Show that

$$\int (a+bx)(x+c)^n\,dx = b\frac{(x+c)^{n+2}}{n+2} + (a-bc)\frac{(x+c)^{n+1}}{n+1}.$$

9. Show that

$$\int_a^b (1+x)^n\,dx = (b-a) + \binom{n}{1}\frac{b^2-a^2}{2} + \binom{n}{2}\frac{b^3-a^3}{3} + \dots + \binom{n}{n}\frac{b^{n+1}-a^{n+1}}{n+1}.$$

10. If $\phi(x)$ be a rational integral algebraic function of x, show that

$$\int_c^x \phi(x)(x-c)^n\,dx = (x-c)^{n+1}\left[\frac{\phi(c)}{n+1} + \frac{\phi'(c)}{n+2}\frac{x-c}{\underline{|1}} + \frac{\phi''(c)}{n+3}\frac{(x-c)^2}{\underline{|2}} + \dots\right].$$

11. By considering $\int (x-a)^p (x-b)^q\,dx$, show that the difference of the series (p and q being positive integers)

$$\frac{(x-b)^{p+q+1}}{p+q+1} + p(b-a)\frac{(x-b)^{p+q}}{p+q} + \frac{p(p-1)}{1\,.\,2}(b-a)^2\frac{(x-b)^{p+q-1}}{p+q-1} + \dots,$$

$$\frac{(x-a)^{p+q+1}}{p+q+1} - q(b-a)\frac{(x-a)^{p+q}}{p+q} + \frac{q(q-1)}{1\,.\,2}(b-a)^2\frac{(x-a)^{p+q-1}}{p+q-1} - \dots$$

is independent of x.

12. Verify by differentiation that

$$(1)\ \int \frac{dx}{1+x^4} = \frac{1}{4\sqrt{2}}\log\frac{1+x\sqrt{2}+x^2}{1-x\sqrt{2}+x^2} + \frac{1}{2\sqrt{2}}\tan^{-1}\frac{x\sqrt{2}}{1-x^2},$$

$$(2)\ \int \frac{\sqrt{1-x^2}-x}{\sqrt{1-x^2}\,[1+x\sqrt{1-x^2}]}\,dx = 2\tan^{-1}(x+\sqrt{1-x^2}).$$

13. If $\phi(x)=A_0x^n+A_1x^{n-1}+A_2x^{n-2}+\ldots+A_n$, prove that

$$\int\frac{\phi(x)}{x-h}dx=B_0\frac{x^n}{n}+B_1\frac{x^{n-1}}{n-1}+B_2\frac{x^{n-2}}{n-2}+\ldots+\phi(h)\log(x-h)$$

where $B_r=hB_{r-1}+A_r$. Write down the values of B_0, B_1, B_2.

14. Show that $\int\frac{\phi(x)}{x-h}dx$ for rational integral algebraic forms of $\phi(x)$ may also be expressed as

$$\phi(h)\log(x-h)+\phi'(h)\frac{(x-h)}{1\lfloor 1}+\phi''(x)\frac{(x-h)^2}{2\lfloor 2}+\phi'''(h)\frac{(x-h)^3}{3\lfloor 3}+\ldots.$$

Prove that

$$\int\frac{e^x-e^h}{x-h}dx=e^h\left[\frac{x-h}{1^2}+\frac{(x-h)^2}{2^2\lfloor 1}+\frac{(x-h)^3}{3^2\lfloor 2}+\frac{(x-h)^4}{4^2\lfloor 3}+\ldots\right]$$

15. Prove that $\int_a^b\frac{\log x}{x}dx=\frac{1}{2}\log(ab)\log\left(\frac{b}{a}\right)$.

16. If

$$J_n(x)=\frac{x^n}{2^n n!}\left[1-\frac{x^2}{2(2n+2)}+\frac{x^4}{2\,.\,4(2n+2)(2n+4)}-\ldots\right]$$

