CHAPTER VII.

FURTHER REDUCTION FORMULAE.

208. Several "**Formulae of Reduction**" have already been established, and the student will have gathered some information as to their nature, mode of construction and use.

The nature of these formulae is that a connection, in general linear, is found between two or more integrals, so that when all but one have been found, the remaining one can be inferred.

209. It will be useful to summarize those which have already occurred. They are as follows:

1. The rule for integration by parts, Art. 90, and for continued integration by parts, Art. 95.

2. Reduction formulae for $\int x^m \binom{\sin}{\cos} nx\,dx$, Art. 102.

3. Reduction formulae for $\int e^{ax} \binom{\sin}{\cos}^m bx\,dx$, Art. 104.

4. Reduction formulae for $\int x^m (\log x)^n\,dx$, Art. 106.

5. Reduction formulae for $\int \sec^n x\,dx.\int \operatorname{cosec}^n x\,dx$, Art. 120, etc.

$\int \tan^n x\,dx$, $\int \cot^n x\,dx$, Art. 125, etc.

6. Reduction formulae for $\int \frac{dx}{(a+b\cos x+c\sin x)^n}$, etc., Arts. 185 to 199.

7. Reduction formulae for $\int \frac{\sin^p x \cos^q x}{(a+b\cos x)^n}\,dx$, etc., Arts. 201 to 203.

210. General Remarks.

The subject of the present chapter will be the construction of such further reduction formulae as may be necessary for present or future uses in the book, and a general indication to the student of the mode of procedure to facilitate their speedy production. It will be noted also that two distinct modes of procedure have been exhibited:

(i) That of integration by parts, or, what comes to the same thing, a proper choice of "P," with a differentiation and subsequent arrangement of the result as a linear function of the expressions whose integrals are to be connected, as exemplified in Arts. 185 to 188.

(ii) A change of the variable, taking the integrand itself, or some function of it, or of some essential part of it, as a new variable, as exemplified in Arts. 194 to 198

We shall also complete the discussion of such integrations as are to be considered, both for the general cases when reduction formulae are required and for *the particular cases in which it is convenient to avoid their use.*

211. Integration of $\int x^{m-1}X^p\,dx$, where $X\equiv a+bx^n$.

In three cases this admits of direct integration, and no reduction formulae is required:

I. When p **is a positive integer.**

II. When $\dfrac{m}{n}$ **is an integer:** (i) Positive. (ii) Negative.

III. When $\dfrac{m}{n}+p$ **is an integer:** (i) Positive. (ii) Negative.

In other cases a *reduction formula is necessary.*

212. I. If *p be a positive integer* we can expand $(a+bx^n)^p$ in a finite series by the binomial theorem and integrate each term.

Thus

$$\int x^{m-1}(a+bx^n)^p\,dx=a^p\frac{x^m}{m}+{}^pC_1a^{p-1}b\frac{x^{m+n}}{m+n}+\dots+b^p\frac{x^{m+pn}}{m+pn}.$$

If p be *fractional or negative*, the binomial expansion is *non-terminating*, and therefore the integration after expansion would not express the result *in finite terms*. Expansion therefore in such cases *should not be resorted to if avoidable.*

213. II. *Let* $p=\frac{r}{s}$ *where* r *and* s *are integers*, and s, at least, positive (which covers *all commensurable fractional or negative values of* p).

Put $X\equiv a+bx^n=z^s$,

$\therefore\quad bnx^{n-1}dx=sz^{s-1}dz;$

$$\therefore \int x^{m-1}X^p\,dx=\frac{s}{bn}\int x^{m-1}z^r\frac{z^{s-1}\,dz}{x^{n-1}}=\frac{s}{bn}\int x^{m-n}z^{r+s-1}\,dz$$

$$=\frac{s}{bn}\int z^{r+s-1}\left(\frac{z^s-a}{b}\right)^{\frac{m}{n}-1}dz$$

$$=\frac{s}{nb^{\frac{m}{n}}}\int z^{r+s-1}(z^s-a)^{\frac{m}{n}-1}dz.$$

(i) Hence *when* $\frac{m}{n}$ *is a positive integer* >0, a finite expression may be found for the integral by *expanding this binomial*, integrating each term, and finally substituting back for z its value, viz. $(a+bx^n)^{\frac{1}{s}}$.

(ii) And *when* $\frac{m}{n}$ *is a negative integer or zero*,

$$\frac{z^{r+s-1}}{(z^s-a)^{-\frac{m}{n}+1}}$$

may be put into *partial fractions* by the rules explained in Chapter V., and the integration can then be effected in finite terms.

214. III. Again, we may write the integral

$$\int x^{m-1}(a+bx^n)^{\frac{r}{s}}dx \quad\text{as}\quad \int x^{m+\frac{rn}{s}-1}(b+ax^{-n})^{\frac{r}{s}}dx;$$

and therefore by case II. this is integrable in finite terms if $\dfrac{m+\frac{rn}{s}}{-n}$ *be an integer, positive or negative, i.e. if* $\frac{m}{n}+\frac{r}{s}$ *be an integer negative or positive, and the proper substitution is*

$b+ax^{-n}=z^s$, leading to a finite expansion if $\frac{m}{n}+\frac{r}{s}$ be a negative integer, or to partial fractions if $\frac{m}{n}+\frac{r}{s}$ be a positive integer or zero.

215. To sum up:

Case I. p a positive integer: **Expand.**

Case II. $\frac{m}{n}$ an integer: **Substitute $a+bx^n=z^s$; then expand, or partial fractions, as the case may require.**

Case III. $\frac{m}{n}+p$ an integer: **Substitute $ax^{-n}+b=z^s$; then expand, or partial fractions, as the case may require.**

216. **Illustrative Examples.**

1. p a positive integer.

Consider $I\equiv\int x^5(1+x^7)^3\,dx=\int(x^5+3x^{12}+3x^{19}+x^{26})\,dx$

$$=\frac{x^6}{6}+\frac{3x^{13}}{13}+\frac{3x^{20}}{20}+\frac{x^{27}}{27}.$$

2. $\frac{m}{n}$ a positive integer.

Consider $I\equiv\int x^{13}\,(1+x^7)^{\frac{3}{5}}\,dx.$ Here $\frac{m}{n}=\frac{14}{7}=2.$

Let $1+x^7=z^5$; $\therefore\ dx=\frac{5}{7}\frac{z^4}{x^6}dz.$

$$I=\int x^{13}z^3\,.\,\frac{5}{7}\frac{z^4}{x^6}\,dz=\frac{5}{7}\int z^7\,(z^5-1)\,dz=\frac{5}{7}\left(\frac{z^{13}}{13}-\frac{z^8}{8}\right)$$

$$=\frac{5}{7}\left[\frac{1}{13}(1+x^7)^{\frac{13}{5}}-\frac{1}{8}(1+x^7)^{\frac{8}{5}}\right].$$

3. $\frac{m}{n}$ a negative integer.

Consider $I\equiv\int x^{-8}\,(1+x^7)^{\frac{1}{3}}\,dx.$ Here $\frac{m}{n}=-\frac{7}{7}=-1.$

Let $1+x^7=z^3$; $\therefore\ dx=\frac{3}{7}\frac{z^2}{x^6}dz.$

$$I=\int x^{-8}\,.\,z\frac{3}{7}\frac{z^2}{x^6}\,dz=\frac{3}{7}\int\frac{z^3}{(z^3-1)^2}\,dz.$$

Following the rules of Arts. 155-156, we may express $\frac{z^3}{(z^3-1)^2}$ as

$$\frac{1}{9}\frac{1}{z-1}+\frac{1}{9}\frac{1}{(z-1)^2}-\frac{1}{18}\frac{2z+1+5}{(z+\frac{1}{2})^2+\frac{3}{4}}+\frac{1}{6}\frac{2z+1+1}{(z^2+z+1)^2},$$

whence

$$\int \frac{z^3}{(z^3-1)^2}\,dz = \frac{1}{9}\log(z-1) - \frac{1}{9}\cdot\frac{1}{z-1} - \frac{1}{18}\log(z^2+z+1) - \frac{5}{9\sqrt{3}}\tan^{-1}\frac{2z+1}{\sqrt{3}}$$
$$-\frac{1}{6}\,\frac{1}{z^2+z+1} + \frac{1}{6}\int\frac{dz}{[(z+\frac{1}{2})^2+\frac{3}{4}]^2}.$$

In the last term, put $z+\frac{1}{2}=\frac{\sqrt{3}}{2}\tan\theta$; $\quad\therefore\; dz=\frac{\sqrt{3}}{2}\sec^2\theta\,d\theta$;

$$\therefore\; \int\frac{dz}{(z^2+z+1)^2} = \frac{\sqrt{3}}{2}\int\frac{\sec^2\theta\,d\theta}{\frac{9}{16}\sec^4\theta} = \frac{8}{3\sqrt{3}}\int\cos^2\theta\,d\theta$$
$$= \frac{4}{3\sqrt{3}}\int(1+\cos 2\theta)\,d\theta = \frac{4}{3\sqrt{3}}(\theta+\sin\theta\cos\theta)$$
$$= \frac{4}{3\sqrt{3}}\tan^{-1}\frac{2z+1}{\sqrt{3}} + \frac{1}{3}\,\frac{2z+1}{z^2+z+1}.$$

Hence $\int x^{-8}(1+x^7)^{\frac{1}{3}}\,dx$

$$=\frac{3}{7}\left[\frac{1}{9}\log(z-1) - \frac{1}{9}\,\frac{1}{z-1} - \frac{1}{18}\log(z^2+z+1) + \frac{1}{9}\,\frac{z-1}{z^2+z+1} - \frac{1}{3\sqrt{3}}\tan^{-1}\frac{2z+1}{\sqrt{3}}\right],$$

where $z=\sqrt[3]{1+x^7}$.

4. $\frac{m}{n}+\frac{r}{s}$ a positive integer.

Consider $I\equiv\int x^{\frac{1}{2}}(1+x^3)^{\frac{1}{2}}\,dx$. Here $\frac{m}{n}+\frac{r}{s}=\frac{\frac{3}{2}}{3}+\frac{1}{2}=1.$

Then $$I=\int x^2(1+x^{-3})^{\frac{1}{2}}\,dx.$$

Let $$1+x^{-3}=z^2;\quad dx=-\frac{2}{3}x^4z\,dz;$$

$$\therefore\; I=-\frac{2}{3}\int x^6z^2\,dz = -\frac{2}{3}\int\frac{z^2}{(z^2-1)^2}\,dz,$$

which can be put into partial fractions. In this case, however, the labour can be avoided by the substitution $z=\sec\theta$, and then

$$\int\frac{z^2}{(z^2-1)^2}\,dz = \int\frac{\sec^2\theta\sec\theta\tan\theta\,d\theta}{\tan^4\theta} = \int\operatorname{cosec}^3\theta\,d\theta$$
$$= -\int\sqrt{1+\cot^2\theta}\,d\cot\theta$$
$$= -\left[\frac{\cot\theta\sqrt{1+\cot^2\theta}}{2} + \frac{1}{2}\log\left(\cot\theta+\sqrt{1+\cot^2\theta}\right)\right];$$

$$\therefore\; I=\frac{1}{3}\left[\cot\theta\operatorname{cosec}\theta + \log(\cot\theta+\operatorname{cosec}\theta)\right],$$

where $\cos\theta=\frac{1}{z}=\frac{x^{\frac{3}{2}}}{(1+x^3)^{\frac{1}{2}}}$.

5. $\dfrac{m}{n}+\dfrac{r}{s}$ a negative integer.

Consider $I\equiv\int x^2(1+x^5)^{-\frac{13}{5}}\,dx.$ Here $\dfrac{m}{n}+\dfrac{r}{s}=\dfrac{3}{5}-\dfrac{13}{5}=-2.$

Then $$I=\int x^{-11}(1+x^{-5})^{-\frac{13}{5}}\,dx.$$

Put $$1+x^{-5}=z^5\,;\quad dx=-x^6z^4\,dz\,;$$

$$\therefore\ I=-\int x^{-5}z^{-9}\,dz$$
$$=+\int z^{-9}(1-z^5)\,dz$$
$$=-\frac{z^{-8}}{8}+\frac{z^{-3}}{3}=\frac{1}{3}\,\frac{x^3}{(1+x^5)^{\frac{3}{5}}}-\frac{1}{8}\,\frac{x^8}{(1+x^5)^{\frac{8}{5}}}$$
$$=\frac{1}{24}\,\frac{x^3(8+5x^5)}{(1+x^5)^{\frac{8}{5}}}.$$

217. The Six Connections Possible.

When $X\equiv a+bx^n$ and $\int x^{m-1}X^p\,dx$ is not immediately integrable by one of the foregoing rules, it may be shown that, by integration by parts, it can be connected with any of six other integrals.

Thus, for instance,
$$\int x^{m-1}X^p\,dx=\frac{x^mX^p}{m}-\int\frac{npb}{m}x^{m+n-1}X^{p-1}\,dx,$$
and by different modes of treatment we may show that the six integrals, with any one of which
$$\int x^{m-1}X^p\,dx$$
can be linearly connected, are
$$\int x^{m-1}X^{p-1}\,dx,\qquad \int x^{m-1}X^{p+1}\,dx,$$
$$\int x^{m-n-1}X^p\,dx,\qquad \int x^{m+n-1}X^p\,dx,$$
$$\int x^{m-n-1}X^{p+1}\,dx,\qquad \int x^{m+n-1}X^{p-1}\,dx,$$
that is, the index of X can be decreased or increased by 1, leaving the index of x unaltered;

the index of x can be decreased or increased by n, leaving the index of X unaltered;

the index of x can be decreased by n, and that of X increased by 1;

or, the index of x can be increased by n, and that of X decreased by 1.

That is, either index can be increased or decreased, leaving the other unaltered, that of x by n, that of X by **1**;

or, the one increased and the other decreased in that way (but *not both increased or both decreased* at the same operation).

The **rule for effecting this connection** may be put into the following handy form:

Let $P=x^{\lambda+1}X^{\mu+1}$, where λ and μ are the *smaller indices of x and X respectively*, in the two expressions whose integrals are to be connected. Find $\dfrac{dP}{dx}$. Rearrange this if necessary *as a linear function of the expressions whose integrals are to be connected.* Integrate, and the connection is complete.

In the rearrangement it may be necessary to substitute $a+bx^n$ for X, or $\dfrac{X-a}{b}$ for x^n, as may be required for the particular case in hand.

The rearrangement can always be performed. It will be unnecessary to integrate by parts. The advantage derivable from the use of the rule of "The Smaller Index $+1$" will be that it will enable us to connect at once with the particular one of the six possible integrals which may appear desirable.

218. **Proof of the Rule of "The Smaller $+1$."**

For proof it is sufficient to verify the rule in each case. Thus to connect

$$\int x^{m-1}X^p\,dx \quad \text{with} \quad \int x^{m-1}X^{p-1}\,dx,$$

put $P=x^mX^p$.

$$\begin{aligned}\therefore\ \frac{dP}{dx} &= mx^{m-1}X^p+x^m pX^{p-1}\frac{dX}{dx}\\ &= mx^{m-1}X^p+pbnx^{m+n-1}X^{p-1}\\ &= mx^{m-1}X^p+pnx^{m-1}(X-a)X^{p-1},\end{aligned}$$

(note the rearrangement "as a linear function, etc."),

$$= (m+pn)x^{m-1}X^p-apnx^{m-1}X^{p-1}.$$

Hence, $$P=(m+pn)\int x^{m-1}X^p\,dx-apn\int x^{m-1}X^{p-1}\,dx;$$

or, $$\int x^{m-1}X^p\,dx=\frac{x^mX^p}{m+pn}+\frac{apn}{m+pn}\int x^{m-1}X^{p-1}\,dx.$$

The *advantage in this reduction* lies in the fact that the *index of the often troublesome factor X^p may be lowered* if p be positive, or *raised* if p be negative, and by *successive applications of the same formula*, if necessary, we may ultimately *reduce the integral* to one which has been previously obtained, or which can be managed with greater ease.

219. **List of the Six Connections.**

The student should verify all six connections by the above rule, and also by integration by parts.

They are as follow:

$$(1)\quad \int x^{m-1}X^p\,dx=\frac{x^mX^p}{m+pn}+\frac{apn}{m+pn}\int x^{m-1}X^{p-1}\,dx.$$

$$(2)\quad \int x^{m-1}X^p\,dx=-\frac{x^mX^{p+1}}{an(p+1)}+\frac{m+pn+n}{an(p+1)}\int x^{m-1}X^{p+1}\,dx.$$

$$(3)\quad \int x^{m-1}X^p\,dx=\frac{x^{m-n}X^{p+1}}{b(m+pn)}-\frac{(m-n)a}{b(m+pn)}\int x^{m-n-1}X^p\,dx.$$

$$(4)\quad \int x^{m-1}X^p\,dx=\frac{x^mX^{p+1}}{am}-\frac{(m+pn+n)b}{am}\int x^{m+n-1}X^p\,dx.$$

$$(5)\quad \int x^{m-1}X^p\,dx=\frac{x^{m-n}X^{p+1}}{bn(p+1)}-\frac{m-n}{bn(p+1)}\int x^{m-n-1}X^{p+1}\,dx.$$

$$(6)\quad \int x^{m-1}X^p\,dx=\frac{x^mX^p}{m}-\frac{bpn}{m}\int x^{m+n-1}X^{p-1}\,dx.$$

We have written $m-1$ as the index of x in the primary integral. This is merely for the convenience of making the several coefficients on the right-hand side smaller and more compact than they would be with an index m.

220. **Special Cases.**

The case where $m+pn=0$ comes under the heading $\frac{m}{n}+p=$ integer, already discussed (Art. 211), and needs no reduction formula.

The case $p=0$ integrates at once; as also the case $n=0$.

The case $p+1=0$ integrates by partial fractions.

The case $m=0$ needs no reduction formula, coming under the heading of Case II. Art. 213, (ii).

When the student is convinced of the truth of the rule *in all cases,* the *six possibilities* of connection and the *method* of connection are all that need be remembered.

That the increase or decrease in the index of x should be "n at a time," whilst that of X is only "1 at a time," is to be expected, since $X \equiv a+bx^n$.

221. An integral of form

$$\int x^{n-1}(ax^p+bx^q)^r\,dx$$

can be written as $$\int x^{n+pr-1}(a+bx^{q-p})^r\,dx,$$

or as $$\int x^{n+qr-1}(b+ax^{p-q})^r\,dx,$$

and therefore is reduced at once to the form considered.

222. **Integrals of form**

$$\int\frac{x^m}{(a+bx^n)^p}\,dx, \quad\text{or}\quad \int\frac{(a+bx^n)^p}{x^m}\,dx, \quad\text{or}\quad \int\frac{dx}{x^m(a+bx^n)^p},$$

are obviously included in the same rules, as there has been no limitation as to the signs of the indices in the formulae discussed.

223. **Illustrative Examples.**

Ex. 1. Find the value of $I\equiv\int(x^2+a^2)^{\frac{5}{2}}\,dx.$

We may connect with $\int(x^2+a^2)^{\frac{3}{2}}\,dx$, and this again with $\int(x^2+a^2)^{\frac{1}{2}}\,dx$, and this last is a standard form.

As the reduction is to be used *more than once*, we will connect

$$\int(x^2+a^2)^{\frac{n}{2}}dx \quad\text{with}\quad \int(x^2+a^2)^{\frac{n}{2}-1}\,dx.$$

Let $P=x(x^2+a^2)^{\frac{n}{2}}$.

Then $$\frac{dP}{dx}=(x^2+a^2)^{\frac{n}{2}}+nx^2(x^2+a^2)^{\frac{n}{2}-1}$$

$$=(x^2+a^2)^{\frac{n}{2}}+n(x^2+a^2-a^2)(x^2+a^2)^{\frac{n}{2}-1}$$

(note this preparatory step, which *might be performed mentally*)

$$=(n+1)(x^2+a^2)^{\frac{n}{2}}-na^2(x^2+a^2)^{\frac{n}{2}-1}$$

(which is now arranged as a *linear function of the two expressions whose integrals were to be connected*).

Integrating, $$P=(n+1)\int(x^2+a^2)^{\frac{n}{2}}dx-na^2\int(x^2+a^2)^{\frac{n}{2}-1}dx,$$

i.e. $$\int(x^2+a^2)^{\frac{n}{2}}dx=\frac{x(x^2+a^2)^{\frac{n}{2}}}{n+1}+\frac{na^2}{n+1}\int(x^2+a^2)^{\frac{n}{2}-1}dx.$$

Putting $n=5$ and then $n=3$,

$$\int(x^2+a^2)^{\frac{5}{2}}dx=\frac{x(x^2+a^2)^{\frac{5}{2}}}{6}+\frac{5a^2}{6}\int(x^2+a^2)^{\frac{3}{2}}dx$$

and $$\int(x^2+a^2)^{\frac{3}{2}}dx=\frac{x(x^2+a^2)^{\frac{3}{2}}}{4}+\frac{3a^2}{4}\int(x^2+a^2)^{\frac{1}{2}}dx$$

and $$\int(x^2+a^2)^{\frac{1}{2}}dx=\frac{x(x^2+a^2)^{\frac{1}{2}}}{2}+\frac{a^2}{2}\sinh^{-1}\frac{x}{a}.$$

Thus $$\int(x^2+a^2)^{\frac{5}{2}}dx=\frac{1}{6}x(x^2+a^2)^{\frac{5}{2}}+\frac{5}{6.4}a^2x(x^2+a^2)^{\frac{3}{2}}$$
$$+\frac{5.3}{6.4.2}a^4x(x^2+a^2)^{\frac{1}{2}}+\frac{5.3}{6.4.2}a^6\sinh^{-1}\frac{x}{a}.$$

This result might have been obtained more quickly by substituting $x=a\tan\theta$ and using the reduction formula

$$\int\sec^{n+2}\theta\,d\theta=\frac{1}{n+1}\tan\theta\sec^n\theta+\frac{n}{n+1}\int\sec^n\theta\,d\theta \text{ (Art. 122)},$$

whence we get

$$I=\int(x^2+a^2)^{\frac{5}{2}}dx=a^6\int\sec^7\theta\,d\theta$$
$$=a^6[\tfrac{1}{6}\tan\theta\sec^5\theta+\tfrac{5}{6}\{\tfrac{1}{4}\tan\theta\sec^3\theta+\tfrac{3}{4}(\tfrac{1}{2}\tan\theta\sec\theta+\tfrac{1}{2}\log\overline{\sec\theta+\tan\theta})\}],$$

which gives the same result as before.

Ex. 2. Find the value of $I\equiv\int\frac{dx}{(x^2+a^2)^{\frac{5}{2}}}$.

First connect $\int(x^2+a^2)^{-\frac{n}{2}}dx$ with $\int(x^2+a^2)^{-\frac{n}{2}+1}dx$.

Put $P=x(x^2+a^2)^{-\frac{n}{2}+1}$.

$$\frac{dP}{dx}=(x^2+a^2)^{-\frac{n}{2}+1}-(n-2)x^2(x^2+a^2)^{-\frac{n}{2}}$$
$$=(x^2+a^2)^{-\frac{n}{2}+1}-(n-2)(x^2+a^2-a^2)(x^2+a^2)^{-\frac{n}{2}}$$
$$=(3-n)(x^2+a^2)^{-\frac{n}{2}+1}+(n-2)a^2(x^2+a^2)^{-\frac{n}{2}};$$

$$\therefore\int(x^2+a^2)^{-\frac{n}{2}}dx=\frac{x(x^2+a^2)^{-\frac{n}{2}+1}}{(n-2)a^2}+\frac{n-3}{(n-2)a^2}\int(x^2+a^2)^{-\frac{n}{2}+1}dx.$$

Putting $n=5$ and then $n=3$,

$$\int (x^2+a^2)^{-\frac{5}{2}}\,dx=\frac{1}{3}\,\frac{x(x^2+a^2)^{-\frac{3}{2}}}{a^2}+\frac{2}{3a^2}\int (x^2+a^2)^{-\frac{3}{2}}\,dx$$

and $$\int (x^2+a^2)^{-\frac{3}{2}}\,dx=\frac{1}{1}\cdot\frac{x(x^2+a^2)^{-\frac{1}{2}}}{a^2}+0;$$

$$\therefore\ \int (x^2+a^2)^{-\frac{5}{2}}\,dx=\frac{1}{3}\,\frac{x(x^2+a^2)^{-\frac{3}{2}}}{a^2}+\frac{2}{3a^4}\,x(x^2+a^2)^{-\frac{1}{2}}.$$

This again would have been shortened by the substitution $x=a\tan\theta$, which is specially suited for functions involving $\sqrt{x^2+a^2}$.

Thus $$\int \frac{dx}{(x^2+a^2)^{\frac{5}{2}}}=\frac{1}{a^4}\int\frac{\sec^2\theta\,d\theta}{\sec^5\theta}=\frac{1}{a^4}\int\cos^3\theta\,d\theta$$

$$=\frac{1}{a^4}\int(1-\sin^2\theta)\,d\sin\theta$$

$$=\frac{1}{a^4}\left(\sin\theta-\frac{\sin^3\theta}{3}\right),\quad\text{where }\sin\theta=\frac{x}{\sqrt{a^2+x^2}},$$

$$=\frac{1}{a^4}\left\{\frac{x}{(a^2+x^2)^{\frac{1}{2}}}-\frac{1}{3}\,\frac{x^3}{(a^2+x^2)^{\frac{3}{2}}}\right\},$$

which is the same as the previous result, though in a different form.

Ex. 3. Find the value of $I_n\equiv\int (x^2+a^2)^{\frac{n}{2}}\,dx$, n being a positive odd integer. Let $x(x^2+a^2)^{\frac{n}{2}}\equiv P_n$.

Since $$I_n=\frac{P_n}{n+1}+\frac{na^2}{n+1}\,I_{n-2}\quad\text{(Ex. 1)},$$

$$I_{n-2}=\frac{P_{n-2}}{n-1}+\frac{n-2}{n-1}\,a^2 I_{n-4},$$

etc.,

and $$I_1=\int\sqrt{x^2+a^2}\,dx=\frac{x\sqrt{a^2+x^2}}{2}+\frac{a^2}{2}\sinh^{-1}\frac{x}{a},$$

we have

$$I_n=\frac{P_n}{n+1}+\frac{n}{(n+1)(n-1)}\,a^2P_{n-2}+\frac{n(n-2)}{(n+1)(n-1)(n-3)}\,a^4P_{n-4}+\dots$$
$$+\frac{n(n-2)(n-4)\dots3}{(n+1)(n-1)\dots4.2}\,a^{n-1}P_1$$
$$+\frac{n(n-2)(n-4)\dots3.1}{(n+1)(n-1)\dots4.2}\,a^{n+1}\sinh^{-1}\frac{x}{a}.$$

Ex. 4. Find the value of $I_n\equiv\int\frac{dx}{(x^2+a^2)^{\frac{n}{2}}}$, n being a positive integer. Let $\dfrac{x}{(x^2+a^2)^{\frac{n-2}{2}}}\equiv P_n$.

Since $$I_n=\frac{P_n}{(n-2)\,a^2}+\frac{n-3}{n-2}\,\frac{1}{a^2}\,I_{n-2},$$

we have $$I_{n-2}=\frac{P_{n-2}}{(n-4)\,a^2}+\frac{n-5}{n-4}\,\frac{1}{a^2}\,I_{n-4}$$

etc.

When n is an odd positive integer, we ultimately arrive at I_3, and

$$I_3=\int\frac{dx}{(x^2+a^2)^{\frac{3}{2}}}=\frac{x(x^2+a^2)^{-\frac{1}{2}}}{a^2}=\frac{P_3}{a^2};$$

$$\therefore\ I_n=\frac{1}{n-2}\frac{P_n}{a^2}+\frac{n-3}{(n-2)(n-4)}\frac{P_{n-2}}{a^4}+\frac{(n-3)(n-5)}{(n-2)(n-4)(n-6)}\frac{P_{n-4}}{a^6}+\dots$$

$$+\frac{(n-3)(n-5)\dots 2}{(n-2)(n-4)\dots 3.1}\frac{P_3}{a^{r-1}},\quad \text{where } P_n\equiv\frac{x}{(x^2+a^2)^{\frac{n-2}{2}}}$$

In the case when n is an even integer, $\equiv 2m$ say,

$$I_{2m}=\int\frac{dx}{(x^2+a^2)^m}=\frac{P_{2m}}{(2m-2)a^2}+\frac{2m-3}{2m-2}\cdot\frac{1}{a^2}I_{2(m-1)},\quad \text{where } P_{2m}=\frac{x}{(x^2+a^2)^{m-1}}$$

$$=\frac{1}{2m-2}\cdot\frac{P_{2m}}{a^2}+\frac{2m-3}{(2m-2)(2m-4)}\frac{P_{2(m-1)}}{a^4}+\frac{(2m-3)(2m-5)}{(2m-2)(2m-4)(2m-6)}\frac{P_{2(m-2)}}{a^6}$$

$$+\dots+\frac{(2m-3)(2m-5)\dots 1}{(2m-2)(2m-4)(2m-6)\dots 2}\frac{1}{a^{2m-1}}\tan^{-1}\frac{x}{a}.$$

In integration between limits 0 and ∞,

$$I_{2m}\Big]_0^\infty=\frac{(2m-3)(2m-5)\dots 1}{(2m-2)(2m-4)\dots 2}\cdot\frac{1}{a^{2m-1}}\cdot\frac{\pi}{2}.$$

M. Bertrand* shows a very ingenious deduction from this result, viz. putting $a=1$ and $x=\frac{z}{\sqrt{m}}$,

$$\frac{1}{\sqrt{m}}\int_0^\infty\frac{dz}{\left(1+\frac{z^2}{m}\right)^m}=\frac{(2m-3)(2m-5)\dots 1}{(2m-2)(2m-4)\dots 2}\frac{\pi}{2}.$$

Take the case when m is indefinitely increased; then

$$Lt_{m=\infty}\left(1+\frac{z^2}{m}\right)^m=e^{z^2}.$$

Hence
$$\int_0^\infty e^{-z^2}dz=\frac{\pi}{2}Lt_{m=\infty}\frac{1.3.5\dots(2m-3)}{2.4.6\dots(2m-2)}\sqrt{m},$$

and by Wallis's Theorem (Hobson, *Trigonometry*, p. 331),

$$\frac{2.4.6\dots(2m-2)}{1.3.5\dots(2m-3)}\quad\text{and}\quad\sqrt{\frac{\pi}{2}(2m-1)}$$

become infinite in a ratio of equality.

Hence
$$\frac{\pi}{2}Lt_{m=\infty}\frac{1.3.5\dots(2m-3)}{2.4.6\dots(2m-2)}\sqrt{m}$$

$$=\frac{\pi}{2}Lt\frac{\sqrt{m}}{\sqrt{\frac{\pi}{2}(2m-1)}}=\frac{\sqrt{\pi}}{2};$$

$$\therefore\ \int_0^\infty e^{-z^2}dz=\tfrac{1}{2}\sqrt{\pi}.$$

Consider also $I_m=\int_0^\infty x^m e^{-x^2}dx$, m being a positive integer.

* Bertrand, *Calc. Diff.* p. 130: see also Hall, *D. and I. C.*, p. 330.

Integrating by parts,

$$I_m = -\frac{1}{2}\int_0^\infty x^{m-1}(-2xe^{-x^2})\,dx$$

$$= -\frac{1}{2}\left\{\left[x^{m-1}e^{-x^2}\right]_0^\infty - (m-1)\int_0^\infty x^{m-2}e^{-x^2}\,dx\right\}$$

$$= \frac{m-1}{2}I_{m-2} \quad (m>1).$$

Now $$I_0 = \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

and $$I_1 = \int_0^\infty xe^{-x^2}\,dx = -\frac{1}{2}\left[e^{-x^2}\right]_0^\infty = \frac{1}{2};$$

$$\therefore \int_0^\infty x^{2n}e^{-x^2}\,dx = \frac{(2n-1)(2n-3)\ldots 1}{2^{n+1}}\sqrt{\pi},$$

$$\int_0^\infty x^{2n+1}e^{-x^2}\,dx = \frac{2n.(2n-2)\ldots 4.2}{2^{n+1}} = \frac{n!}{2},$$ n being a positive integer.

Note also that if the integration extends from $-\infty$ to $+\infty$,

$$\int_{-\infty}^\infty e^{-x^2}\,dx = \sqrt{\pi},$$

$$\int_{-\infty}^\infty x^{2n}e^{-x^2}\,dx = \frac{(2n-1)(2n-3)\ldots 1}{2^n}\sqrt{\pi},$$

but $$\int_{-\infty}^\infty x^{2n+1}e^{-x^2}\,dx = 0,$$

for to any positive element of the integrand in the third integral there is always an equal negative element.

Ex. 5. Calculate the value of $\int_0^{2a} x^m\sqrt{2ax-x^2}\,dx$, m being a positive integer.

We proceed to connect

$$\int x^m\sqrt{2ax-x^2}\,dx \text{ with } \int x^{m-1}\sqrt{2ax-x^2}\,dx,$$

i.e. $$\int x^{m+\frac{1}{2}}(2a-x)^{\frac{1}{2}}dx \text{ with } \int x^{m-\frac{1}{2}}(2a-x)^{\frac{1}{2}}\,dx.$$

Let $P = x^{m+\frac{1}{2}}(2a-x)^{\frac{3}{2}}$, according to the rule; then

$$\frac{dP}{dx} = (m+\tfrac{1}{2})x^{m-\frac{1}{2}}(2a-x)^{\frac{3}{2}} - \tfrac{3}{2}x^{m+\frac{1}{2}}(2a-x)^{\frac{1}{2}}$$

$$= (2m+1)ax^{m-\frac{1}{2}}(2a-x)^{\frac{1}{2}} - (m+2)x^{m+\frac{1}{2}}(2a-x)^{\frac{1}{2}}.$$

Hence

$$(m+2)\int x^{m+\frac{1}{2}}(2a-x)^{\frac{1}{2}}\,dx = -x^{m+\frac{1}{2}}(2a-x)^{\frac{3}{2}} + (2m+1)a\int x^{m-\frac{1}{2}}(2a-x)^{\frac{1}{2}}\,dx,$$

i.e.

$$\int_0^{2a} x^m\sqrt{2ax-x^2}\,dx = -\left[\frac{x^{m-1}(2ax-x^2)^{\frac{3}{2}}}{m+2}\right]_0^{2a} + \frac{2m+1}{m+2}a\int_0^{2a} x^{m-1}\sqrt{2ax-x^2}dx$$

$$= \frac{2m+1}{m+2}a\int_0^{2a} x^{m-1}\sqrt{2ax-x^2}\,dx\,;$$

$\therefore$ if $I_m = \int_0^{2a} x^m \sqrt{2ax - x^2}\,dx$,

$$I_m = \frac{2m+1}{m+2}\,a I_{m-1} = \frac{2m+1}{m+2}\cdot\frac{2m-1}{m+1}\,a^2 I_{m-2}$$

$$= \frac{2m+1}{m+2}\cdot\frac{2m-1}{m+1}\cdot\frac{2m-3}{m}\,a^3 I_{m-3} = \text{etc.}$$

$$= \frac{2m+1}{m+2}\cdot\frac{2m-1}{m+1}\cdot\frac{2m-3}{m}\cdots\frac{5}{4}\cdot\frac{3}{3}\cdot a^m I_0.$$

Now, to find I_0 or $\int_0^{2a}\sqrt{2ax-x^2}\,dx$, put $x = a(1-\cos\theta)$.

Then $$dx = a\sin\theta\,d\theta$$

and $$\sqrt{2ax-x^2} = a\sin\theta.$$

Also, when $x=0$ we have $\theta=0$; when $x=2a$ we have $\theta=\pi$.

Hence

$$I_0 = \int_0^\pi a^2\sin^2\theta\,d\theta = \frac{a^2}{2}\int_0^\pi (1-\cos 2\theta)\,d\theta = \frac{a^2}{2}\Big[\theta - \tfrac{1}{2}\sin 2\theta\Big]_0^\pi = \frac{\pi a^2}{2}.$$

Hence $I_m = \dfrac{(2m+1)(2m-1)\ldots 3}{(m+2)(m+1)\ldots 3}\,a^{m+2}\dfrac{\pi}{2} = \dfrac{(2m+1)\,!}{m\,!\,(m+2)\,!}\cdot\dfrac{\pi a^{m+2}}{2^m}.$

Examples.

Prove that

1. $\displaystyle\int x^{m-1}(a+bx)^p\,dx = \frac{x^m(a+bx)^p}{m+p} + \frac{ap}{m+p}\int x^{m-1}(a+bx)^{p-1}\,dx.$

2. $\displaystyle\int \frac{(a+bx)^p}{x^{m+1}}\,dx = -\frac{(a+bx)^{p+1}}{max^m} + \frac{(p-m+1)}{m}\,\frac{b}{a}\int\frac{(a+bx)^p}{x^m}\,dx.$

3. $\displaystyle\int\frac{dx}{x^m(a+bx)} = -\frac{1}{m-1}\cdot\frac{1}{a}\cdot\frac{1}{x^{m-1}} + \frac{1}{m-2}\cdot\frac{b}{a^2}\cdot\frac{1}{x^{m-2}} - \frac{1}{m-3}\cdot\frac{b^2}{a^3}\cdot\frac{1}{x^{m-3}}$

$$+\ldots+(-1)^{m-1}\frac{1}{1}\cdot\frac{b^{m-2}}{a^{m-1}}\cdot\frac{1}{x} + (-1)^m\frac{b^{m-1}}{a^m}\log\frac{a+bx}{x}.$$

4. $\displaystyle\int\frac{(a+bx)^p}{x}\,dx = \frac{(a+bx)^p}{p} + a\int\frac{(a+bx)^{p-1}}{x}\,dx.$ [Bertrand.]

5. $\displaystyle\int\frac{dx}{(a+bx^2)^{p+1}} = \frac{x}{2ap(a+bx^2)^p} + \frac{2p-1}{2ap}\int\frac{dx}{(a+bx^2)^p}.$ [Bertrand.]

6. $\displaystyle\int\frac{x^n\,dx}{(a+bx^3)^{p+1}} = \frac{x^{n+1}}{3ap(a+bx^3)^p} - \frac{n-3p+1}{3ap}\int\frac{x^n}{(a+bx^3)^p}\,dx,$ [Bertrand.]

and evaluate

$$\int\frac{x^7}{a+bx^3}\,dx,\quad \int\frac{x^3}{(a+bx^3)^2}\,dx,\quad \int\frac{dx}{x^3(a+bx^3)}.$$

7. $\int x^n(a+bx^4)^p\,dx = \frac{x^{n+1}(a+bx^4)^p}{n+1} - \frac{4bp}{n+1}\int x^{n+4}(a+bx^4)^{p-1}\,dx,$

$$\int \frac{x^n}{a+bx^4}\,dx = \frac{x^{n-3}}{(n-3)b} - \frac{a}{b}\int \frac{x^{n-4}}{a+bx^4}\,dx,$$

$$\int \frac{dx}{(a+bx^4)^{p+1}} = \frac{x}{4ap(a+bx^4)^p} + \frac{4p-1}{4ap}\int \frac{dx}{(a+bx^4)^p}, \qquad \text{[BERTRAND.]}$$

and evaluate $\displaystyle\int \frac{dx}{(a+bx^4)^4},\quad \int \frac{dx}{x^2(a+bx^4)^3}.$

224. Reduction formulae for $\int \sin^p x \cos^q x\,dx$.

Integrals of this form also conform to the rule of "the smaller index $+1$," explained in Art. 217.

Connection can be effected with any of the following six integrals:

$$\int \sin^{p-2}x\cos^q x\,dx, \qquad \int \sin^{p+2}x\cos^q x\,dx,$$

$$\int \sin^p x\cos^{q-2}x\,dx, \qquad \int \sin^p x\cos^{q+2}x\,dx,$$

$$\int \sin^{p-2}x\cos^{q+2}x\,dx, \qquad \int \sin^{p+2}x\cos^{q-2}x\,dx,$$

by the following rule:

Put $P = \sin^{\lambda+1}x\cos^{\mu+1}x$, where λ and μ are the smaller indices of $\sin x$ and $\cos x$ respectively in the two expressions whose integrals are to be connected.

Find $\frac{dP}{dx}$, and rearrange as a *linear function of the expressions whose integrals are to be connected.* This rearrangement can always be performed.

Integrate, and the connection is effected.

Each of these connections might be effected by integration by parts, but the advantage to be gained by the present rule is the same as has been explained in Art. 217.