(*i.e.* Bessel's function), prove that

$$(1)\quad \int_0^x J_1(x)dx=1-J_0(x),$$

$$(2)\quad \int_0^x\left[J_{n-1}(x)-J_{n+1}(x)\right]dx=2J_n(x)\quad (n>0).$$

17. Prove that

$$\lambda\xi^{-\lambda}\int_0^\xi x^{\lambda-1}(1-x^\mu)^\nu dx=F\left(-\nu,\frac{\lambda}{\mu},\frac{\lambda}{\mu}+1,\xi^\mu\right)$$

when $F(\alpha,\beta,\gamma,x)$ denotes the hypergeometric series

$$1+\frac{\alpha\,.\,\beta}{1\,.\,\gamma}x+\frac{\alpha\,\overline{\alpha+1}}{1\,.\,2}\frac{\beta\,\overline{\beta+1}}{\gamma\,\overline{\gamma+1}}x^2+\frac{\alpha\,\overline{\alpha+1}\,\overline{\alpha+2}}{1\,.\,2\,.\,3}\cdot\frac{\beta\,\overline{\beta+1}\,\overline{\beta+2}}{\gamma\,\overline{\gamma+1}\,\overline{\gamma+2}}x^3+\ldots.$$

[I. C. S., 1898.]

18. Assuming that the speed of the current in a river at a distance x from the bank follows the law

$$v=v_0+kx(a-x),$$

where a is the breadth of the river and v_0 and k are constants, find by integration how far down stream a man will be carried who rows 4 miles an hour, pointing the boat's head always straight at the opposite bank, so as to cross in the least time possible: the width of the river being half a mile, the banks being straight and parallel, and the speed of the current being 2 miles an hour near the banks, and 3 miles an hour in mid-stream. [I. C. S., 1905.]

19. Find the moment of inertia of a rectangle of sides $2a$, $2b$ about a line joining the mid-points of the opposite sides of length $2a$.

The section of a ship at the water line is 120 feet long. If the middle line be divided into six equal portions, the ordinates of the boundary of the area at the middle points of the segments are given by the following table:

Distances from end	10	30	50	70	90	110
Ordinates - -	10	20	20	18	14	8

Draw a figure showing the section, remembering that it is symmetrical on both sides of the middle line. By approximate methods of summation (treating the segments as rectangles) find the value of Ak^2, the moment of inertia of the section about the middle line, and the height of the metacentre above the centre of buoyancy if the ship displaces 36,000 cubic feet of water.

$$\left[\text{The height required} = \frac{\text{Moment of Inertia}}{\text{Displacement.}}.\right]$$

[I. C. S., 1908.]

20. A substance A transforms into a substance B, the rate of transformation in grammes per second at the time t being equal to ax, where x denotes the number of grammes of A existing at that instant. In like manner B transforms into a third substance C, the rate of transformation being by, where y is the number of grammes of B existing at time t.

Write down the relations between x, y and their differential coefficients with respect to the time, and show that these equations are satisfied by putting

$$x = Pe^{-at}, \quad y = Qe^{-at} + Re^{-bt},$$

provided Q and R are properly determined in terms of P, y being zero when $t = 0$.

Also show that the quantity of B existing will be greatest at

$$\frac{\log \text{Nap}\, a - \log \text{Nap}\, b}{a - b} \text{ seconds after the zero of time.}$$

[I. C. S., 1908.]

21. Prove that the integral of the sum of an infinite series taken over a range within which the series is absolutely convergent, is equal to the sum of the integrals of the terms.

Employ this theorem to find an expansion of $\log(1+z)$ in ascending powers of z, pointing out the range for which the expansion is valid.

Having given $\log_e 2 = 0{\cdot}69315$, prove that $\log_e 61 = 4{\cdot}1109$ to five significant figures. [I. C. S., 1904.]

22. OK is the diameter bounding a semicircle of radius r, P any point on OK, and PQ an ordinate to the diameter OK. If x denote the length OP and z the area which PQ cuts off the semicircle, interpret $\frac{dz}{dx}$ and $\frac{d^2z}{dx^2}$.

Find a curve for which the area bounded by the curve, the axes of x and y and the ordinate at a distance x from the axis of y, is $a^2 \tan \frac{x}{a}$. [I. C. S., 1902.]

23. From the equation

$$\frac{1}{y}\left(ah + \int_h^x y\,dx\right) = h,$$

where a and h are constants, find y in terms of x.