For example, let us connect the integrals

$$\int \sin^p x\cos^q x\,dx \quad\text{and}\quad \int \sin^{p-2}x\cos^q x\,dx.$$

Let $P=\sin^{p-1}x\cos^{q+1}x$.

$$\frac{dP}{dx}=(p-1)\sin^{p-2}x\cos^{q+2}x-(q+1)\sin^p x\cos^q x$$
$$=(p-1)\sin^{p-2}x\cos^q x(1-\sin^2 x)-(q+1)\sin^p x\cos^q x$$
$$=(p-1)\sin^{p-2}x\cos^q x-(p+q)\sin^p x\cos^q x.$$

[Note the last two lines of *rearrangement as a linear function* of $\sin^p x\cos^q x$ and $\sin^{p-2}x\cos^q x$.]

Hence

$$P=(p-1)\int\sin^{p-2}x\cos^q x\,dx-(p+q)\int\sin^p x\cos^q x\,dx$$

and

$$\int\sin^p x\cos^q x\,dx=-\frac{\sin^{p-1}x\cos^{q+1}x}{p+q}+\frac{p-1}{p+q}\int\sin^{p-2}x\cos^q x\,dx.$$

225. List of the Six Connections.

The student should note carefully the *possibilities* of connection for $\int\sin^p x\cos^q x\,dx$.

The indices of either $\sin x$ or $\cos x$ may be increased or diminished by 2, the other index being unaltered; or, the index of the one lowered by 2 and the other increased by 2. Writing s for $\sin x$ and c for $\cos x$, the six connections are:

$$(1)\quad \int s^p c^q\,dx=-\frac{s^{p-1}c^{q+1}}{p+q}+\frac{p-1}{p+q}\int s^{p-2}c^q\,dx.$$

$$(2)\quad \int s^p c^q\,dx=\frac{s^{p+1}c^{q+1}}{p+1}+\frac{p+q+2}{p+1}\int s^{p+2}c^q\,dx.$$

$$(3)\quad \int s^p c^q\,dx=\frac{s^{p+1}c^{q-1}}{p+q}+\frac{q-1}{p+q}\int s^p c^{q-2}\,dx.$$

$$(4)\quad \int s^p c^q\,dx=-\frac{s^{p+1}c^{q+1}}{q+1}+\frac{p+q+2}{q+1}\int s^p c^{q+2}\,dx.$$

$$(5)\quad \int s^p c^q\,dx=-\frac{s^{p-1}c^{q+1}}{q+1}+\frac{p-1}{q+1}\int s^{p-2}c^{q+2}\,dx.$$

$$(6)\quad \int s^p c^q\,dx=\frac{s^{p+1}c^{q-1}}{p+1}+\frac{q-1}{p+1}\int s^{p+2}c^{q-2}\,dx.$$

Each of these should be verified by the student by means of the rule given, viz. "Put $P=\sin^{\lambda+1}x\cos^{\mu+1}x$, where λ, μ are, etc. ...," and also by integration by parts.

226. **Special Cases.**

When $p+q=0$, the integral is $\int \tan^p x\,dx$, and is integrated by the reduction formulae of Art. 125.

When $p+1=0$,

$$\int \sin^p x \cos^q x\,dx = \int \frac{\cos^q x}{\sin x}dx = -\int \frac{\cos^q x}{1-\cos^2 x}d(\cos x),$$

and then we write $\cos x = z$, and use the method of partial fractions, or proceed as in Art. 228.

When $q+1=0$,

$$\int \sin^p x \cos^q x\,dx = \int \frac{\sin^p x}{\cos x}dx = \int \frac{\sin^p x}{1-\sin^2 x}d(\sin x),$$

and then we again use partial fractions, or proceed as in Art. 228.

227. The student is again reminded that **when either** p **or** q **is odd,** or when $p+q$ **is a negative even integer,** *there is an easier mode of procedure* (Art. 114). Also that *in any case we have the method of multiple angles* when the indices are positive and integral; and in general this will be a more speedy method of obtaining the *indefinite* integral than the employment of a reduction formula. The results, however, will be necessarily produced in a different form by such processes.

228. We must also notice that, in the formulae of Art. 225, *either p or q, or both of them, may be negative.* Hence we now have reduction formulae for integrals such as

$$\int \frac{\sin^p x}{\cos^q x}dx, \quad \int \frac{\cos^q x}{\sin^p x}dx, \quad \text{or} \quad \int \frac{dx}{\sin^p x \cos^q x},$$

and to these the "multiple-angle method" of Art. 112 would not apply, by reason of the non-termination of the binomial expansion used for the purpose of conversion.

Thus, putting $-q$ for q in formula (5) of Art. 225,

$$\int \frac{\sin^p x}{\cos^q x}dx = \frac{\sin^{p-1} x}{(q-1)\cos^{q-1} x} - \frac{p-1}{q-1}\int \frac{\sin^{p-2} x}{\cos^{q-2} x}dx.$$

Putting $-p$ for p in formula (6),

$$\int \frac{\cos^q x}{\sin^p x}dx = -\frac{\cos^{q-1} x}{(p-1)\sin^{p-1} x} - \frac{q-1}{p-1}\int \frac{\cos^{q-2} x}{\sin^{p-2} x}dx.$$

Putting $-p$ for p and $-q$ for q in (2) and (4),

$$\int\frac{dx}{\sin^p x\cos^q x}=-\frac{1}{(p-1)\sin^{p-1}x\cos^{q-1}x}+\frac{p+q-2}{p-1}\int\frac{dx}{\sin^{p-2}x\cos^q x}$$

or
$$=\frac{1}{(q-1)\sin^{p-1}x\cos^{q-1}x}+\frac{p+q-2}{q-1}\int\frac{dx}{\sin^p x\cos^{q-2}x}.$$

etc.

If, however, $p=1$ or $q=1$ in these results, *i.e.* for integrals of form $\int\frac{\sin^p x}{\cos x}dx$ or $\int\frac{\cos^q x}{\sin x}dx$, these reductions obviously fail.

In the case $\int\frac{\sin^p x}{\cos x}dx$, we may put $q=-1$ in formula (1), Art. 225.

Then
$$\int\frac{\sin^p x}{\cos x}dx=-\frac{\sin^{p-1}x}{p-1}+\int\frac{\sin^{p-2}x}{\cos x}dx,$$

and repeating the operation, we presently arrive at $\int\frac{\sin^2 x}{\cos x}dx$ if p be an even integer, or at $\int\frac{\sin x}{\cos x}dx$ if p be odd, giving respectively $\log\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)-\sin x$ or $\log\sec x$ in the two cases.

Similarly, for $\int\frac{\cos^q x}{\sin x}dx$, put $p=-1$ in formula (3);

$$\therefore \int\frac{\cos^q x}{\sin x}dx=\frac{\cos^{q-1}x}{q-1}+\int\frac{\cos^{q-2}x}{\sin x}dx,$$

finally arriving at $\int\frac{\cos^2 x}{\sin x}dx$ or at $\int\frac{\cos x}{\sin x}dx$,

i.e. $\log\tan\frac{x}{2}+\cos x$ or $\log\sin x$ as the case may be.

229. The cases when p or q vanishes, *i.e.* the integrals

$$\int\sin^n x\,dx \quad\text{and}\quad \int\cos^n x\,dx$$

are of primary importance.

Connect $\int\sin^n x\,dx$ with $\int\sin^{n-2}x\,dx$.

Let $P=\sin^{n-1}x\cos x$, according to rule; then

$$\frac{dP}{dx}=(n-1)\sin^{n-2}x\cos^2 x-\sin^n x$$
$$=(n-1)\sin^{n-2}x-n\sin^n x;$$

$$\therefore \int\sin^n x\,dx=-\frac{\sin^{n-1}x\cos x}{n}+\frac{n-1}{n}\int\sin^{n-2}x\,dx.$$

Similarly,

$$\int \cos^n x\,dx = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx.$$

230. To calculate

$$S_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx \quad \text{and} \quad C_n = \int_0^{\frac{\pi}{2}} \cos^n x\,dx.$$

Since $\sin^{n-1}x \cos x$ vanishes when n is an integer, not less than 2, at both limits, $x=0$ and $x=\frac{\pi}{2}$, we have

$$S_n = \frac{n-1}{n} S_{n-2} = \frac{n-1}{n} \cdot \frac{n-3}{n-2} S_{n-4}$$

$$= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} S_{n-6} = \text{etc.}$$

If n be *even* this ultimately comes to

$$S_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \int_0^{\frac{\pi}{2}} 1\,dx,$$

i.e. $$S_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}.$$

If n be *odd* we similarly get

$$S_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{4}{5} \cdot \frac{2}{3} \int_0^{\frac{\pi}{2}} \sin x\,dx,$$

and since $$\int_0^{\frac{\pi}{2}} \sin x\,dx = \Big[-\cos x\Big]_0^{\frac{\pi}{2}} = 1,$$

we have $$S_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{4}{5} \cdot \frac{2}{3}.$$

In a similar way it may be seen that $\int_0^{\frac{\pi}{2}} \cos^n x\,dx$ has *precisely the same value* as the above integral in each case, n odd, n even. This may be shown, too, from other considerations.

We thus have

$$\int_0^{\frac{\pi}{2}} \binom{\sin}{\cos}^n x\,dx = \frac{n-1}{n}\,\frac{n-3}{n-2}\,\frac{n-5}{n-4} \cdots \frac{3}{4}\,\frac{1}{2}\,\frac{\pi}{2}, \quad n \text{ even};$$

or $$= \frac{n-1}{n}\,\frac{n-3}{n-2}\,\frac{n-5}{n-4} \cdots \frac{4}{5}\,\frac{2}{3}\,1, \quad n \text{ odd}.$$

231. The student should notice that these formulae are written down most easily by *beginning with the denominator.* We then have the ordinary sequence of the natural numbers written backwards,

$$(n \text{ under } \overline{n-1}) \times (\overline{n-2} \text{ under } \overline{n-3}) \times (\overline{n-4} \text{ under } \overline{n-5}) \ldots \text{etc.},$$

stopping at (2 under 1) if n be even, and *writing a factor* $\frac{\pi}{2}$; or stopping at (3 under 2) if n be odd, with *no extra factor*.

Thus (1) $\int_0^{\frac{\pi}{2}} \sin^{12}\theta\, d\theta = \frac{11}{12}\frac{9}{10}\frac{7}{8}\frac{5}{6}\frac{3}{4}\frac{1}{2}\frac{\pi}{2}.$

(2) $\int_0^{\frac{\pi}{2}} \sin^{11}\theta\, d\theta = \frac{10}{11}\frac{8}{9}\frac{6}{7}\frac{4}{5}\frac{2}{3}.$

(3) $\int_0^{\frac{\pi}{4}} \cos^6 2\theta \cos^2\theta\, d\theta = \int_0^{\frac{\pi}{2}} \cos^6\phi \frac{1+\cos\phi}{2}\, \tfrac{1}{2}\, d\phi,$ where $\phi = 2\theta$.

$$= \frac{1}{4}\left[\frac{5}{6}\frac{3}{4}\frac{1}{2}\frac{\pi}{2} + \frac{6}{7}\frac{4}{5}\frac{2}{3}\right].$$

(4) $\int_0^{\frac{\pi}{6}} \cos^7 3\theta \sin^4 6\theta\, d\theta = 2^4 \int_0^{\frac{\pi}{6}} \sin^4 3\theta \cos^{11} 3\theta\, d\theta$

$$= \frac{2^4}{3}\int_0^{\frac{\pi}{2}} \sin^4\phi \cos^{11}\phi\, d\phi, \text{ where } \phi = 3\theta,$$

$$= \frac{2^4}{3}\int_0^{\frac{\pi}{2}} (\cos^{11}\phi - 2\cos^{13}\phi + \cos^{15}\phi)\, d\phi$$

$$= \frac{2^4}{3}\frac{2}{3}\frac{4}{5}\frac{6}{7}\frac{8}{9}\frac{10}{11}\left(1 - 2\frac{12}{13} + \frac{12}{13}\frac{14}{15}\right)$$

$$= \text{etc.}$$

Examples.

1. Prove that

$$\int \sin^p\theta \cos^q\theta\, d\theta = \frac{\sin^{p+1}\theta \cos^{q-1}\theta}{p+q} - \frac{q-1}{(p+q)(p+q-2)} \sin^{p-1}\theta \cos^{q-1}\theta + \frac{(p-1)(q-1)}{(p+q)(p+q-2)} \int \sin^{p-2}\theta \cos^{q-2}\theta\, d\theta$$

the indices *being both diminished.*

2. Prove that $\int \frac{\sin^p\theta}{\cos^q\theta}\, d\theta = \frac{\sin^{p-1}\theta}{(q-1)\cos^{q-1}\theta} - \frac{p-1}{q-1}\int \frac{\sin^{p-2}\theta}{\cos^{q-2}\theta}\, d\theta.$

3. Prove that $\int \frac{\cos^p\theta}{\sin^q\theta}\, d\theta = -\frac{\cos^{p-1}\theta}{(q-1)\sin^{q-1}\theta} - \frac{p-1}{q-1}\int \frac{\cos^{p-2}\theta}{\sin^{q-2}\theta}\, d\theta.$

4. Prove that

$$\int \frac{d\theta}{\sin^p\theta\cos^q\theta} = \frac{1}{(q-1)\sin^{p-1}\theta\cos^{q-1}\theta} + \frac{p+q-2}{q-1}\int\frac{d\theta}{\sin^p\theta\cos^{q-2}\theta}$$

$$= -\frac{1}{(p-1)\sin^{p-1}\theta\cos^{q-1}\theta} + \frac{p+q-2}{p-1}\int\frac{d\theta}{\sin^{p-2}\theta\cos^q\theta}.$$

5. $\displaystyle\int\frac{\sin^p\theta}{\cos\theta}d\theta = -\frac{\sin^{p-1}\theta}{p-1} + \int\frac{\sin^{p-2}\theta\,d\theta}{\cos\theta}.$

6. $\displaystyle\int\frac{\cos^p\theta}{\sin\theta}d\theta = \quad \frac{\cos^{p-1}\theta}{p-1} + \int\frac{\cos^{p-2}\theta}{\sin\theta}d\theta.$

7. (a) $$\int\sin^{2n}\theta\,d\theta = -\frac{c}{2n}\left[s^{2n-1} + \frac{2n-1}{2n-2}s^{2n-3} + \frac{(2n-1)(2n-3)}{(2n-2)(2n-4)}s^{2n-5} + \dots + \frac{1.3\dots(2n-1)}{2.4\dots(2n-2)}s\right] + \frac{1.3\dots(2n-1)}{2.4\dots2n}\theta.$$

(b) $$\int\sin^{2n+1}\theta\,d\theta = -\frac{c}{2n+1}\left[s^{2n} + \frac{2n}{2n-1}s^{2n-2} + \frac{2n(2n-2)}{(2n-1)(2n-3)}s^{2n-4} + \dots + \frac{2.4\dots2n}{1.3\dots(2n-1)}\right],$$

where c and s stand respectively for $\cos\theta$ and $\sin\theta$. [BERTRAND.]

8. (a) $$\int\cos^{2n}\theta\,d\theta = \frac{s}{2n}\left[c^{2n-1} + \frac{2n-1}{2n-2}c^{2n-3} + \frac{(2n-1)(2n-3)}{(2n-2)(2n-4)}c^{2n-5} + \dots + \frac{1.3\dots(2n-1)}{2.4\dots(2n-2)}c\right] + \frac{1.3\dots(2n-1)}{2.4\dots2n}\theta.$$

(b) $$\int\cos^{2n+1}\theta\,d\theta = \frac{s}{2n+1}\left[c^{2n} + \frac{2n}{2n-1}c^{2n-2} + \dots + \frac{2.4\dots2n}{1.3\dots(2n-1)}\right],$$

c and s being respectively $\cos\theta$ and $\sin\theta$. [BERTRAND.]

9. Prove

(a) $$\int\operatorname{cosec}^5\theta\,d\theta = -\frac{1}{4}\frac{c}{s^4} - \frac{1.3}{2.4}\frac{c}{s^2} + \frac{1.3}{2.4}\log\tan\frac{\theta}{2}.$$

(b) $$\int\sec^5\theta d\theta \;=\; \frac{1}{4}\frac{s}{c^4} + \frac{1.3}{2.4}\frac{s}{c^2} + \frac{3}{8}\log\tan\left(\frac{\theta}{2}+\frac{\pi}{4}\right),$$

where $c\equiv\cos\theta$, $s\equiv\sin\theta$.

10. Prove that

(a) $$\int\cos^{2n}\theta\,d\theta = \frac{1}{2^{2n-1}}\left[\frac{\sin 2n\theta}{2n} + 2n\frac{\sin(2n-2)\theta}{2n-2} + \frac{2n(2n-1)}{1.2}\frac{\sin(2n-4)\theta}{2n-4} + \dots + \frac{2n(2n-1)\dots(n+1)}{1.2\dots n}\theta\right].$$

(b) $$\int \cos^{2n+1}\theta\, d\theta = \frac{1}{2^{2n}}\left[\frac{\sin(2n+1)\theta}{2n+1} + (2n+1)\frac{\sin(2n-1)\theta}{2n-1} + \frac{(2n+1)2n}{1.2}\frac{\sin(2n-3)\theta}{2n-3} + \ldots + \frac{(2n+1)2n\ldots(n+2)}{1.2\ldots n}\sin\theta\right].$$

(c) $$\int \sin^{2n}\theta\, d\theta = \frac{(-1)^n}{2^{2n-1}}\left[\frac{\sin 2n\theta}{2n} - 2n\frac{\sin(2n-2)\theta}{2n-2} + \frac{2n(2n-1)}{1.2}\frac{\sin(2n-4)\theta}{2n-4} - \ldots + (-1)^{n-1}\frac{2n(2n-1)\ldots(n+2)}{1.2\ldots(n-1)}\frac{\sin 2\theta}{2} + (-1)^n\frac{2n(2n-1)\ldots(n+1)}{1.2\ldots n}\frac{\theta}{2}\right].$$

(d) $$\int \sin^{2n+1}\theta\, d\theta = \frac{(-1)^{n+1}}{2^{2n}}\left[\frac{\cos(2n+1)\theta}{2n+1} - (2n+1)\frac{\cos(2n-1)\theta}{2n-1} + \ldots + (-1)^n\frac{(2n+1)\ldots(n+2)}{1.2\ldots n}\cos\theta\right].$$

[BERTRAND.]

232. INTRODUCTION OF THE GAMMA FUNCTION.

For what follows we shall require a new function $\Gamma(n+1)$, which will be *defined sufficiently for present purposes* by the equations

$$\Gamma(n+1) = n\Gamma(n), \quad \Gamma(1) = 1, \quad \Gamma(\tfrac{1}{2}) = \sqrt{\pi}.$$

This will be enough to find its value whenever n is a positive integer, or of the form $\dfrac{2k+1}{2}$, where k is a positive integer.

For instance

$$\Gamma(6) = 5\,\Gamma(5) = 5.4\,\Gamma(4) = 5.4.3\,\Gamma(3)$$
$$= 5.4.3.2\,\Gamma(2) = 5.4.3.2.1\,\Gamma(1) = 5\,!,$$

$$\Gamma(\tfrac{11}{2}) = \tfrac{9}{2}\,\Gamma(\tfrac{9}{2}) = \tfrac{9}{2}\cdot\tfrac{7}{2}\,\Gamma(\tfrac{7}{2}) = \tfrac{9}{2}\cdot\tfrac{7}{2}\cdot\tfrac{5}{2}\,\Gamma(\tfrac{5}{2}) = \tfrac{9}{2}\cdot\tfrac{7}{2}\cdot\tfrac{5}{2}\cdot\tfrac{3}{2}\,\Gamma(\tfrac{3}{2})$$
$$= \tfrac{9}{2}\cdot\tfrac{7}{2}\cdot\tfrac{5}{2}\cdot\tfrac{3}{2}\cdot\tfrac{1}{2}\,\Gamma(\tfrac{1}{2}) = \tfrac{9}{2}\cdot\tfrac{7}{2}\cdot\tfrac{5}{2}\cdot\tfrac{3}{2}\cdot\tfrac{1}{2}\sqrt{\pi}.$$

This function is called a Gamma function. We shall define it more generally later and investigate its properties. For the present, it is temporarily introduced to secure facility in the rapid evaluation of a class of integrals to be discussed.

233. It will be noted that the products of the first n odd numbers $1.3.5.7\ldots(2n-1)$ and of the first n even numbers $2.4.6\ldots 2n$ can be expressed in terms of this function, for

$$\Gamma\left(\frac{2n+1}{2}\right)=\frac{2n-1}{2}\cdot\frac{2n-3}{2}\cdot\frac{2n-5}{2}\ldots\frac{1}{2}\sqrt{\pi}$$

and
$$\Gamma\left(\frac{2n+2}{2}\right)=\frac{2n}{2}\cdot\frac{2n-2}{2}\cdot\frac{2n-4}{2}\ldots\frac{2}{2};$$

$$\therefore\ 1.3.5\ldots(2n-1)=\frac{2^n}{\sqrt{\pi}}\Gamma\left(\frac{2n+1}{2}\right),$$

and
$$2.4.6\ldots 2n=2^n\,\Gamma\left(\frac{2n+2}{2}\right)=2^n\Gamma(n+1).$$

234. To investigate a formula for $\int_0^{\frac{\pi}{2}}\sin^p\theta\cos^q\theta\,d\theta$, p and q being positive integers.

Let this integral be denoted by $f(p, q)$; then since

$$\int\sin^p\theta\cos^q\theta\,d\theta=-\frac{\sin^{p-1}\theta\cos^{q+1}\theta}{p+q}+\frac{p-1}{p+q}\int\sin^{p-2}\theta\cos^q\theta\,d\theta,$$

we have, if p and q be positive integers and p not less than 2,

$$f(p, q)=\frac{p-1}{p+q}f(p-2, q).$$

Case I. Let p be even, $=2m$, and q be also even, $=2n$.

Then
$$f(2m, 2n)=\frac{2m-1}{2m+2n}f(2m-2, 2n)$$
$$=\frac{(2m-1)(2m-3)}{(2m+2n)(2m+2n-2)}f(2m-4, 2n)=\text{etc.}$$
$$=\frac{(2m-1)(2m-3)\ldots 1}{(2m+2n)(2m+2n-2)\ldots(2n+2)}f(0, 2n),$$

and
$$(0, 2n)=\int_0^{\frac{\pi}{2}}\cos^{2n}\theta\,d\theta=\frac{2n-1}{2n}\,\frac{2n-3}{2n-2}\ldots\frac{1}{2}\cdot\frac{\pi}{2};$$

$$\therefore f(2m, 2n)=\frac{[1.3.5\ldots(2m-1)][1.3.5\ldots(2n-1)]}{2.4.6\ldots(2m+2n)}\cdot\frac{\pi}{2}$$

$$=\frac{\frac{2^m}{\sqrt{\pi}}\Gamma\left(\frac{2m+1}{2}\right)\frac{2^n}{\sqrt{\pi}}\Gamma\left(\frac{2n+1}{2}\right)}{2^{m+n}\Gamma\left(\frac{2m+2n+2}{2}\right)}\cdot\frac{\pi}{2}$$

$$=\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q}{2}+1\right)}.$$

CASE II. Let p be even, $=2m$, and q be odd, $=2n-1$. Then

$$f(2m,\,2n-1)=\frac{2m-1}{2m+2n-1}f(2m-2,\,2n-1)=\text{etc.}$$

$$=\frac{(2m-1)(2m-3)\dots 1}{(2m+2n-1)(2m+2n-3)\dots(2n+1)}f(0,\,2n-1)$$

and $$f(0,\,2n-1)=\int_0^{\frac{\pi}{2}}\cos^{2n-1}\theta\,d\theta=\frac{2n-2}{2n-1}\cdot\frac{2n-4}{2n-3}\dots\frac{2}{3},$$

i.e. $$f(2m,\,2n-1)=\frac{[1.3.5\dots(2m-1)][2.4.6\dots(2n-2)]}{1.3.5\dots(2m+2n-1)}$$

$$=\frac{\dfrac{2^m}{\sqrt{\pi}}\Gamma\left(\dfrac{2m+1}{2}\right)2^{n-1}\Gamma\left(\dfrac{2n}{2}\right)}{\dfrac{2^{m+n}}{\sqrt{\pi}}\Gamma\left(\dfrac{2m+2n+1}{2}\right)}$$

$$=\frac{\Gamma\left(\dfrac{p+1}{2}\right)\Gamma\left(\dfrac{q+1}{2}\right)}{2\Gamma\left(\dfrac{p+q}{2}+1\right)}.$$

CASE III. Let p be odd, $=2m-1$, and q be even, $=2n$. In this case we obtain similarly

$$f(2m-1,\,2n)=\frac{[2.4.6\dots(2m-2)][1.3.5\dots(2n-1)]}{1.3.5\dots(2m+2n-1)}.$$

But this may also be deduced at once from Case II. by putting

$$\theta=\frac{\pi}{2}-\phi,$$

for then $$\int_0^{\frac{\pi}{2}}\sin^p\theta\cos^q\theta\,d\theta=\int_{\frac{\pi}{2}}^0\cos^p\phi\sin^q\phi\,(-1)\,d\phi$$

$$=\int_0^{\frac{\pi}{2}}\sin^q\phi\cos^p\phi\,d\phi,$$

so that $$f(p,\,q)=f(q,\,p).$$

Hence the result is again

$$\frac{\Gamma\left(\dfrac{p+1}{2}\right)\Gamma\left(\dfrac{q+1}{2}\right)}{2\Gamma\left(\dfrac{p+q}{2}+1\right)}.$$

CASE IV. Let p be odd, $=2m-1$, and q be odd, $=2n-1$.

$$f(2m-1,\,2n-1)=\frac{2m-2}{2m+2n-2}f(2m-3,\,2n-1)=\text{etc.}$$

$$=\frac{(2m-2)(2m-4)\dots 2}{(2m+2n-2)(2m+2n-4)\dots(2n+2)}f(1,\,2n-1)$$

and $$f(1,\,2n-1)=\int_0^{\frac{\pi}{2}}\sin\theta\cos^{2n-1}\theta\,d\theta=\left[-\frac{\cos^{2n}\theta}{2n}\right]_0^{\frac{\pi}{2}}=\frac{1}{2n};$$

$$\therefore f(2m-1,\ 2n-1)=\frac{[2.4.6\ldots(2m-2)][2.4.6\ldots(2n-2)]}{2.4.6\quad(2m+2n-2)}$$

$$=\frac{2^{m-1}\Gamma\left(\frac{2m}{2}\right)2^{n-1}\Gamma\left(\frac{2n}{2}\right)}{2^{m+n-1}\Gamma\left(\frac{2m+2n}{2}\right)}$$

$$=\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q}{2}+1\right)}.$$

235. Hence, in every case we have the same result, viz.

$$\int_0^{\frac{\pi}{2}}\sin^p\theta\cos^q\theta\,d\theta=\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q}{2}+1\right)},$$

and it will be noticed that the $\frac{p+q}{2}+1$ occurring in the denominator is the *sum of the* $\frac{p+1}{2}$ *and the* $\frac{q+1}{2}$ in the numerator.

236. As it has been assumed that p is not <2 we must consider the particular cases $p=1$, $p=0$ separately.

When $p=1$, $\int_0^{\frac{\pi}{2}}\sin\theta\cos^q\theta\,d\theta=\left[-\frac{\cos^{q+1}\theta}{q+1}\right]_0^{\frac{\pi}{2}}=\frac{1}{q+1}.$

Now $$\frac{\Gamma\left(\frac{1+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{q+3}{2}\right)}=\frac{1}{2\frac{q+1}{2}}=\frac{1}{q+1}$$

Hence, this case conforms to the general rule.

When $p=0$, $$\int_0^{\frac{\pi}{2}}\binom{\sin}{\cos}^n\theta\,d\theta=\frac{(n-1)(n-3)\ldots1}{n(n-2)\ldots2}\frac{\pi}{2}\quad(n\text{ even})$$

$$\frac{(n-1)(n-3)\ldots2}{n(n-2)\ldots3}\quad(n\text{ odd}).$$

In the case n even, the above result may be written

$$\frac{\frac{2^{\frac{n}{2}}}{\sqrt{\pi}}\Gamma\left(\frac{n+1}{2}\right)}{2^{\frac{n}{2}}\Gamma\left(\frac{n+2}{2}\right)}\frac{\pi}{2},\quad i.e.\ \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{n+2}{2}\right)},$$

and in the case n odd, the result is

$$\frac{2^{\frac{n-1}{2}}\Gamma\left(\frac{n+1}{2}\right)}{\frac{2^{\frac{n+1}{2}}}{\sqrt{\pi}}\Gamma\left(\frac{n+2}{2}\right)}=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{n+2}{2}\right)}.$$

Hence these cases also conform to the general rule

$$\int_0^{\frac{\pi}{2}}\sin^p\theta\cos^q\theta\,d\theta=\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)},$$

which may therefore be assumed in all cases where p and q are positive integers.

237. This, then, is a very convenient formula for evaluating quickly integrals of the above form.

Thus,
$$\int_0^{\frac{\pi}{2}}\sin^6\theta\cos^8\theta\,d\theta=\frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{9}{2}\right)}{2\Gamma(8)}$$
$$=\frac{\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\sqrt{\pi}.\frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\sqrt{\pi}}{2.7.6.5.4.3.2.1}$$
$$=\frac{5\pi}{2^{12}}.$$

If, however, the limits be other than 0 *and an integral multiple of* $\frac{\pi}{2}$, *we must find the indefinite integral either by a reduction formula or by the method of Arts.* 114-117 *before inserting the limits.*

238. **Integrals of form**

$$I_{m,p}\equiv\int x^mX^p\,dx,\ \text{where}\ X\equiv a+bx+cx^2.$$

[This form obviously includes all such cases as

$$\int\frac{dx}{(a+bx+cx^2)^p},\quad\int\frac{x^m}{(a+bx+cx^2)^p}dx,\quad\int\frac{(a+bx+cx^2)^p}{x^m}dx,$$

$$\int\frac{dx}{x^m(a+bx+cx^2)^p},\quad\int\frac{x^m\,dx}{\sqrt{a+bx+cx^2}},\quad\int\frac{(x-p)^n}{\sqrt{x^2+ax+b}}dx,\ \text{etc.}]$$

I. Consider the case when $m=0$, *i.e.* $I_{0,p}\equiv\int X^p\,dx$.

Put $P=(b+2cx)X^p$.

Then
$$\frac{dP}{dx}=2cX^p+p(b+2cx)^2X^{p-1}$$
$$=2cX^p+p(b^2-4ac+4cX)X^{p-1}$$
$$=(2p+1)2cX^p+p(b^2-4ac)X^{p-1};$$
$$\therefore\ (b+2cx)X^p=(2p+1)2cI_{0,p}+p(b^2-4ac)I_{0,p-1},$$
i.e.
$$\int X^p\,dx=\frac{(b+2cx)X^p}{2(2p+1)c}-\frac{p(b^2-4ac)}{2(2p+1)c}\int X^{p-1}\,dx.\ \ldots\ldots\ldots\text{(A)}$$

This reduction *fails when* $2p+1=0$, but in that case the integral is $\int\frac{dx}{\sqrt{a+bx+cx^2}}$, and has been considered in Art. 80. The formula (A) will finally reduce the integration of $\int X^p\,dx$ to that of something of the form $\int X^s\,dx$, where s lies between 0 and 1. If $s=0$ or $\frac{1}{2}$, the integration can be written down. Hence, in all cases where p is integral or of form $\frac{2k+1}{2}$, where k is a positive integer, the integration of $\int X^p dx$ can be effected.

If p be a negative integer or of form $-\frac{2k+1}{2}$, we can apply the same formula to lower the index in the denominator, viz.
$$\int X^{p-1}\,dx=\frac{(b+2cx)X^p}{p(b^2-4ac)}-\frac{2(2p+1)c}{p(b^2-4ac)}\int X^p\,dx,$$
or writing $-p$ for p,
$$\int\frac{dx}{X^{p+1}}=-\frac{(b+2cx)}{p(b^2-4ac)}\frac{1}{X^p}-2\frac{(2p-1)c}{p(b^2-4ac)}\int\frac{dx}{X^p}.$$

II. Next, consider the case when $m=1$, *i.e.* $I_{1,p}\equiv\int xX^p\,dx$.

Put $P=X^{p+1}$.
$$\frac{dP}{dx}=(p+1)(b+2cx)X^p;$$
$$\therefore\ X^{p+1}=(p+1)b\int X^p\,dx+2(p+1)c\int xX^p\,dx;$$
$$\therefore\ I_{1,p}=\frac{X^{p+1}}{2(p+1)c}-\frac{b}{2c}I_{0,p},\ \ldots\ldots\ldots\ldots\ldots\text{(B)}$$
and the last integral has been considered.

This reduction fails when $p=-1$.

But this case is $I_{1,-1} \equiv \int \frac{x\,dx}{a+bx+cx^2}$, and no reduction is required.

239. In the case when $m=-1$, *i.e.* $I_{-1,p} \equiv \int \frac{X^p}{x}\,dx$, put $P=X^p$.

Then
$$\frac{dP}{dx}=p(b+2cx)X^{p-1}$$
$$=\frac{p}{x}[bx+2(X-a-bx)]X^{p-1}$$
$$=p\left[-\frac{2a}{x}-b+\frac{2X}{x}\right]X^{p-1}$$
$$=-2ap\frac{X^{p-1}}{x}-pbX^{p-1}+\frac{2pX^p}{x};$$
$$\therefore\ \int\frac{X^p}{x}\,dx=\frac{X^p}{2p}+\frac{b}{2}\int X^{p-1}\,dx+a\int\frac{X^{p-1}}{x}\,dx,$$
that is
$$I_{-1,p}=\frac{X^p}{2p}+\frac{b}{2}I_{0,p-1}+aI_{-1,p-1}. \quad\text{.......................(C)}$$

240. **In the case** $I_n \equiv \int \frac{x^m}{\sqrt{a+bx+cx^2}}\,dx$, **put**
$$P=x^{m-1}\sqrt{a+bx+cx^2}.$$
$$\frac{dP}{dx}=(m-1)x^{m-2}\sqrt{a+bx+cx^2}+\frac{x^{m-1}(b+2cx)}{2\sqrt{a+bx+cx^2}}$$
$$=\frac{2(m-1)(a+bx+cx^2)+bx+2cx^2}{2\sqrt{a+bx+cx^2}}x^{m-2};$$
$$\therefore\ P=(m-1)a\,I_{m-2}+\frac{(2m-1)}{2}b\,I_{m-1}+mc\,I_m,$$

which connects I_m with I_{m-1} and I_{m-2} (unless $m=0$).

Now
$$I_1\equiv\int\frac{x\,dx}{\sqrt{a+bx+cx^2}}$$
$$=\frac{1}{2c}\int\left(\frac{b+2cx}{\sqrt{a+bx+cx^2}}-\frac{b}{\sqrt{a+bx+cx^2}}\right)dx$$
$$=\frac{1}{c}\sqrt{a+bx+cx^2}-\frac{b}{2c}I_0,$$

and I_0 is discussed in Arts. 80. 81.

241. III. In the general case $I_{m,p} \equiv \int x^m X^p\,dx$, **since**

$$a + bx + cx^2 \equiv X,$$

we have $\quad x^{m-2}X^{p+1} = x^{m-2}(a+bx+cx^2)X^p,$

and therefore

$$I_{m-2,\,p+1} = aI_{m-2,\,p} + bI_{m-1,\,p} + cI_{m,\,p}. \quad\text{...........}(D)$$

Again, let $P = x^{m-1}X^{p+1}$. Then we have

$$\begin{aligned}\frac{dP}{dx} &= (m-1)x^{m-2}X^{p+1} + (p+1)x^{m-1}(b+2cx)X^p \\ &= x^{m-2}X^p[(m-1)(a+bx+cx^2)+(p+1)(bx+2cx^2)] \\ &= (m-1)ax^{m-2}X^p + (m+p)bx^{m-1}X^p + (m+2p+1)cx^mX^p;\end{aligned}$$

$\therefore\ x^{m-1}X^{p+1}$

$$= (m-1)aI_{m-2,\,p} + (m+p)bI_{m-1,\,p} + (m+2p+1)cI_{m,\,p}. \quad\text{...}(E)$$

Eliminating $I_{m-2,\,p}$ between (D) and (E),

$$x^{m-1}X^{p+1} - (m-1)I_{m-2,\,p+1} = (p+1)bI_{m-1,\,p} + 2(p+1)cI_{m,\,p}. \quad (F)$$

We thus have, collecting the results,

$$\int x^{m-2}X^{p+1}\,dx$$

$$= a\int x^{m-2}X^p\,dx + b\int x^{m-1}X^p\,dx + c\int x^mX^p\,dx, \quad\text{....................}(D)$$

$$(m+2p+1)\,c\int x^mX^p\,dx$$

$$= x^{m-1}X^{p+1} - (m-1)a\int x^{m-2}X^p\,dx - (m+p)b\int x^{m-1}X^p\,dx, \quad (E)$$

$$\int x^mX^p\,dx$$

$$= \frac{x^{m-1}X^{p+1}}{2(p+1)c} - \frac{(m-1)}{2(p+1)c}\int x^{m-2}X^{p+1}\,dx - \frac{b}{2c}\int x^{m-1}X^p\,dx; \quad\text{......}(F)$$

or, writing $-p$ for p to adapt them to the use of cases in which the index of X is negative,

$$\int \frac{x^{m-2}}{X^{p-1}}\,dx$$

$$= a\int\frac{x^{m-2}}{X^p}\,dx + b\int\frac{x^{m-1}}{X^p}\,dx + c\int\frac{x^m}{X^p}\,dx, \quad\text{.............................}(D')$$

$$(m-2p+1)c\int\frac{x^m}{X^p}dx$$

$$=\frac{x^{m-1}}{X^{p-1}}-(m-1)a\int\frac{x^{m-2}}{X^p}dx-(m-p)b\int\frac{x^{m-1}}{X^p}dx, \quad\text{.............(E}')$$

$$\int\frac{x^m}{X^p}dx$$

$$=-\frac{1}{2(p-1)c}\frac{x^{m-1}}{X^{p-1}}+\frac{(m-1)}{2(p-1)c}\int\frac{x^{m-2}}{X^{p-1}}dx-\frac{b}{2c}\int\frac{x^{m-1}}{X^p}dx. \quad\text{....(F}')$$

242. Remarks.

The case of $p=-1$, in which formula (F) fails, is

$$I_{m,\,-1}=\int\frac{x^m}{a+bx+cx^2}dx.$$

But in this case we proceed to partial fractions, and no reduction is required.

Equation (D) (p positive) expresses one integral in terms of three others, with a lower power of X at the expense of introducing higher powers of x; and

Equation (D′) raises the power of X in the denominators.

Equations (E) and (E′) reduce to integrations with the same powers of X but lower powers of x.

Equation (F) connects with two integrals, in both of which the index of x is lowered, whilst that of X is raised in one integral and remains the same in the other.

Equation (F′) plays a similar part for the negative index of X.

243. Integrals of form

$$\int(px+q)^m(a+bx+cx^2)^n\,dx, \quad\text{or}\quad \int\frac{(px+q)^m}{(a+bx+cx^2)^n}dx,$$

obviously come under the heading discussed, after transformation, by making $px+q=y$, which transforms $a+bx+cx^2$ to the form $A+By+Cy^2$, where

$$Ap^2=ap^2-bpq+cq^2, \quad Bp^2=bp-2cq, \quad Cp^2=c,$$

and
$$\int(px+q)^m(a+bx+cx^2)^n\,dx$$

becomes
$$\frac{1}{p}\int y^m(A+By+Cy^2)^n\,dy,$$

and similarly in other cases.

The particular cases where $b=0$ or $c=0$ come under the heading of those discussed as $\int x^{m-1}(a+bx^r)^p\,dx$ in Art. 217.

244. Integrals of form $I_n \equiv \int \dfrac{dx}{(q+px)^n\sqrt{a+bx+cx^2}}$ may be regarded as coming under the head of those discussed in Art. 241, for the substitution $q+px=y$ immediately reduces them to that form. But as this form occurs very frequently and is of considerable importance, it is desirable to consider it independently.