The value of a being 2 feet, and of h 10 feet, evaluate y when x is 30 feet. [I. C. S., 1910.]

24. Denoting by A the area between the curve $y = f(x)$ and the axis of x, from the value zero to the value a of x, show that, when $f(x)$ is a rational integral algebraic function of the *third* degree,

$$A = \frac{a}{6}(y_0 + 4y_1 + y_2),$$

where $\quad y_0 = f(0), \quad y_1 = f\left(\frac{a}{2}\right), \quad y_2 = f(a).$

Compare the result given by this rule with the true value, taken to three places of decimals, for the curve $y = \sin x$, between the values 0 and 0·5 of x reckoned in radians. [I. C. S., 1912.]

25. Verify that the area of the curve

$$y = A + Bx + Cx^2 + Dx^3$$

between the limits $x = h$ and $x = -h$ is equal to the product of h and the sum of the ordinates at

$$x = h/\sqrt{3} \quad \text{and} \quad x = -h/\sqrt{3}.$$

In the case of the curve

$$y = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 \equiv f(x),$$

verify in like manner that the area between $x = h$ and $x = -h$ is equal to

$$\{5f(h\sqrt{3/5}) + 8f(0) + 5f(-h\sqrt{3/5})\}\, h/9.$$

[I. C. S., 1913.]

26. Find the differential coefficient of

$$e^{-x}(1 + x + x^2/2\,! + x^3/3\,! + x^4/4\,!)\,;$$

and deduce that the sum of the first five terms of the exponential series is less than e^x by the quantity

$$e^x \int_0^x \frac{\alpha^4 e^{-\alpha}}{4\,!}\, d\alpha.$$

What would be the corresponding result if the series were taken to n terms instead of to five terms? [I. C. S., 1913.]

27. Weddle's Rule for finding, approximately, the area bounded by a curve, two ordinates, and a base forming part of the axis of x, is: "Divide the base into six equal parts and draw the ordinates at the points of division, making, with the extreme ordinates, seven in all. Of these ordinates add the first, third, fourth, fifth and seventh, and five times the second, fourth and sixth. Multiply the sum by one-twentieth of the base."

Prove that the rule gives the true result when the limits are 0 and 1 and the curve has any of the forms

$$y=a, \quad y=ax^2, \quad y=ax^4, \quad y=a(x-1/2)^n,$$

where a is a constant and n is an odd positive integer.

Find, by the rule, the value of $6 \times \int_0^{\frac{1}{\sqrt{3}}} \frac{dx}{1+x^2}$ to seven places of decimals. Check the result by integration. [I. C. S., 1911.]

28. Show that the work done by a gas in altering its volume from v_1 to v_2 according to the Adiabatic Law

$$pv^\gamma = p_0 v_0^\gamma$$

is

$$\frac{p_0 v_0^\gamma}{\gamma-1}\left[\frac{1}{v_1^{\gamma-1}} - \frac{1}{v_2^{\gamma-1}}\right].$$

If the law be Isothermal ($\gamma=1$), show that this becomes

$$p_0 v_0 \log \frac{v_2}{v_1}.$$

If a gas expands isothermally from state p_1, v_1 to state p_2, v_2 (Operation I.),

then expands adiabatically from state p_2, v_2 to state p_3, v_3 (Operation II.),

then contracts isothermally from state p_3, v_3 to state p_4, v_4 (Operation III.),

then contracts adiabatically from state p_4, v_4 to state p_1, v_1 (Operation IV.),

(1) find the amounts of work done by or upon the gas during each of these four operations, drawing a graph of the whole cycle of changes;

(2) show that the work done in the whole cycle of operations is measured by

$$(p_1v_1 - p_3v_3)\log\frac{v_2}{v_1} \text{ (the adiabatic portions cancelling)};$$

(3) that $$v_1v_3 = v_2v_4;$$

(4) that, writing

$$p_1v_1 = p_2v_2 = \alpha_1, \qquad p_3v_3 = p_4v_4 = \alpha_2,$$
$$p_2v_2{}^\gamma = p_3v_3{}^\gamma = \beta_1, \qquad p_4v_4{}^\gamma = p_1v_1{}^\gamma = \beta_2,$$

the above expression for the work may be written

$$\frac{\alpha_1 - \alpha_2}{\gamma - 1}\log\frac{\beta_1}{\beta_2}.$$

[This cycle of operations is known as a Carnot's cycle for a perfect heat engine.]