Let $P \equiv \dfrac{\sqrt{a+bx+cx^2}}{(q+px)^{n-1}}$

Then

$$\frac{dP}{dx}=\frac{b+2cx}{2(q+px)^{n-1}\sqrt{a+bx+cx^2}}-\frac{(n-1)p\sqrt{a+bx+cx^2}}{(q+px)^n}$$

$$=\frac{(b+2cx)(q+px)-2(n-1)p(a+bx+cx^2)}{2(q+px)^n\sqrt{a+bx+cx^2}}$$

$$=\frac{1}{2}\frac{\lambda+\mu(q+px)+\nu(q+px)^2}{(q+px)^n\sqrt{a+bx+cx^2}}, \text{ say,}$$

where

$$\lambda+\mu q+\nu q^2=qb-2(n-1)pa,$$
$$\mu p+2\nu pq=2qc+pb-2(n-1)pb,$$
$$\nu p^2=2pc-2(n-1)pc,$$

from which we obtain

$$\lambda=-2(n-1)(ap^2-bpq+cq^2)/p,$$
$$\mu=-(2n-3)(bp-2cq)/p,$$
$$\nu=-2(n-2)c/p.$$

And $2P=\lambda I_n+\mu I_{n-1}+\nu I_{n-2}$ is the formula sought.

That is

$$2(n-1)\frac{ap^2-bpq+cq^2}{p}I_n$$

$$=-\frac{2\sqrt{a+bx+cx^2}}{(q+px)^{n-1}}-(2n-3)\frac{bp-2cq}{p}I_{n-1}-2(n-2)\frac{c}{p}I_{n-2}.$$

The case where $n=1$ is given in Art. 287, whence I_2 can be found from the present formula, in which the coefficient of I_{n-2} vanishes when $n=2$. Then I_3, I_4, ... can be successively derived.

245. The integral

$$J_n = \int \frac{Mx+N}{(px+q)^n} \frac{dx}{\sqrt{ax^2+bx+c}}$$

may be written as

$$J_n = \int \frac{\frac{M}{p}(px+q)+\left(N-\frac{Mq}{p}\right)}{(px+q)^n \sqrt{ax^2+bx+c}} dx$$

$$= \frac{M}{p} I_{n-1} + \frac{Np-Mq}{p} I_n,$$

where I_n is the integral discussed in Art. 244.

This therefore constitutes a reduction formula for J_n.

But both this integral and the more general integral

$$J_n' = \int \frac{Mx+N}{(Ax^2+Bx+C)^n} \frac{dx}{\sqrt{ax^2+bx+c}}$$

are more conveniently evaluated by differentiation with regard to one of the constants involved, q in the one case, C in the other, as explained subsequently (see Art. 364).

246. The integrable cases.

Denote $I_{m,p} \equiv \int x^m X^p\, dx$ for shortness by (m, p).

The special cases

$$(0, -1), \quad (0, -\tfrac{1}{2}), \quad (0, \tfrac{1}{2}), \quad (0, 1)$$

are all simple elementary integrals whose values have been discussed.

Formula (A), which connects $(0, p)$ and $(0, p-1)$, will therefore continue the series both ways and yield

$$(0, \pm\tfrac{3}{2}), \quad (0, \pm 2), \quad (0, \pm\tfrac{5}{2}), \quad (0, \pm 3), \quad (0, \pm\tfrac{7}{2}), \quad \text{etc.},$$

$$\text{i.e. } (0, \pm k) \quad \text{or} \quad \left(0, \pm\frac{2k+1}{2}\right),$$

where k is any integer.

Formula (B) connects $(1, p)$ with $(0, p)$, and therefore contributes the integrals

$$(1, \pm k), \quad \left(1, \pm\frac{2k+1}{2}\right),$$

where k is any integer.

Formula (C) connects $(-1, p)$ with $(-1, p-1)$; and $(-1, -\frac{1}{2})$ and $(-1, \pm 1)$ are simple cases already discussed;

$$\left.\begin{array}{ll} \therefore & (-1, -\tfrac{3}{2}), \quad (-1, -\tfrac{5}{2}), \quad (-1, -\tfrac{7}{2}), \text{ etc.,} \\ \text{and} & (-1, +\tfrac{1}{2}), \quad (-1, +\tfrac{3}{2}), \quad (-1, +\tfrac{5}{2}), \text{ etc.,} \end{array}\right\} \text{are contributed;}$$

as also $(-1, \pm 2)$, $(-1, \pm 3)$, $(-1, \pm 4)$, etc.

i.e. $(-1, \pm k)$, $\left(-1, \pm\frac{2k+1}{2}\right)$, are contributed where k is any integer.

Formula (D) connects $(m-2, p+1)$, $(m-2, p)$, $(m-1, p)$, (m, p)

$\therefore$ $(0, p+1)$, $(0, p)$, $(1, p)$, $(2, p)$ are connected,

$(1, p+1)$, $(1, p)$, $(2, p)$, $(3, p)$ are connected,

etc.;

$$\therefore \left.\begin{array}{ll} (2, \pm k), & \left(2, \pm\dfrac{2k+1}{2}\right), \\ (3, \pm k), & \left(3, \pm\dfrac{2k+1}{2}\right), \\ (4, \pm k), & \left(4, \pm\dfrac{2k+1}{2}\right), \end{array}\right\} \text{are contributed,}$$

etc.;

Formula (E) connects $(m-2, p)$, $(m-1, p)$, (m, p);

therefore also $$\left.\begin{array}{ll} (-2, \pm k), & \left(-2, \pm\dfrac{2k+1}{2}\right), \\ (-3, \pm k), & \left(-3, \pm\dfrac{2k+1}{2}\right), \end{array}\right\} \text{are contributed,}$$

etc.

Hence *all integrals of form*

$$\int x^m X^p\,dx, \quad \text{where } X \equiv a+bx+cx^2,$$

can be integrated in finite terms when p is of form $\pm k$ or $\pm\dfrac{2k+1}{2}$, and m, k are integers positive or negative.

EXAMPLES.

247. 1. Taking

$$\int\frac{dx}{X^{p+1}} = \frac{b+2cx}{pkX^p} + \frac{2(2p-1)c}{pk}\int\frac{dx}{X^p},$$

where $X \equiv a+bx+cx^2$ and $k \equiv 4ac-b^2$,

prove
$$\int\frac{dx}{X^2} = \frac{b+2cx}{kX} + \frac{2c}{k}\int\frac{dx}{X},$$

$$\int\frac{dx}{X^3} = \frac{(b+2cx)}{k}\left(\frac{1}{2X^2}+\frac{3c}{kX}\right) + \frac{6c^2}{k^2}\int\frac{dx}{X},$$

$$\int\frac{dx}{X^4} = \frac{b+2cx}{k}\left(\frac{1}{3X^3}+\frac{5c}{3kX^2}+\frac{10c^2}{k^2X}\right) + \frac{20c^3}{k^3}\int\frac{dx}{X}.$$

[BERTRAND.]

2. Show that if $I_n \equiv \int\frac{x^n}{X}dx$, then $cI_n + bI_{n-1} + aI_{n-2} = \frac{x^{n-1}}{n-1}$, and prove

$$\int\frac{x\,dx}{X} = \frac{1}{2c}\log X - \frac{b}{2c}\int\frac{dx}{X}.$$

Deduce $$\int \frac{x^2\,dx}{X} = \frac{x}{c} - \frac{b}{2c^2}\log X + \frac{b^2-2ac}{2c^2}\int\frac{dx}{X},$$

$$\int \frac{x^3\,dx}{X} = \frac{x^2}{2c} - \frac{bx}{c^2} + \frac{b^2-ac}{2c^3}\log X + \frac{3ac-b^2}{2c^3}\,b\int\frac{dx}{X}.$$

[BERTRAND.]

3 Prove

$$\int\frac{dx}{xX} = \frac{1}{2a}\log\frac{x^2}{X} - \frac{b}{2a}\int\frac{dx}{X},$$

and deduce

$$\int\frac{dx}{x^3X} = \frac{b}{a^2x} - \frac{1}{2ax^2} + \frac{b^2-ac}{2a^3}\log\frac{x^2}{X} + \frac{b(3ac-b^2)}{2a^3}\int\frac{dx}{X}.$$

[BERTRAND.]

(The value of $\int\frac{dx}{X}$ occurring in each of these results is given in Art. 80.)

4. If $X \equiv a+bx+cx^2$, prove that

$$\int\frac{x^{m+1}}{X^{p+1}}\,dx = -\frac{x^m}{2cpX^p} + \frac{m}{2cp}\int\frac{x^{m-1}\,dx}{X^p} - \frac{b}{2c}\int\frac{x^m}{X^{p+1}}\,dx.$$

[BERTRAND.]

5. Prove that if $X \equiv x^2+ax+a^2$,

$$\int X^{\frac{n}{2}}\,dx = \frac{2x+a}{2(n+1)}X^{\frac{n}{2}} + \frac{3na^2}{4(n+1)}\int X^{\frac{n}{2}-1}\,dx.$$

[ST. JOHN'S, 1889.]

6. Prove that if $X \equiv x^2+x+1$,

$$(a)\ \int\frac{dx}{x^4X} = \frac{1}{2}\log\frac{x^2}{X} - \frac{1}{3x^3} + \frac{1}{2x^2} - \frac{1}{\sqrt{3}}\tan^{-1}\frac{2x+1}{\sqrt{3}}.$$

$$(b)\ \int\frac{dx}{X^{p+1}} = \frac{1+2x}{3pX^p} + \frac{2(2p-1)}{3p}\int\frac{dx}{X^p}.$$

7. Show that if p be a positive integer and $X \equiv x^2+x+1$,

$$(a)\ \int\frac{dx}{(x^2+x+1)^{p+1}}$$
$$= \frac{(1+2x)}{3}\Big[\frac{1}{pX^p} + \frac{(2p-1)}{p(p-1)}\left(\frac{2}{3}\right)\frac{1}{X^{p-1}} + \frac{(2p-1)(2p-3)}{p(p-1)(p-2)}\left(\frac{2}{3}\right)^2\frac{1}{X^{p-2}}$$
$$+\ldots+\frac{(2p-1)(2p-3)\ldots3.1}{p(p-1)\ldots2.1}\left(\frac{2}{3}\right)^{p-1}\frac{1}{X}\Big]$$
$$+\frac{(2p-1)(2p-3)\ldots3.1}{p(p-1)(p-2)\ldots2.1}\,\frac{2}{\sqrt{3}}\left(\frac{2}{3}\right)^p\tan^{-1}\frac{2x+1}{\sqrt{3}}.$$

(b) $\int\frac{dx}{(x^2+x+1)^{\frac{2n+1}{2}}}$, n being a positive integer,

$$= \frac{1+2x}{3}\Big[\frac{1}{2n-1}\,\frac{2}{X^{\frac{2n-1}{2}}} + \frac{(n-1)}{(2n-1)(2n-3)}\left(\frac{2}{3}\right)\frac{2^3}{X^{\frac{2n-3}{2}}}$$
$$+\frac{(n-1)(n-2)}{(2n-1)(2n-3)(2n-5)}\left(\frac{2}{3}\right)^2\frac{2^5}{X^{\frac{2n-5}{2}}}$$
$$+\ldots+\frac{(n-1)!}{(2n-1)(2n-3)\ldots1}\left(\frac{2}{3}\right)^{n-1}\frac{2^{2n-1}}{X^{\frac{1}{2}}}\Big].$$

248. Reduction of $I_n \equiv \int \frac{x^n}{\sqrt{a+bx^2+cx^4}}\,dx$.

Let $X \equiv a+bx^2+cx^4$, and put $P = x^{n-3}\sqrt{X}$.

Then
$$\frac{dP}{dx} = (n-3)x^{n-4}\sqrt{X} + x^{n-3}\frac{(bx+2cx^3)}{\sqrt{X}}$$
$$= \frac{(n-3)x^{n-4}(a+bx^2+cx^4)+bx^{n-2}+2cx^n}{\sqrt{X}}$$
$$= \frac{(n-1)cx^n+(n-2)bx^{n-2}+(n-3)ax^{n-4}}{\sqrt{X}};$$
$$\therefore\ x^{n-3}\sqrt{X} = (n-1)cI_n + (n-2)bI_{n-2} + (n-3)aI_{n-4}.$$

249. **Integrations of**

(i) $\int \cos px \cos^n qx\, dx$, (ii) $\int \cos px \sin^n qx\, dx$,

(iii) $\int \sin px \cos^n qx\, dx$, (iv) $\int \sin px \sin^n qx\, dx$,

including $\int \frac{\cos px}{\cos^n qx}\,dx$, **etc.**

There are two classes of reduction formulae for such integrals.

We may connect
$$\int \cos px \cos^n qx\, dx \text{ with } \int \cos px \cos^{n-2} qx\, dx,$$
or we may connect
$$\int \cos px \cos^n qx\, dx \text{ with } \int \cos(p-q)x \cos^{n-1} qx\, dx,$$
and the like with the other three cases.

250. First, we consider the **former class of reduction.**

(i) Let $I_n \equiv \int \cos px \cos^n qx\, dx$.

Then
$$I_n = \frac{\sin px}{p}\cos^n qx + \frac{nq}{p}\int \sin px \cos^{n-1} qx \sin qx\, dx$$
$$= \frac{\sin px}{p}\cos^n qx + \frac{nq}{p}\Big[-\frac{\cos px}{p}\cos^{n-1} qx \sin qx$$
$$+\int \frac{\cos px}{p}\{-(n-1)q\cos^{n-2}qx(1-\cos^2 qx)+q\cos^n qx\}\,dx\Big]$$

$$=\frac{\sin px}{p}\cos^n qx-\frac{nq}{p^2}\cos px\cos^{n-1}qx\sin qx$$

$$+\frac{nq^2}{p^2}\int\cos px\{-(n-1)\cos^{n-2}qx+n\cos^n qx\}\,dx;$$

$$\therefore\ \left(1-\frac{n^2q^2}{p^2}\right)I_n=\frac{\cos^{n-1}qx}{p^2}(p\sin px\cos qx-nq\cos px\sin qx)$$
$$-\frac{n(n-1)q^2}{p^2}I_{n-2};$$

$$\therefore\ I_n=\cos^{n-1}qx\,\frac{p\sin px\cos qx-nq\cos px\sin qx}{p^2-n^2q^2}$$
$$-\frac{n(n-1)q^2}{p^2-n^2q^2}I_{n-2}.$$

Now $\quad \dfrac{d}{dx}\dfrac{\cos^n qx}{\cos px}=\cos^{n-1}qx\dfrac{p\sin px\cos qx-nq\cos px\sin qx}{\cos^2 px}.$

Hence the reduction formula may be written more compactly as

$$I_n=\frac{\cos^2 px}{p^2-n^2q^2}\frac{d}{dx}\frac{\cos^n qx}{\cos px}-\frac{n(n-1)q^2}{p^2-n^2q^2}I_{n-2}.$$

By successive reduction, the power-factor $\cos^n qx$ may be reduced either to $\cos qx$ or to unity, when n is a positive integer, and the integration can then be completed.

If n be negative $(=-m)$, we can, by solving for I_{n-2}, express the same formula as

$$I_{-m-2}=\frac{\cos^2 px}{m(m+1)q^2}\frac{d}{dx}\frac{\sec^m qx}{\cos px}-\frac{p^2-m^2q^2}{m(m+1)q^2}I_{-m},$$

and therefore a reduction formula for $\displaystyle\int\frac{\cos px}{\cos^m qx}dx$ is also furnished.

Similar work and remarks apply to the other three cases, (ii), (iii) and (iv), but it is desirable to consider them in detail.

251. (ii) Let $I_n\equiv\int\cos px\sin^n qx\,dx$.

Then

$$I_n=\frac{\sin px}{p}\sin^n qx-\frac{nq}{p}\int\sin px\sin^{n-1}qx\cos qx\,dx$$

$$=\frac{\sin px}{p}\sin^n qx-\frac{nq}{p}\Big[-\frac{\cos px}{p}\sin^{n-1}qx\cos qx$$

$$+\int\frac{\cos px}{p}\{(n-1)q\sin^{n-2}qx(1-\sin^2 qx)-q\sin^n qx\}\,dx\Big]$$

$$=\frac{\sin px}{p}\sin^n qx+\frac{nq}{p^2}\cos px\sin^{n-1}qx\cos qx$$

$$-\frac{nq^2}{p^2}\int\cos px\{(n-1)\sin^{n-2}qx-n\sin^n qx\}\,dx;$$

$$\therefore\ \left(1-\frac{n^2q^2}{p^2}\right)I_n=\frac{\sin^{n-1}qx}{p^2}(p\sin px\sin qx+nq\cos px\cos qx)$$

$$-n(n-1)\frac{q^2}{p^2}I_{n-2};$$

$$\therefore\ I_n=\sin^{n-1}qx\frac{(p\sin px\sin qx+nq\cos px\cos qx)}{p^2-n^2q^2}$$

$$-\frac{n(n-1)q^2}{p^2-n^2q^2}I_{n-2},$$

i.e.
$$I_n=\frac{\cos^2 px}{p^2-n^2q^2}\frac{d}{dx}\frac{\sin^n qx}{\cos px}-\frac{n(n-1)q^2}{p^2-n^2q^2}I_{n-2}.$$

252. (iii) Let $I_n\equiv\int\sin px\cos^n qx\,dx.$

Then

$$I_n=-\frac{\cos px}{p}\cos^n qx-\frac{nq}{p}\int\cos px\cos^{n-1}qx\sin qx\,dx$$

$$=-\frac{\cos px}{p}\cos^n qx-\frac{nq}{p}\Big[\frac{\sin px}{p}\cos^{n-1}qx\sin qx$$

$$-\int\frac{\sin px}{p}\{-(n-1)q\cos^{n-2}qx(1-\cos^2 qx)+q\cos^n qx\}\,dx\Big];$$

$$\therefore\ \left(1-\frac{n^2q^2}{p^2}\right)I_n=-\frac{\cos^{n-1}qx}{p^2}(p\cos px\cos qx+nq\sin px\sin qx)$$

$$-n(n-1)\frac{q^2}{p^2}I_{n-2};$$

$$\therefore\ I_n=-\cos^{n-1}qx\frac{(p\cos px\cos qx+nq\sin px\sin qx)}{p^2-n^2q^2}$$

$$-\frac{n(n-1)q^2}{p^2-n^2q^2}I_{n-2},$$

i.e.
$$I_n=\frac{\sin^2 px}{p^2-n^2q^2}\frac{d}{dx}\frac{\cos^n qx}{\sin px}-\frac{n(n-1)q^2}{p^2-n^2q^2}I_{n-2}.$$

253. (iv) Let $I_n \equiv \int \sin px \sin^n qx \, dx$.

Then

$$I_n = -\frac{\cos px}{p}\sin^n qx + \frac{nq}{p}\int \cos px \sin^{n-1} qx \cos qx \, dx$$

$$= -\frac{\cos px}{p}\sin^n qx + \frac{nq}{p}\left[\frac{\sin px}{p}\sin^{n-1} qx \cos qx\right.$$

$$\left. -\int \frac{\sin px}{p}\{(n-1)q \sin^{n-2} qx(1-\sin^2 qx) - q \sin^n qx\}\, dx\right];$$

$$\therefore \left(1-\frac{n^2q^2}{p^2}\right) I_n = -\frac{\sin^{n-1} qx}{p^2}(p\cos px \sin qx - nq \sin px \cos qx)$$

$$-n(n-1)\frac{q^2}{p^2} I_{n-2};$$

$$\therefore I_n = -\sin^{n-1} qx \frac{p\cos px \sin qx - nq \sin px \cos qx}{p^2-n^2q^2}$$

$$-\frac{n(n-1)q^2}{p^2-n^2q^2} I_{n-2};$$

i.e.
$$I_n = \frac{\sin^2 px}{p^2-n^2q^2}\frac{d}{dx}\frac{\sin^n qx}{\sin px} - \frac{n(n-1)q^2}{p^2-n^2q^2} I_{n-2}.$$

254. The four results are therefore

$$\int \cos px \cos^n qx \, dx = \frac{\cos^2 px}{p^2-n^2q^2}\frac{d}{dx}\frac{\cos^n qx}{\cos px}$$

$$-\frac{n(n-1)q^2}{p^2-n^2q^2}\int \cos px \cos^{n-2} qx \, dx.$$

$$\int \cos px \sin^n qx \, dx = \frac{\cos^2 px}{p^2-n^2q^2}\frac{d}{dx}\frac{\sin^n qx}{\cos px}$$

$$-\frac{n(n-1)q^2}{p^2-n^2q^2}\int \cos px \sin^{n-2} qx \, dx.$$

$$\int \sin px \cos^n qx \, dx = \frac{\sin^2 px}{p^2-n^2q^2}\frac{d}{dx}\frac{\cos^n qx}{\sin px}$$

$$-\frac{n(n-1)q^2}{p^2-n^2q^2}\int \sin px \cos^{n-2} qx \, dx.$$

$$\int \sin px \sin^n qx \, dx = \frac{\sin^2 px}{p^2-n^2q^2}\frac{d}{dx}\frac{\sin^n qx}{\sin px}$$

$$-\frac{n(n-1)q^2}{p^2-n^2q^2}\int \sin px \sin^{n-2} qx \, dx.$$

That is, if A stands for the first factor and P the second, or power-factor, *i.e.* $I_n = \int AP\,dx$, we have, *in all cases*,

$$(p^2-n^2q^2)\,I_n = A^2\frac{d}{dx}\left(\frac{P}{A}\right)-n(n-1)\,q^2I_{n-2}$$

or
$$(p^2-n^2q^2)\,I_n = A\frac{dP}{dx}-P\frac{dA}{dx}-n(n-1)\,q^2I_{n-2}.$$

Writing $-m$ for n, for the cases where n is negative, we may write this as

$$m(m+1)q^2I_{-m-2} = A\frac{dP}{dx}-P\frac{dA}{dx}-(p^2-m^2q^2)\,I_{-m}.$$

255. Such formulae are more particularly useful for negative indices of the power factor. For if the integral sought be, say,

$$\int \cos 4x \sin^5 3x\,dx,$$

the "multiple angle" process for $\sin^5 3x$ will be more convenient than a reduction.