29. If dQ be the whole heat absorbed by a body of uniform temperature whilst its temperature changes continuously from θ to $\theta + d\theta$, and if ϕ be a function of the independent variables which define the state of the body and such that

$$d\phi = \frac{dQ}{\theta},$$

ϕ is called the Entropy of the body (Clausius).

Show that if a graph be drawn to represent θ as a function of ϕ, the area between the graph, the ϕ-axis and the ordinates corresponding to the initial and final states represents on some scale the heat absorbed.

In the case of a perfect gas satisfying the law $\frac{pv}{\theta} = \text{const.} = R$, assume the Thermodynamic Equation

$$dQ = C_v d\theta + p\,dv,$$

where C_v is the specific heat at constant volume, and show that in changing from state θ_1, v_1, ϕ_1 to state θ_2, v_2, ϕ_2,

$$\left(\frac{\theta_2}{\theta_1}\right)^{C_v}\left(\frac{v_2}{v_1}\right)^{R} = \frac{e^{\phi_2}}{e^{\phi_1}}.$$

Taking the temperature as a function of the entropy and simultaneous values of ϕ and θ as given in the following table:

ϕ	1·60	1·70	1·75	1·85
θ	450	400	370	340

and assuming that one unit of area indicates one unit of heat, what is the total heat received during these changes?

[There is a brief sketch of the fundamental formulae of Thermodynamics on pages 56 and 57 of *Solutions of Senate House Problems for* 1878 which may be found useful. Students may also read Tait's *Thermodynamics* or Parker's *Thermodynamics* for detailed accounts of the theory; other useful books are Zeuner, *Théorie Mécanique de la Chaleur*; Briot, *Théorie Mécanique de la Chaleur*.]

30. In the case of a saturated vapour, if C' be the specific heat of the vapour, *i.e.* the heat imparted to one gramme of the saturated vapour to keep it constantly in the saturated state when slowly compressed till the temperature rises one degree Fahrenheit; C that of the liquid from which it is derived at the same pressure and temperature, L the latent heat, then it can be shown that

$$C' = C + \frac{dL}{d\theta} - \frac{L}{\theta},$$

where θ is the absolute temperature.

Let C_p be the specific heat of the liquid at constant pressure, which, as liquids are practically incompressible, is so nearly equal to C that no appreciable error results from regarding them as identical.

Then Regnault has shown experimentally that the sum of the free and latent heat, viz. $L + \int_{273}^{\theta} C_p d\theta$, is not a constant as had been supposed by Watt in his earlier experiments, but is a function of the temperature θ, viz. putting $\theta = 273 + \theta'$ and J being the number of ergs in one calorie (41,539,739·8 ergs or about 3 foot-lbs.), he obtained the equations

$$\left.\begin{aligned} &(1)\quad L + \int_{273}^{\theta} C_p d\theta = J(\alpha + \beta\theta' + \gamma\theta'^2), \\ &(2)\quad \qquad\qquad\quad C_p = J(\alpha' + \beta'\theta' + \gamma'\theta'^2), \end{aligned}\right\}$$

experimentally, determining the constants α, β, γ; α', β', γ' for several vapours.

Using these data, prove that

$$(1)\quad \frac{dL}{d\theta} + C_p = J(\beta + 2\gamma\theta'),$$

$$(2)\quad L = J\left[\alpha + (\beta - \alpha')\theta' + (2\gamma - \beta')\frac{\theta'^2}{2} - \gamma'\frac{\theta'^3}{3}\right],$$

$$(3)\quad c' \equiv \frac{C'}{J} = \beta + 2\gamma\theta' - \frac{\alpha + (\beta - \alpha')\theta' + (2\gamma - \beta')\dfrac{\theta'^2}{2} - \gamma'\dfrac{\theta'^3}{3}}{273 + \theta'}.$$

31. Show that the integral equivalent of the equation

$$\int_0^x \frac{dx}{1+x+x^2} + \int_0^y \frac{dy}{1+y+y^2} + \int_0^z \frac{dz}{1+z+z^2} = 0$$

is of the form

$$xyz + a(yz+zx+xy) + b(x+y+z) + c = 0,$$

where a, b, c are certain constants. [Oxf. I. P., 1913.]