Thus, $$\sin^5 3x = \frac{1}{2^4}(\sin 15x - 5\sin 9x + 10\sin 3x);$$

$$\therefore\ \cos 4x\sin^5 3x = \frac{1}{2^5}[(\sin 19x+\sin 11x)-5(\sin 13x+\sin 5x) + 10(\sin 7x - \sin x)],$$

and the integral is

$$-\frac{1}{2^5}\left[\frac{\cos 19x}{19}+\frac{\cos 11x}{11}-\frac{5\cos 13x}{13}-\frac{5\cos 5x}{5}+\frac{10\cos 7x}{7}-\frac{10\cos x}{1}\right].$$

But to integrate $\int\frac{\cos 4x}{\sin^5 3x}dx$, this process is useless. Therefore we change n to $-n$ in the second of the formulae of Art. 254.

Then

$$\int\frac{\cos px}{\sin^{n+2}qx}\,dx = \frac{\cos^2 px}{n(n+1)\,q^2}\,\frac{d}{dx}\,\frac{\sec px}{\sin^n qx}-\frac{p^2-n^2q^2}{n(n+1)q^2}\int\frac{\cos px}{\sin^n qx}\,dx;$$

whence

$$\int\frac{\cos 4x}{\sin^5 3x}\,dx = \frac{\cos^2 4x}{3\,.\,4\,.\,3^2}\,\frac{d}{dx}\,\frac{\sec 4x}{\sin^3 3x}+\frac{13\,.\,5}{3\,.\,4\,.\,3^2}\int\frac{\cos 4x}{\sin^3 3x}\,dx$$

and $$\int\frac{\cos 4x}{\sin^3 3x}dx = \frac{\cos^2 4x}{1\,.\,2\,.\,3^2}\,\frac{d}{dx}\,\frac{\sec 4x}{\sin 3x}-\frac{7\,.\,1}{1\,.\,2\,.\,3^2}\int\frac{\cos 4x}{\sin 3x}\,dx;$$

whilst

$$\int\frac{\cos 4x}{\sin 3x}\,dx = \frac{1}{2}\int\left(\frac{1}{\sin x}-\frac{1}{\sin 3x}-4\sin x\right)dx$$
$$=\frac{1}{2}\log\tan\frac{x}{2}-\frac{1}{6}\log\tan\frac{3x}{2}+2\cos x;$$

hence

$$\int \frac{\cos 4x}{\sin^5 3x}\,dx = \frac{\cos^2 4x}{3.4.3^2}\frac{d}{dx}\frac{\sec 4x}{\sin^3 3x} + \frac{13.5}{3.4.3^3}\left\{\frac{\cos^2 4x}{1.2.3^2}\frac{d}{dx}\frac{\sec 4x}{\sin 3x} - \frac{7.1}{1.2.3^2}\left(\frac{1}{2}\log\tan\frac{x}{2} - \frac{1}{6}\log\tan\frac{3x}{2} + 2\cos x\right)\right\}$$

$$= \text{etc.}$$

256. For the **second mode of reduction**, mentioned above in Art. 249, we may connect $I_{p,n}$, that is $\int \cos px \cos^n qx\,dx$ or one of the other cases with

$$I_{p-q,n-1} \quad \text{or with} \quad I_{p-2q,n-2},$$

To shorten the expressions we shall use the notation

$$c_p \text{ for } \cos px, \quad s_p \text{ for } \sin px, \quad \text{etc.}$$

The mode of procedure is the same in all cases, viz.:

Put P = the power factor × the complementary function of the other factor. Differentiate and rearrange.

(i) $$I_{p,n} = \int c_p c_q^n\,dx.$$

Let $$P_1 = s_p c_q^n.$$

Then $$\frac{dP_1}{dx} = pc_p c_q^n - nq s_p c_q^{n-1} s_q$$

$$= c_q^{n-1}[(p+nq)\,c_p c_q - nq c_{p-q}];$$

$$\therefore\ P_1 = (p+nq)\int c_p c_q^n\,dx - nq\int c_{p-q} c_q^{n-1}\,dx.$$

(ii) $$I_{p,n} = \int c_p s_q^n\,dx.$$

Let $$P_2 = s_p s_q^n.$$

Then $$\frac{dP_2}{dx} = pc_p s_q^n + nq s_p s_q^{n-1} c_q$$

$$= s_q^{n-1}[(p+nq)\,c_p s_q + nq s_{p-q}];$$

$$\therefore\ P_2 = (p+nq)\int c_p s_q^n dx + nq\int s_{p-q} s_q^{n-1}\,dx.$$

(iii) $$I_{p,n} = \int s_p c_q^n\,dx.$$

Let $$P_3 = c_p c_q^n.$$

Then $$\frac{dP_3}{dx} = -ps_p c_q^n - nq c_p c_q^{n-1} s_q$$

$$= -c_q^{n-1}[(p+nq)\,s_p c_q - nq s_{p-q}];$$

$$\therefore\ P_3 = -(p+nq)\int s_p c_q^n\,dx + nq\int s_{p-q} c_q^{n-1}\,dx$$

(iv) $$I_{p,n}=\int s_p s_q{}^n\,dx.$$

Let $$P_4=c_p s_q{}^n.$$

Then $$\frac{dP_4}{dx}=-ps_ps_q{}^n+nqc_ps_q{}^{n-1}c_q$$

$$=-s_q{}^{n-1}[(p+nq)s_ps_q-nqc_{p-q}];$$

$$\therefore\ P_4=-(p+nq)\int s_ps_q{}^n\,dx+nq\int c_{p-q}s_q{}^{n-1}\,dx.$$

We thus obtain the four results:

$$(1)\ (p+nq)\int c_pc_q{}^n\,dx=\ \ s_pc_q{}^n+nq\int c_{p-q}c_q{}^{n-1}\,dx.$$

$$(2)\ (p+nq)\int c_ps_q{}^n\,dx=\ \ s_ps_q{}^n-nq\int s_{p-q}s_q{}^{n-1}\,dx.$$

$$(3)\ (p+nq)\int s_pc_q{}^n\,dx=-c_pc_q{}^n+nq\int s_{p-q}c_q{}^{n-1}\,dx.$$

$$(4)\ (p+nq)\int s_ps_q{}^n\,dx=-c_ps_q{}^n+nq\int c_{p-q}s_q{}^{n-1}\,dx.$$

Thus an integral of the first kind connects directly with a lower order integral of the first kind;
an integral of the second kind connects directly with a lower order integral of the fourth kind;
an integral of the third kind connects directly with a lower order integral of the third kind;
an integral of the fourth kind connects directly with a lower order integral of the second kind.

Thus to connect an integral of the second or fourth kind with one of its own kind, a second operation is necessary.

For example,

$$(p+nq)\int c_ps_q{}^n\,dx=s_ps_q{}^n-nq\int s_{p-q}s_q{}^{n-1}dx$$

$$=s_ps_q{}^n-\frac{nq}{p-q+(n-1)q}\left[-c_{p-q}s_q{}^{n-1}+(n-1)q\int c_{p-2q}s_q{}^{n-2}\,dx\right],$$

which connects $\int c_ps_q{}^n\,dx$ with $\int c_{p-2q}s_q{}^{n-2}\,dx$,

and similarly for $\int s_ps_q{}^n\,dx$ with $\int s_{p-2q}s_q{}^{n-2}\,dx$.

257. Avoidance of a Reduction Formula.

For integrals of the classes under discussion, viz.

$$\int \cos px \cos^n qx\, dx, \text{ etc.,}$$

it is often convenient to avoid a reduction formula altogether so long as n is a positive integer, when we shall require to put the power-factor ($\cos^n qx$ in this case) into cosines or sines of multiples of qx, as seen in the example in Art. 255.

Proceeding as in Art. 112, the formulae required are:

$$2^n \cos^n\theta = \left(y+\frac{1}{y}\right)^n = \text{etc.}$$

$$= 2[\cos n\theta + {}^nC_1 \cos(n-2)\theta + {}^nC_2\cos(n-4)\theta + \dots + K],$$

where $K = \dfrac{1}{2}\dfrac{n!}{\left(\dfrac{n}{2}!\right)^2}$ if n be even, ...(A)

or $= \dfrac{n!}{\dfrac{n-1}{2}!\,\dfrac{n+1}{2}!}\cos\theta$ if n be odd, ...(B)

$$2^{n-1}(-1)^{\frac{n}{2}}\sin^n\theta = \cos n\theta - {}^nC_1\cos(n-2)\theta$$

$$+ \dots + (-1)^{\frac{n}{2}}\frac{n!}{2\left(\dfrac{n}{2}!\right)^2} \text{ if } n \text{ be even; (C)}$$

$$2^{n-1}(-1)^{\frac{n-1}{2}}\sin^n\theta = \sin n\theta - {}^nC_1\sin(n-2)\theta$$

$$+ \dots + (-1)^{\frac{n-1}{2}}\frac{n!}{\dfrac{n-1}{2}!\,\dfrac{n+1}{2}!}\sin\theta$$

if n be odd. (D)

Then taking $\theta = qx$,

$2^n\cos^n qx\cos px =$ a series of form $2\Sigma K_r \cos rx \cos px$, say,

$$= \sum K_r(\cos\overline{r+p}x + \cos\overline{r-p}x),$$

and $$\int\cos px\cos^n qx\,dx = \frac{1}{2^n}\left(\sum K_r\frac{\sin\overline{r+p}x}{r+p} + \sum K_r\frac{\sin\overline{r-p}x}{r-p}\right),$$

taking due account of the final terms.

Similarly we may proceed in the other cases.

The formulae (A), (B), (C), (D) can be readily reproduced as explained previously in Art. 112 for any particular value of n for which they may be required.

Ex. $\int \sin 2x \sin^6 5x\, dx.$

$$2^6 \iota^6 \sin^6 \theta = \left(y - \frac{1}{y}\right)^6 = \quad y^6 + \frac{1}{y^6} - 6\left(y^4 + \frac{1}{y^4}\right) + 15\left(y^2 + \frac{1}{y^2}\right) - 20$$
$$= \quad 2\cos 6\theta - 12\cos 4\theta + 30\cos 2\theta - 20\,;$$

$\therefore$ taking $\theta = 5x$,

$$\sin^6 5x \sin 2x = -\frac{1}{2^6}[2\sin 2x \cos 30x - 12 \sin 2x \cos 20x$$
$$+ 30 \sin 2x \cos 10x - 20 \sin 2x]$$
$$= -\frac{1}{2^6}[\sin 32x - \sin 28x - 6(\sin 22x - \sin 18x)$$
$$+ 15(\sin 12x - \sin 8x) - 20 \sin 2x],$$

and
$$\int \sin 2x \sin^6 5x\, dx = \quad \frac{1}{2^6}\left[\frac{\cos 32x}{32} - \frac{\cos 28x}{28} - 6\left(\frac{\cos 22x}{22} - \frac{\cos 18x}{18}\right)\right.$$
$$\left. + 15\left(\frac{\cos 12x}{12} - \frac{\cos 8x}{8}\right) - 20\,\frac{\cos 2x}{2}\right].$$

258. **The Integrals**

$$(1)\ \int \frac{\cos nx}{\cos^p x}\, dx. \qquad (2)\ \int \frac{\sin nx}{\cos^p x}\, dx.$$
$$(3)\ \int \frac{\cos nx}{\sin^p x}\, dx. \qquad (4)\ \int \frac{\sin nx}{\sin^p x}\, dx.$$

In case (1),

$$I_{n,p} \equiv \int \frac{\cos nx}{\cos^p x}\, dx = \int \frac{2\cos x \cos(n-1)x - \cos(n-2)x}{\cos^p x}\, dx$$
$$= 2I_{n-1,\,p-1} - I_{n-2,\,p}.$$

In case (2),

$$I_{n,p} \equiv \int \frac{\sin nx}{\cos^p x}\, dx = \int \frac{2\cos x \sin(n-1)x - \sin(n-2)x}{\cos^p x}\, dx$$
$$= 2I_{n-1,\,p-1} - I_{n-2,\,p}.$$

For cases (3) and (4), let

$$I_{n,p} \equiv \int \frac{\cos nx}{\sin^p x}\, dx, \quad J_{n,p} \equiv \int \frac{\sin nx}{\sin^p x}\, dx.$$

In case (3),

$$I_{n,p} \equiv \int \frac{\cos nx}{\sin^p x}\, dx = \int \frac{-2\sin x \sin(n-1)x + \cos(n-2)x}{\sin^p x}\, dx$$
$$= -2J_{n-1,\,p-1} + I_{n-2,\,p}.$$

In case (4),

$$J_{n,p} \equiv \int \frac{\sin nx}{\sin^p x}\, dx = \int \frac{2\sin x \cos(n-1)x + \sin(n-2)x}{\sin^p x}\, dx$$
$$= 2I_{n-1,\,p-1} + J_{n-2,\,p}.$$

The cases (1) and (2), therefore, reduce to lower order integrals of the same form.

The cases (3) and (4) reduce to lower order integrals, but in each case the forms are partly interchanged.

It may be worth noting that in the form $\int \frac{\cos nx}{\cos^p x}\,dx$ we might as an alternative method express $\cos nx$ as a series of powers of $\cos x$ and integrate each term by methods already discussed.

If n be odd $\int \frac{\sin nx}{\sin^p x}\,dx$ may be treated similarly by expressing $\sin nx$ as a series of powers of $\sin x$ and integrating each term.

If n be even $\sin nx$ contains a factor $\cos x$, and the integral is immediately obtainable; *e.g.*

$$\int \frac{\sin 4x}{\sin^5 x}\,dx = \int \frac{4\sin x(1-2\sin^2 x)}{\sin^5 x}\cos x\,dx$$

$$= \int (4s^{-4} - 8s^{-2})\,ds = -\frac{4}{3}\frac{1}{\sin^3 x} + \frac{8}{\sin x}.$$

Similar remarks apply in the other cases.

259. Ex. 1. $\int \frac{\cos 5x}{\cos^3 x}\,dx = 2\int \frac{\cos 4x}{\cos^2 x}\,dx - \int \frac{\cos 3x}{\cos^3 x}\,dx$

$$= 2\left[2\int \frac{\cos 3x}{\cos x}\,dx - \int \frac{\cos 2x}{\cos^2 x}\,dx\right] - \int \frac{\cos 3x}{\cos^3 x}\,dx$$

$$= 4\int (4\cos^2 x - 3)\,dx - 2\int (2 - \sec^2 x)\,dx$$

$$- \int (4 - 3\sec^2 x)\,dx$$

$$= 8\left(x + \frac{\sin 2x}{2}\right) - 12x - 4x + 2\tan x - 4x + 3\tan x$$

$$= 4\sin 2x - 12x + 5\tan x;$$

or otherwise, and more readily, *without a reduction*,

$$\int \frac{\cos 5x}{\cos^3 x}\,dx = \int \frac{16\cos^5 x - 20\cos^3 x + 5\cos x}{\cos^3 x}\,dx$$

$$= \int \{8(1+\cos 2x) - 20 + 5\sec^2 x\}\,dx$$

$$= 4\sin 2x - 12x + 5\tan x, \text{ as before.}$$

Ex. 2. $\displaystyle\int \frac{\cos 5x}{\sin^3 x}\,dx = -2\int \frac{\sin 4x}{\sin^2 x}\,dx + \int \frac{\cos 3x}{\sin^3 x}\,dx$

$$= -2\left[2\int \frac{\cos 3x}{\sin x}\,dx + \int \frac{\sin 2x}{\sin^2 x}\,dx\right]$$

$$+\left[-2\int \frac{\sin 2x}{\sin^2 x}\,dx + \int \frac{\cos x}{\sin^3 x}\,dx\right]$$

$$= -4\int (\cot x - 4\sin x\cos x)\,dx - 8\int \cot x\,dx$$

$$+\int \frac{\cos x}{\sin^3 x}\,dx$$

$$= \quad 16\frac{\sin^2 x}{2} - 12\log\sin x - \frac{1}{2\sin^2 x}$$

$$= \quad 8\sin^2 x - 12\log\sin x - \tfrac{1}{2}\operatorname{cosec}^2 x\,;$$

or otherwise, and more readily, *without a reduction*,

$$\int \frac{\cos 5x}{\sin^3 x}\,dx = \int \frac{1 - 12\sin^2 x + 16\sin^4 x}{\sin^3 x}\,d\sin x$$

$$= -\tfrac{1}{2}\operatorname{cosec}^2 x - 12\log\sin x + 8\sin^2 x, \text{ as before.}$$

260. Integrals $I_n \equiv \displaystyle\int \frac{\sin^n px}{\cos px}\,dx, \quad J_n \equiv \int \frac{\cos^n px}{\sin px}\,dx.$

$$I_n \equiv \int \frac{\sin^n px}{\cos px}\,dx = \int \frac{\sin^{n-2} px(1-\cos^2 px)}{\cos px}\,dx$$

$$= -\int \cos px \sin^{n-2} px\,dx + I_{n-2}\,;$$

$$\therefore\ I_n = -\frac{\sin^{n-1} px}{(n-1)p} + I_{n-2}.$$

$$J_n = \int \frac{\cos^n px}{\sin px}\,dx = \int \frac{\cos^{n-2} px(1-\sin^2 px)}{\sin px}\,dx$$

$$= -\int \sin px \cos^{n-2} px\,dx + J_{n-2}\,;$$

$$\therefore\ J_n = \frac{\cos^{n-1} px}{(n-1)p} + J_{n-2}.$$

Also since

$$I_1 = \int \tan px\,dx = \frac{1}{p}\log\sec px,$$

$$I_2 = \int (\sec px - \cos px)\,dx = -\frac{\sin px}{p} + \frac{1}{p}\log\tan\left(\frac{px}{2} + \frac{\pi}{4}\right),$$

$$J_1 = \int \cot px\,dx = \frac{1}{p}\log\sin px,$$

$$J_2 = \int (\operatorname{cosec} px - \sin px)\,dx = \frac{\cos px}{p} + \frac{1}{p}\log\tan\frac{px}{2},$$

we have

$$p\int\frac{\sin^{2n}px}{\cos px}dx = -\frac{\sin^{2n-1}px}{2n-1}-\frac{\sin^{2n-3}px}{2n-3}-\dots$$
$$-\frac{\sin^3 px}{3}-\frac{\sin px}{1}+\log\tan\left(\frac{px}{2}+\frac{\pi}{4}\right),$$

$$p\int\frac{\sin^{2n+1}px}{\cos px}dx=-\frac{\sin^{2n}px}{2n}-\frac{\sin^{2n-2}px}{2n-2}-\dots$$
$$-\frac{\sin^4 px}{4}-\frac{\sin^2 px}{2}+\log\sec px,$$

$$p\int\frac{\cos^{2n}px}{\sin px}dx = \frac{\cos^{2n-1}px}{2n-1}+\frac{\cos^{2n-3}px}{2n-3}+\dots$$
$$+\frac{\cos^3 px}{3}+\frac{\cos px}{1}+\log\tan\frac{px}{2},$$

$$p\int\frac{\cos^{2n+1}px}{\sin px}dx=\frac{\cos^{2n}px}{2n}+\frac{\cos^{2n-2}px}{2n-2}+\dots$$
$$+\frac{\cos^4 px}{4}+\frac{\cos^2 px}{2}+\log\sin px.$$

261. Integration of

$$\int\frac{\cos px}{\cos qx}dx,\quad \int\frac{\cos px}{\sin qx}dx,\quad \int\frac{\sin px}{\cos qx}dx,\quad \int\frac{\sin px}{\sin qx}dx.$$

(i) We may regard p, q as integral and prime to each other

For if p, q be fractional, $=\frac{r_1}{s_1}$ and $\frac{r_2}{s_2}$ respectively, let

$$\frac{r_1}{s_1} \text{ and } \frac{r_2}{s_2}$$

be reduced to the forms $\frac{R_1}{S}$, $\frac{R_2}{S}$,

where S is the L.C.M. of s_1 and s_2 and R_1, R_2 are integers.

Let $x=Sy$. Then

$$\int\frac{\begin{matrix}\cos\\ \sin\end{matrix}(px)}{\begin{matrix}\cos\\ \sin\end{matrix}(qx)}dx=\int\frac{\begin{matrix}\cos\\ \sin\end{matrix}\left(\frac{R_1}{S}\right)x}{\begin{matrix}\cos\\ \sin\end{matrix}\left(\frac{R_2}{S}\right)x}dx=S\int\frac{\begin{matrix}\cos\\ \sin\end{matrix}(R_1y)}{\begin{matrix}\cos\\ \sin\end{matrix}(R_2y)}dy.$$

Hence we only need to consider the case where p and q are integers.

The signs of p and q are also immaterial to the discussion.

Again, if p and q were not prime to each other, let G be the G.C.M., and let $p=Gp'$, $q=Gq'$, and let $x=\frac{y}{G}$. Then

$$\int \frac{\begin{matrix}\cos\\ \sin\end{matrix}(Gp'x)}{\begin{matrix}\cos\\ \sin\end{matrix}(Gq'x)}\,dx=\frac{1}{G}\int \frac{\begin{matrix}\cos\\ \sin\end{matrix}(p'y)}{\begin{matrix}\cos\\ \sin\end{matrix}(q'y)}\,dy,$$

where p' and q' are prime to each other.

Therefore we shall need only to consider the case where p, q are *positive integers, prime to each other*

(ii) Supposing $p>q$.

Since
$$\begin{aligned}
\cos px+\cos(p-2q)x &= 2\cos(p-q)x\cos qx,\\
\cos px-\cos(p-2q)x &= -2\sin(p-q)x\sin qx,\\
\sin px+\sin(p-2q)x &= 2\sin(p-q)x\cos qx,\\
\sin px-\sin(p-2q)x &= 2\cos(p-q)x\sin qx,
\end{aligned}$$

we have

$$\begin{aligned}
\int\frac{\cos px}{\cos qx}dx &= 2\,\frac{\sin(p-q)x}{p-q}-\int\frac{\cos(p-2q)x}{\cos qx}dx\\
\int\frac{\cos px}{\sin qx}dx &= 2\,\frac{\cos(p-q)x}{p-q}+\int\frac{\cos(p-2q)x}{\sin qx}dx,\\
\int\frac{\sin px}{\cos qx}dx &= -2\,\frac{\cos(p-q)x}{p-q}-\int\frac{\sin(p-2q)x}{\sin qx}dx,\\
\int\frac{\sin px}{\sin qx}dx &= 2\,\frac{\sin(p-q)x}{p-q}+\int\frac{\sin(p-2q)x}{\sin qx}dx.
\end{aligned}$$

Hence, by a sufficient number of reductions of this kind, we can reduce the integration of

$$\int \frac{\begin{matrix}\cos\\ \sin\end{matrix}(px)}{\begin{matrix}\cos\\ \sin\end{matrix}(qx)}\,dx$$

to that of another integral of the same form, say

$$\int \frac{\begin{matrix}\cos\\ \sin\end{matrix}Px}{\begin{matrix}\cos\\ \sin\end{matrix}qx}\,dx,$$

where P lies between q and $-q$.

Hence we shall introduce no limitation upon our method in the discussion of such integrals in assuming $p < q$.

(iii) We take, then, p and q positive, integral, prime to each other, and $p < q$. The case p and q, both even, need not be considered, being a reducible case as shown.

Now

$$\left.\begin{array}{l} \text{if } n \text{ be even,} \\ \qquad \cos nx = \prod_1^{\frac{n}{2}} \dfrac{\sin^2 a_r - \sin^2 x}{\sin^2 a_r}, \\ \text{if } n \text{ be odd,} \\ \qquad \cos nx = \cos x \prod_1^{\frac{n-1}{2}} \dfrac{\sin^2 a_r - \sin^2 x}{\sin^2 a_r}. \end{array}\right\} \text{where } a_r = (2r-1)\frac{\pi}{2n}.$$

$$\left.\begin{array}{l} \text{if } n \text{ be even,} \\ \sin nx = n \sin x \cos x \prod_1^{\frac{n-2}{2}} \dfrac{\sin^2 a_r - \sin^2 x}{\sin^2 a_r}, \\ \text{if } n \text{ be odd,} \\ \qquad \sin nx = n \sin x \prod_1^{\frac{n-1}{2}} \dfrac{\sin^2 a_r - \sin^2 x}{\sin^2 a_r}, \end{array}\right\} \text{where } a_r = \frac{r\pi}{n}.$$

And where necessary a factor $\sin^2 a_r - \sin^2 x$ can be written as $\cos^2 x - \cos^2 a_r$. (See Hobson, *Trigonometry*, p. 114.)

Factorizing both numerator and denominator of

$$\frac{\begin{matrix}\cos \\ \sin\end{matrix}(px)}{\begin{matrix}\cos \\ \sin\end{matrix}(qx)},$$

the number of factors in the numerator is less than that in the denominator, and in all cases the integrand can be thrown into partial fractions by the ordinary rules (factors not repeated) and expressed in one of the forms,

$$\sum \frac{A}{\sin^2 a - \sin^2 x}, \quad \sum \frac{A \cos x}{\sin^2 a - \sin^2 x}, \quad \sum \frac{A \sin x}{\cos^2 x - \cos^2 a},$$

and the particular fractions

$$\frac{A \sin x}{\cos^2 x - 1}, \quad \frac{A \cos x}{1 - \sin^2 x}, \quad \frac{A}{\cos^2 x} \text{ or } \frac{A}{\sin^2 x}$$

may occur.

(iv) Finally,

$$\int\frac{\sin\alpha\cos\alpha\,dx}{\sin^2\alpha-\sin^2x}=\tanh^{-1}\left(\frac{\tan x}{\tan\alpha}\right),$$

$$\int\frac{\sin\alpha\cos x\,dx}{\sin^2\alpha-\sin^2x}=\tanh^{-1}\left(\frac{\sin x}{\sin\alpha}\right),$$

$$\int\frac{\sin x\cos\alpha\,dx}{\cos^2x-\cos^2\alpha}=\coth^{-1}\left(\frac{\cos x}{\cos\alpha}\right),$$

and

$$\int\frac{\sin x}{\cos^2x-1}\,dx=-\log\tan\frac{x}{2},\quad \int\frac{\cos x}{1-\sin^2x}\,dx=\log\tan\left(\frac{x}{2}+\frac{\pi}{4}\right).$$

Hence in all such cases the integration can be performed.

It is not *essential* that the numerator $\frac{\cos}{\sin}(px)$ should be factorized. It might be expanded in powers of $\cos x$ or $\sin x$, as the case may be. But the *factorization is convenient, presents no difficulty, and as a rule is simpler in application*, as it indicates in factorized form the values of the constants occurring in the partial fractions.

262. Ex. Find the integral $I=\int\frac{\cos\frac{2}{3}x}{\cos\frac{1}{2}x}\,dx.$

Let $x=6y$. Then

$$I=6\int\frac{\cos 4y}{\cos 3y}\,dy \quad\text{and}\quad \int\frac{\cos 4y}{\cos 3y}\,dy=2\sin y-\int\frac{\cos 2y}{\cos 3y}\,dy,$$

by the first reduction formula, (Art. 261, ii).

Also $$\int\frac{\cos 2y}{\cos 3y}\,dy=\int\frac{\dfrac{\sin^2\frac{\pi}{4}-\sin^2y}{\sin^2\frac{\pi}{4}}}{\cos y\,\dfrac{\sin^2\frac{\pi}{6}-\sin^2y}{\sin^2\frac{\pi}{6}}}\,dy$$

$$=\frac{\sin^2\frac{\pi}{6}}{\sin^2\frac{\pi}{4}}\int\frac{\sin^2\frac{\pi}{4}-\sin^2y}{\left(\sin^2\frac{\pi}{2}-\sin^2y\right)\left(\sin^2\frac{\pi}{6}-\sin^2y\right)}\cos y\,dy$$

$$=\frac{\sin^2\frac{\pi}{6}}{\sin^2\frac{\pi}{4}}\int\left[\frac{\sin^2\frac{\pi}{4}-\sin^2\frac{\pi}{2}}{\sin^2\frac{\pi}{6}-\sin^2\frac{\pi}{2}}\,\frac{\cos y}{1-\sin^2y}\right.$$

$$\left.+\frac{\sin^2\frac{\pi}{4}-\sin^2\frac{\pi}{6}}{\sin^2\frac{\pi}{2}-\sin^2\frac{\pi}{6}}\,\frac{\cos y}{\sin^2\frac{\pi}{6}-\sin^2y}\right]dy$$

$$=\frac{\sin^2\frac{\pi}{6}}{\sin^2\frac{\pi}{4}}\left\{\frac{\sin^2\frac{\pi}{4}-\sin^2\frac{\pi}{2}}{\sin^2\frac{\pi}{6}-\sin^2\frac{\pi}{2}}\log\tan\left(\frac{y}{2}+\frac{\pi}{4}\right)\right.$$

$$\left.+\frac{\sin^2\frac{\pi}{4}-\sin^2\frac{\pi}{6}}{\sin^2\frac{\pi}{2}-\sin^2\frac{\pi}{6}}\operatorname{cosec}\frac{\pi}{6}\tanh^{-1}\frac{\sin y}{\sin\frac{\pi}{6}}\right\}$$

So far, obvious arithmetical simplification is postponed, so that the general process may be exhibited and made clear.

Simplifying the arithmetic, we shall finally get

$$\int\frac{\cos\frac{2}{3}x}{\cos\frac{1}{2}x}dx=12\sin\frac{x}{6}-2\log\tan\left(\frac{x}{12}+\frac{\pi}{4}\right)-2\tanh^{-1}\left(2\sin\frac{x}{6}\right).$$

263. **Integrals of form**

$$\int\frac{\cos^n px}{\cos x}dx,\quad \int\frac{\cos^n px}{\sin x}dx,\quad \int\frac{\sin^n px}{\cos x}dx,\quad \int\frac{\sin^n px}{\sin x}dx,$$

where p and n are integers, n being positive.

These are generally integrated as follows:

First put the power factor in the numerator into the form of a series of cosines or sines of multiples of px, say

$$\Sigma A_r \frac{\cos}{\sin}(rpx).$$

We are then to integrate each term, viz. expressions of type

$$\int\frac{\frac{\cos}{\sin}(rpx)}{\frac{\cos}{\sin}(x)}dx$$

by a reduction formula, a case of Art. 261 (ii), viz.:

$$\int\frac{\cos kx}{\cos x}dx=\quad 2\frac{\sin(k-1)x}{k-1}-\int\frac{\cos(k-2)x}{\cos x}dx,$$

$$\int\frac{\cos kx}{\sin x}dx=\quad 2\frac{\cos(k-1)x}{k-1}+\int\frac{\cos(k-2)x}{\sin x}dx,$$

$$\int\frac{\sin kx}{\cos x}dx=-2\frac{\cos(k-1)x}{k-1}-\int\frac{\sin(k-2)x}{\cos x}dx,$$

$$\int\frac{\sin kx}{\sin x}dx=\quad 2\frac{\sin(k-1)x}{k-1}+\int\frac{\sin(k-2)x}{\sin x}dx,$$

which obviously follow from the trigonometrical formulae

$$\cos kx+\cos(k-2)x=2\cos x\cos(k-1)x,$$

etc.

Ex. Consider $\int \frac{\cos^5 3x}{\cos x}\,dx$.

We have, taking $y=e^{3\iota x}$,

$$2^5\cos^5 3x=\left(y+\frac{1}{y}\right)^5=\text{etc.}=2\cos 15x+10\cos 9x+20\cos 3x$$

$$\therefore\ \frac{\cos^5 3x}{\cos x}=\frac{1}{2^4}\left(\frac{\cos 15x}{\cos x}+\frac{5\cos 9x}{\cos x}+\frac{10\cos 3x}{\cos x}\right).$$

But

$$\int\frac{\cos 15x}{\cos x}\,dx=\frac{2\sin 14x}{14}-\frac{2\sin 12x}{12}+\frac{2\sin 10x}{10}-\frac{2\sin 8x}{8}+\frac{2\sin 6x}{6}-\frac{2\sin 4x}{4}+\frac{2\sin 2x}{2}-x,$$

$$\int\frac{\cos 9x}{\cos x}\,dx=\frac{2\sin 8x}{8}-\frac{2\sin 6x}{6}+\frac{2\sin 4x}{4}-\frac{2\sin 2x}{2}+x,$$

$$\int\frac{\cos 3x}{\cos x}\,dx=\frac{2\sin 2x}{2}-x;$$

$$\therefore\ \int\frac{\cos^5 3x}{\cos x}\,dx=\frac{1}{2^4}\left[\frac{2\sin 14x}{14}-\frac{2\sin 12x}{12}+\frac{2\sin 10x}{10}+\frac{8\sin 8x}{8}-\frac{8\sin 6x}{6}+\frac{8\sin 4x}{4}+\frac{12\sin 2x}{2}-6x\right].$$

264. Integrals of form

$$\int\frac{\cos^n px}{\cos qx}\,dx,\quad \int\frac{\cos^n px}{\sin qx}\,dx,\quad \int\frac{\sin^n px}{\cos qx}\,dx,\quad \int\frac{\sin^n px}{\sin qx}\,dx.$$

These are dealt with in a similar manner to those of the previous article.

First expressing the power factor as

$$\Sigma A_r \,{\textstyle\cos\atop\textstyle\sin}\,(rpx),$$

we reduce the integration in each case to that of a series of terms of type

$$\int\frac{{\cos\atop\sin}(p'x)}{{\cos\atop\sin}(qx)}\,dx,$$

and proceed as explained in Art. 261.

Ex. Integrate $I=\int\frac{\cos^5 5x}{\cos 4x}\,dx$.

We have, taking $y=e^{5\iota x}$,

$$2^5\cos^5 5x=\left(y+\frac{1}{y}\right)^5=\text{etc.}=2\cos 25x+10\cos 15x+20\cos 5x;$$

$$\therefore\ I=\frac{1}{2^4}\left[\int\frac{\cos 25x}{\cos 4x}\,dx+5\int\frac{\cos 15x}{\cos 4x}\,dx+10\int\frac{\cos 5x}{\cos 4x}\,dx\right].$$

The reduction formula

$$\int \frac{\cos px}{\cos qx}\,dx = \frac{2\sin(p-q)x}{p-q} - \int \frac{\cos(p-2q)x}{\cos qx}\,dx$$

gives

$$\int \frac{\cos 25x}{\cos 4x}\,dx = \frac{2\sin 21x}{21} - \int \frac{\cos 17x}{\cos 4x}\,dx$$

$$= \frac{2\sin 21x}{21} - \frac{2\sin 13x}{13} + \int \frac{\cos 9x}{\cos 4x}\,dx$$

$$= \frac{2\sin 21x}{21} - \frac{2\sin 13x}{13} + \frac{2\sin 5x}{5} - \int \frac{\cos x}{\cos 4x}\,dx,$$

$$\int \frac{\cos 15x}{\cos 4x}\,dx = \frac{2\sin 11x}{11} - \frac{2\sin 3x}{3} + \int \frac{\cos(-x)}{\cos 4x}\,dx,$$

and $\displaystyle\int \frac{\cos 5x}{\cos 4x}dx = \frac{2\sin x}{1} - \int \frac{\cos(-3x)}{\cos 4x}\,dx.$

Hence

$$I = \frac{1}{2^4}\left[\frac{2\sin 21x}{21} - \frac{2\sin 13x}{13} + \frac{10\sin 11x}{11} + \frac{2\sin 5x}{5} - \frac{10\sin 3x}{3} + \frac{20\sin x}{1} - K\right],$$

where

$$K = \int \frac{\cos x - 5\cos x + 10\cos 3x}{\cos 4x}\,dx = \int \frac{40\cos^3 x - 34\cos x}{\cos 4x}\,dx$$

$$= 2\int \cos x \frac{3 - 20\sin^2 x}{\left(\sin^2\frac{\pi}{8} - \sin^2 x\right)\left(\sin^2\frac{3\pi}{8} - \sin^2 x\right)} \cdot \sin^2\frac{\pi}{8}\sin^2\frac{3\pi}{8}\,dx$$

$$= \frac{2\sin^2\frac{\pi}{8}\sin^2\frac{3\pi}{8}}{\sin^2\frac{3\pi}{8} - \sin^2\frac{\pi}{8}} \int \cos x \left(\frac{3 - 20\sin^2\frac{\pi}{8}}{\sin^2\frac{\pi}{8} - \sin^2 x} - \frac{3 - 20\sin^2\frac{3\pi}{8}}{\sin^2\frac{3\pi}{8} - \sin^2 x}\right) dx$$

$$= \frac{1}{2\sqrt{2}}\left[\left(3 - 20\sin^2\frac{\pi}{8}\right)\operatorname{cosec}\frac{\pi}{8}\tanh^{-1}\frac{\sin x}{\sin\frac{\pi}{8}} - \left(3 - 20\sin^2\frac{3\pi}{8}\right)\operatorname{cosec}\frac{3\pi}{8}\tanh^{-1}\frac{\sin x}{\sin\frac{3\pi}{8}}\right];$$

$$\therefore \int \frac{\cos^5 5x}{\cos 4x}\,dx$$

$$= \frac{1}{8}\left[\frac{1}{21}\sin 21x - \frac{1}{13}\sin 13x + \frac{5}{11}\sin 11x + \frac{1}{5}\sin 5x - \frac{5}{3}\sin 3x + 10\sin x - \frac{1}{4\sqrt{2}}\left\{\left(3 - 20\sin^2\frac{\pi}{8}\right)\operatorname{cosec}\frac{\pi}{8}\tanh^{-1}\frac{\sin x}{\sin\frac{\pi}{8}} - \left(3 - 20\sin^2\frac{3\pi}{8}\right)\operatorname{cosec}\frac{3\pi}{8}\tanh^{-1}\frac{\sin x}{\sin\frac{3\pi}{8}}\right\}\right].$$

265. **Integrals of form**

$$\int \frac{\sin^p x}{x^q}\,dx, \quad \int \frac{\cos^p x}{x^q}\,dx, \quad \int \frac{x^q}{\sin^p x}\,dx, \quad \int \frac{x^q}{\cos^p x}\,dx.$$

$$I_{p,q} \equiv \int \frac{\sin^p x}{x^q}\,dx = -\frac{\sin^p x}{(q-1)x^{q-1}} + \frac{p}{q-1}\int \frac{\sin^{p-1}x\cos x}{x^{q-1}}\,dx$$

$$= -\frac{\sin^p x}{(q-1)x^{q-1}} + \frac{p}{q-1}\left[\left\{-\frac{\sin^{p-1}x\cos x}{(q-2)x^{q-2}}\right\}\right.$$
$$\left. + \frac{1}{q-2}\int \frac{(p-1)\sin^{p-2}x(1-\sin^2 x)-\sin^p x}{x^{q-2}}\,dx\right]$$

$$= -\frac{\sin^{p-1}x}{(q-1)(q-2)x^{q-1}}[(q-2)\sin x + px\cos x]$$
$$+\frac{p}{(q-1)(q-2)}[(p-1)I_{p-2,q-2} - pI_{p,q-2}].$$

Therefore, provided $q \neq 1$ or 2,

$$(q-1)(q-2)I_{p,q} - p(p-1)I_{p-2,q-2} + p^2 I_{p,q-2}$$
$$= -\frac{\sin^{p-1}x}{x^{q-1}}[(q-2)\sin x + px\cos x]. \quad \text{(A)}$$

This formula will be found useful in evaluating certain definite integrals of the form $\int_0^\infty \frac{\sin^p x}{x^q}\,dx$, in the case where $p \nless q$ and where p and q are either both odd or both even integers >2; for in this case the right-hand side vanishes at both limits. We thus have

$$(q-1)(q-2)\int_0^\infty \frac{\sin^p x}{x^q}\,dx + p^2\int_0^\infty \frac{\sin^p x}{x^{q-2}}\,dx - p(p-1)\int_0^\infty \frac{\sin^{p-2}x}{x^{q-2}}\,dx = 0,$$

where $p \nless q > 2$ (see Chap. XXVI.)