32. If the variables x, y, z be so related that

$$yz = F(x), \quad zx = F(y), \quad xy = F(z),$$

show that $\int_{x_1}^x F(x)dx + \int_{y_1}^y F(y)dy + \int_{z_1}^z F(z)dz = xyz - x_1y_1z_1$.

For example, if $x+y+z=0$,

and $yz+zx+xy = -\frac{1}{4}x^2y^2z^2$,

show that

$$\int_{x_1}^x \frac{\sqrt{1+x^4}}{x^2}dx + \int_{y_1}^y \frac{\sqrt{1+y^4}}{y^2}dy + \int_{z_1}^z \frac{\sqrt{1+z^4}}{z^2}dz = \frac{3}{4}(xyz - x_1y_1z_1).$$

[Bertrand, *Calc. Int.*, p. 383.]

33. If $y = \int_0^x e^{\sin x}\,dx$, expand y in powers of x as far as x^5.

[Oxf. I. P., 1911.]

34. Prove that $\int_0^\infty \frac{x^2+3x+3}{(x+1)^3} e^{-x}\sin x\,dx = \frac{1}{2}$. [Oxf. I. P., 1915.]

35. Integrate $\int x^{-2n-2}(1-x)^n(1-cx)^n dx$,

where n is a positive integer. [Oxf. I. P., 1917.]

36. Prove that the fifth differential coefficient of

$$x^4\int\phi(x)dx - 4x^3\int x\phi(x)dx + 6x^2\int x^2\phi(x)dx - 4x\int x^3\phi(x)dx + \int x^4\phi(x)dx$$

is $24\phi(x)$. [Oxf. I. P., 1917.]

37. Integrate $\int \frac{\cos 8\theta - \cos 7\theta}{1+2\cos 5\theta}d\theta$.

38. If $f(x)$ and $F(x)$ be two functions continuous and finite between 0 and x, such that

$$F(x) \equiv \int_0^x f(t)dt,$$

$$f(x) \equiv 1 - \int_0^x F(t)dt,$$

obtain their expansions in ascending powers of x. [Oxf. I. P., 1915.]

39. Prove that $\int_0^1 \frac{dx}{(1+x)(2+x)} = 0{\cdot}288$ nearly.

[MATH. TRIP. I., 1916.]

40. If $y^2 = a^2x^2 + c$, express in terms of y the differential coefficients of the functions

$$\log(ax+y), \quad xy$$

with regard to x.

Hence evaluate $\int \frac{dx}{y}$ and $\int y\,dx$, and prove that

$$\int_2^{\frac{5}{2}} \sqrt{x^2-4}\,dx = \tfrac{15}{8} - 2\log 2.$$

[MATH. TRIP. I., 1914.]

41. Prove that

$$\{\log(a+\theta h) - \log a\} - \theta\{\log(a+h) - \log a\}$$

can be expressed in the form

$$\theta(1-\theta)\int_0^h \frac{x\,dx}{(a+x)(a+\theta x)}.$$

Deduce that in calculating a logarithm to base 10 by the method of proportional parts from tables which give the logarithms of all integers from 10^4 to 10^5, the error is one of defect and cannot amount to $\frac{1}{8}10^{-8}\mu$, where $\mu = \log_{10} e = {\cdot}43429$. Is this negligible in seven-figure tables?

[MATH TRIP. Pt. II., 1919.]

42. Integrate $\int \sin\theta \sqrt{\frac{1+\cos 9\theta}{1+\cos\theta}}\,d\theta$, and show that

$$\int \sqrt{\frac{1+\cos(4n+1)\theta}{1+\cos\theta}}\,d\theta = \theta - 2\left\{\frac{\sin\theta}{1} - \frac{\sin 2\theta}{2} + \frac{\sin 3\theta}{3} - \dots - \frac{\sin 2n\theta}{2n}\right\},$$

n being a positive integer.