266. In the same way, in the second case, supposing $q \neq 1$ or 2,

$$I_{p,q} \equiv \int \frac{\cos^p x}{x^q}\,dx = -\frac{\cos^p x}{(q-1)x^{q-1}} - \frac{p}{q-1}\int \frac{\cos^{p-1}x\sin x}{x^{q-1}}\,dx$$

$$= -\frac{\cos^p x}{(q-1)x^{q-1}} - \frac{p}{q-1}\left[\left\{-\frac{\cos^{p-1}x\sin x}{(q-2)x^{q-2}}\right\}\right.$$
$$\left. + \frac{1}{q-2}\int \frac{\cos^p x-(p-1)\cos^{p-2}x(1-\cos^2 x)}{x^{q-2}}\,dx\right]$$

$$= -\frac{\cos^{p-1}x}{(q-1)(q-2)x^{q-1}}[(q-2)\cos x - px\sin x]$$
$$-\frac{p}{(q-1)(q-2)}[pI_{p,q-2} - (p-1)I_{p-2,q-2}];$$

$$\therefore\ (q-1)(q-2)I_{p,q} - p(p-1)I_{p-2,q-2} + p^2 I_{p,q-2}$$
$$= -\frac{\cos^{p-1}x}{x^{q-1}}[(q-2)\cos x - px\sin x]. \quad \text{(B)}$$

267. Again, in the third case,

$$I_{p,q} \equiv \int x^q \operatorname{cosec}^p x\, dx = \frac{x^{q+1}}{q+1}\operatorname{cosec}^p x + \frac{p}{q+1}\int x^{q+1}\operatorname{cosec}^p x \cot x\, dx$$

$$= \frac{x^{q+1}}{q+1}\operatorname{cosec}^p x + \frac{p}{q+1}\left[\frac{x^{q+2}}{q+2}\operatorname{cosec}^p x \cdot \cot x\right.$$

$$\left. + \frac{1}{q+2}\int x^{q+2}(p \operatorname{cosec}^p x \cot^2 x + \operatorname{cosec}^{p+2} x)\, dx\right]$$

$$= \frac{x^{q+1}\operatorname{cosec}^{p+1} x}{(q+1)(q+2)}[(q+2)\sin x + px\cos x]$$

$$+ \frac{p}{(q+1)(q+2)}\int x^{q+2}[(p+1)\operatorname{cosec}^{p+2} x - p\operatorname{cosec}^p x]dx;$$

$$\therefore\ (q+1)(q+2)I_{p,q} - p(p+1)I_{p+2,q+2} + p^2 I_{p,q+2}$$
$$= x^{q+1}\operatorname{cosec}^{p+1}[(q+2)\sin x + px\cos x]. \quad \text{(C)}$$

268. And finally, in the fourth case,

$$I_{p,q} \equiv \int x^q \sec^p x\, dx = \frac{x^{q+1}}{q+1}\sec^p x - \frac{p}{q+1}\int x^{q+1}\sec^p x \tan x\, dx$$

$$= \frac{x^{q+1}}{q+1}\sec^p x - \frac{p}{q+1}\left[\frac{x^{q+2}}{q+2}\sec^p x \tan x\right.$$

$$\left. - \frac{1}{q+2}\int x^{q+2}(p \sec^p x \tan^2 x + \sec^{p+2} x)\, dx\right]$$

$$= \frac{x^{q+1}\sec^{p+1} x}{(q+1)(q+2)}[(q+2)\cos x - px\sin x]$$

$$+ \frac{p}{(q+1)(q+2)}\int x^{q+2}[(p+1)\sec^{p+2} x - p\sec^p x]\, dx;$$

$$\therefore\ (q+1)(q+2)I_{p,q} - p(p+1)I_{p+2,q+2} + p^2 I_{p,q+2}$$
$$= x^{q+1}\sec^{p+1} x\,[(q+2)\cos x - px\sin x]. \quad \text{(D)}$$

It will be seen that formulae (C) and (D) could have been derived from (A) and (B) by changing the signs of p and q.

269. **Integrals of form** $I_n \equiv \int \frac{x}{\cos^n x}dx$ **or** $\int x \sec^n x\, dx$, **included** as the case $q=1$ in Art. 265, may be treated thus:

$$I_n = \int \frac{x}{\cos^n x}dx = \int \cos x \cdot \frac{x}{\cos^{n+1} x}dx$$

$$= \sin x \cdot \frac{x}{\cos^{n+1} x} - \int \sin x \frac{\cos x + (n+1)x\sin x}{\cos^{n+2} x}dx$$

$$= x\frac{\sin x}{\cos^{n+1} x} - \int \frac{\sin x}{\cos^{n+1} x}dx - (n+1)\int x\frac{1-\cos^2 x}{\cos^{n+2} x}dx$$

$$= x\frac{\sin x}{\cos^{n+1} x} - \frac{1}{n}\frac{1}{\cos^n x} - (n+1)(I_{n+2} - I_n).$$

Therefore

$$(n+1)\,I_{n+2}=\frac{nx\sin x-\cos x}{n\cos^{n+1}x}+nI_n$$

or $$I_{n+2}=\frac{nx\sin x-\cos x}{n(n+1)\cos^{n+1}x}+\frac{n}{n+1}I_n;$$

or, changing n to $n-2$,

$$I_n=\frac{(n-2)x\sin x-\cos x}{(n-1)(n-2)\cos^{n-1}x}+\frac{n-2}{n-1}I_{n-2}. \quad\text{............(1)}$$

Now,

$$I_2=\int x\sec^2 x\,dx=x\tan x+\log\cos x$$

and

$$I_1=\int x\sec x\,dx=x\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)-\int\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)dx.$$

Thus, I_4, I_6, ... can be readily written down.

But I_3, I_5, I_7, ... ultimately connect with $\int\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)dx$, which is not expressible in finite terms as an indefinite integral.

270. **Similarly, if** $I_n=\int\frac{x}{\sin^n x}dx$ **or** $\int x\operatorname{cosec}^n x\,dx$,

we have

$$I_n=\int\sin x\,.\,\frac{x}{\sin^{n+1}x}dx$$

$$=(-\cos x)\frac{x}{\sin^{n+1}x}+\int\cos x\frac{\sin x-(n+1)x\cos x}{\sin^{n+2}x}dx$$

$$=(-\cos x)\frac{x}{\sin^{n+1}x}+\int\frac{\cos x}{\sin^{n+1}x}dx-(n+1)\int x\frac{1-\sin^2 x}{\sin^{n+2}x}dx$$

$$=-x\frac{\cos x}{\sin^{n+1}x}-\frac{1}{n}\frac{1}{\sin^n x}-(n+1)(I_{n+2}-I_n);$$

$$\therefore\ (n+1)I_{n+2}=-\frac{nx\cos x+\sin x}{n\sin^{n+1}x}+nI_n;$$

or, changing n to $n-2$,

$$I_n=-\frac{(n-2)x\cos x+\sin x}{(n-1)(n-2)\sin^{n-1}x}+\frac{n-2}{n-1}I_{n-2}. \quad\text{..........(2)}$$

Noting that

$$I_2=\int x\operatorname{cosec}^2 x\,dx=-x\cot x+\log\sin x$$

and $$I_1=\int x\operatorname{cosec}x\,dx=x\log\tan\frac{x}{2}-\int\log\tan\frac{x}{2}\,dx,$$

it is clear that $I_4, I_6, \ldots$ can be successively written down, but that $I_3, I_5, \ldots$, which connect with $\int \log \tan \frac{x}{2}\, dx$, cannot be expressed in finite terms as an indefinite integral.

It is also obvious that these formulae (1) and (2) could be reproduced by taking

$$P = \frac{(n-2)x \sin x - \cos x}{\cos^{n-1} x} \quad \text{and} \quad P = \frac{(n-2)x \cos x + \sin x}{\sin^{n-1} x}$$

respectively, differentiating, and rearranging the terms.

271. **Reduction formula for**

$$I_n = \int \frac{x^{2n}}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}\, dx,$$

***n* being integral.**

Let $R = (1-x^2)(1-k^2x^2)$, and put $P = x^{2n-3}\sqrt{R}$.

Then $\dfrac{dP}{dx} = (2n-3)x^{2n-4}\sqrt{R} + x^{2n-3}\dfrac{-(1+k^2)x + 2k^2x^3}{\sqrt{R}}$

$$= [(2n-3)\{1-(1+k^2)x^2+k^2x^4\} - (1+k^2)x^2 + 2k^2x^4]\frac{x^{2n-4}}{\sqrt{R}}$$

$$= [(2n-3) - 2(n-1)(1+k^2)x^2 + (2n-1)k^2x^4]\frac{x^{2n-4}}{\sqrt{R}}$$

$$= (2n-3)\frac{x^{2n-4}}{\sqrt{R}} - 2(n-1)(1+k^2)\frac{x^{2n-2}}{\sqrt{R}} + (2n-1)k^2\frac{x^{2n}}{\sqrt{R}}.$$

Hence $P = (2n-3)I_{n-2} - 2(n-1)(1+k^2)I_{n-1} + (2n-1)k^2 I_n$,

i.e. $$I_n = \frac{x^{2n-3}\sqrt{R}}{(2n-1)k^2} + 2\frac{n-1}{2n-1}\frac{(1+k^2)}{k^2}I_{n-1} - \frac{2n-3}{2n-1}\cdot\frac{1}{k^2}I_{n-2}.$$

[Serret, p. 44, Tom. ii., *Calc. Diff. et Intégral.*]

By successive reduction I_n may be made to depend upon I_0 and I_1 by putting in succession $n = 2, 3, 4, \ldots$; and I_0, I_1, which are respectively

$$\int \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}} \quad \text{and} \quad \int \frac{x^2\,dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}},$$

are the integrals known as Legendrian Elliptic Integrals, and discussed later.

When $n = 1$, $\qquad k^2 I_1 = x^{-1}\sqrt{R} + I_{-1}$,

i.e. $$I_{-1} = k^2 I_1 - \frac{\sqrt{R}}{x}.$$

When $n=0$, $k^2I_0=-x^{-3}\sqrt{R}+2(1+k^2)I_{-1}-3I_{-2}$,

and putting successively $n=-1$, -2, etc., we can calculate I_{-2}, I_{-3}, etc., in terms of I_0 and I_1.

272. Obviously, if we put $x=\sin\theta$,

$$I_n=\int\frac{\sin^{2n}\theta\,d\theta}{\sqrt{1-k^2\sin^2\theta}},$$

and the same reduction formula applies.

Thus $I_n=\int\frac{\sin^{2n}\theta\,d\theta}{\sqrt{1-k^2\sin^2\theta}}$ and $I_{-n}=\int\frac{\operatorname{cosec}^{2n}\theta\,d\theta}{\sqrt{1-k^2\sin^2\theta}}$

can both be connected linearly with

$$\int\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \quad\text{and}\quad \int\frac{\sin^2\theta\,d\theta}{\sqrt{1-k^2\sin^2\theta}};$$

and the latter being

$$\frac{1}{k^2}\int\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}-\frac{1}{k^2}\int\sqrt{1-k^2\sin^2\theta}\,d\theta,$$

we have connected each of I_n and I_{-n} with

$$\int\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \quad\text{and}\quad \int\sqrt{1-k^2\sin^2\theta}\,d\theta$$

273. Instead of starting with $P=x^{2n-3}\sqrt{R}$, we might have proceeded to form the connection required by means of integration by parts, which presents no difficulty.

Thus $$R=(1-x^2)(1-k^2x^2),$$

$$\frac{dR}{dx}=-2(1+k^2)x+4k^2x^3.$$

Multiply by $\frac{x^{2n-3}}{2\sqrt{R}}$ and integrate

$$\int\frac{x^{2n-3}}{2\sqrt{R}}\frac{dR}{dx}dx=-(1+k^2)I_{n-1}+2k^2I_n.$$

But the left side $=x^{2n-3}\sqrt{R}-(2n-3)\int x^{2n-4}\sqrt{R}\,dx$

$$=x^{2n-3}\sqrt{R}-(2n-3)\int\frac{x^{2n-4}(1-x^2)(1-k^2x^2)}{\sqrt{R}}dx;$$

$$\therefore -(1+k^2)I_{n-1}+2k^2I_n$$
$$=x^{2n-3}\sqrt{R}-(2n-3)[I_{n-2}-(1+k^2)I_{n-1}+k^2I_n],$$

i.e. $x^{2n-3}\sqrt{R}=(2n-1)k^2I_n-2(n-1)(1+k^2)I_{n-1}+(2n-3)I_{n-2}$,

the result already obtained.

274. Reduction formula for

$$I_n=\int\frac{dx}{(1+ax^2)^n\sqrt{(1-x^2)(1-k^2x^2)}},$$

i.e. $$\int\frac{d\theta}{(1+a\sin^2\theta)^n\sqrt{1-k^2\sin^2\theta}},$$

where $x=\sin\theta$.

Let $R\equiv(1-x^2)(1-k^2x^2)$, as before; then

$$\frac{1}{2}\frac{dR}{dx}=-(1+k^2)x+2k^2x^3.$$

Put $P=\dfrac{x\sqrt{R}}{(1+ax^2)^{n-1}}$.

Then

$$\frac{dP}{dx}=\frac{\sqrt{R}+\dfrac{x}{2\sqrt{R}}\dfrac{dR}{dx}}{(1+ax^2)^{n-1}}-\frac{2(n-1)ax^2\sqrt{R}}{(1+ax^2)^n}$$

$$=\frac{(1+ax^2)[(1-x^2)(1-k^2x^2)-(1+k^2)x^2+2k^2x^4]-2(n-1)ax^2(1-x^2)(1-k^2x^2)}{(1+ax^2)^n\sqrt{R}}$$

$$=\frac{(1+ax^2)[1-2(1+k^2)x^2+3k^2x^4]-2(n-1)ax^2(1-x^2)(1-k^2x^2)}{(1+ax^2)^n\sqrt{R}}$$

$$=\frac{A+B(1+ax^2)+C(1+ax^2)^2+D(1+ax^2)^3}{(1+ax^2)^n\sqrt{R}},\text{ say,}$$

where $$A+B+C+D=1,$$
$$aB+2aC+3aD=a-2(1+k^2)-2(n-1)a,$$
$$a^2C+3a^2D=3k^2-2a(1+k^2)+2(n-1)a(1+k^2),$$
$$a^3D=3k^2a-2(n-1)ak^2;$$

whence we obtain

$$a^2A=\quad(2n-2)(a+1)(a+k^2),$$
$$a^2B=-(2n-3)[a(a+2)+(2a+3)k^2],$$
$$a^2C=\quad(2n-4)[a+(a+3)k^2],$$
$$a^2D=-(2n-5)k^2.$$

Then $$P=AI_n+BI_{n-1}+CI_{n-2}+DI_{n-3},$$

and I_n is connected with three integrals of the same form, but lower order. Also, the formula is true whether n is positive or negative.

Now $$I_0=\int\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}},$$

and is Legendre's first elliptic integral (Chaps. XI. and XXXI.).

$$I_1=\int\frac{dx}{(1+ax^2)\sqrt{(1-x^2)(1-k^2x^2)}},$$

and is Legendre's third elliptic integral; and

$$\begin{aligned}I_{-1}&=\int\frac{1+ax^2}{\sqrt{(1-x^2)(1-k^2x^2)}}dx\\&=\left(1+\frac{a}{k^2}\right)\int\frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}-\frac{a}{k^2}\int\frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}dx,\end{aligned}$$

and these are, respectively, Legendre's first and second elliptic integrals.

These integrals I_0, I_1, I_{-1} are therefore all known. Their properties will be discussed in the proper place. We thus have a means of connecting I_n with them for any integral value of n, positive or negative.

The same reduction formula obviously must hold for

$$\int\frac{d\theta}{(1+a\sin^2\theta)^n\sqrt{1-k^2\sin^2\theta}},$$

which is only another form of the same integral.

EXAMPLES.

1. If $X\equiv ax^n+b$, obtain reduction formulae for the integral $u_{p,q}\equiv\int\frac{x^p}{X^q}dx$ of the forms,

$$\text{(i)}\quad Au_{p,q}+Bu_{p-n,q}+R=0,$$
$$\text{(ii)}\quad A'u_{p,q}+B'u_{p,q-1}+R'=0,$$

where A, B, A', B' are constants and R, R' are algebraic functions of x. [MATH. TRIP., 1896]

2. Prove that

(a) $$\int \cos^{2n}\phi\, d\phi = \frac{1}{2n}\tan\phi\cos^{2n}\phi + \left(1-\frac{1}{2n}\right)\int \cos^{2n-2}\phi\, d\phi,$$ [TRINITY, 1891.]

(b) $$\int \sec^{2n+1}\phi\, d\phi = \frac{1}{2n}\tan\phi\sec^{2n-1}\phi + \left(1-\frac{1}{2n}\right)\int \sec^{2n-1}\phi\, d\phi,$$ [I. C. S., 1886.]

3. Prove that

$$\int (a^2+x^2)^{\frac{2n+1}{2}}dx = \frac{x}{2n+2}(a^2+x^2)^{\frac{2n+1}{2}} + \frac{2n+1}{2n+2}a^2\int (a^2+x^2)^{\frac{2n-1}{2}}dx.$$ [I. C. S., 1886.]

4. Investigate a formula of reduction applicable to

$$\int x^m(1+x^2)^{\frac{n}{2}}dx,$$

where m and n are positive integers, and complete the integration if $m=5$, $n=7$. [ST. JOHN'S COLL., 1881.]

5. If $\phi(n) \equiv a^{2n-1}\int_0^\infty \frac{dx}{(a^2+x^2)^n}$, prove that $\phi(n) = \frac{2n-3}{2n-2}\phi(n-1)$. [R. P.]

6. Investigate formulae of reduction for

(a) $\int \frac{dx}{(x^2+a^2)^{\frac{n}{2}}}$. (b) $\int x^n(a+bx)^{p+\frac{1}{2}}dx$.

(c) $\int \frac{x^m}{(a^2+x^2)^{\frac{1}{2}}}dx$. (d) $\int \frac{x^m}{(a^3+x^3)^{\frac{n}{3}}}dx$.

(e) $\int \frac{x^m}{(x^3-1)^{\frac{1}{3}}}dx$. (f) $\int x^{2n}(x^2+a^2)^{\frac{2p+1}{2}}dx$,

and obtain the value of $\int x^8(x^3-1)^{-\frac{1}{3}}dx$. [COLLEGES, CAMB.]

7. Investigate a formula of reduction for

$$\int \frac{x^{2n+1}}{(1-x^2)^{\frac{1}{2}}}dx,$$

and by means of this integral show that

$$\frac{1}{2n+2} + \frac{1}{2}\cdot\frac{1}{2n+4} + \frac{1\,.\,3}{2\,.\,4}\cdot\frac{1}{2n+6} + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\cdot\frac{1}{2n+8} + \ldots \textit{ ad inf.}$$
$$= \frac{2\,.\,4\,.\,6\ldots 2n}{3\,.\,5\,.\,7\ldots(2n+1)}.$$

Sum also the series

$$\frac{1}{2n+1} + \frac{1}{2}\cdot\frac{1}{2n+3} + \frac{1\,.\,3}{2\,.\,4}\cdot\frac{1}{2n+5} + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\cdot\frac{1}{2n+7} + \ldots \textit{ ad inf.}$$

[MATH. TRIP., 1897.]

8. Find a reduction formula for

$$\int_1^x \frac{x^n\,dx}{\sqrt{x-1}}.$$

Show that

$$\frac{2\,.\,4\,.\,6\,\ldots\,2n}{3\,.\,5\,.\,7\,\ldots\,(2n+1)}\left[1+\frac{1}{2}x+\frac{1\,.\,3}{2\,.\,4}x^2+\ldots+\frac{1\,.\,3\,\ldots\,(2n-1)}{2\,.\,4\,\ldots\,2n}x^n\right]$$

$$=1+\frac{a_1}{3}(x-1)+\frac{a_2}{5}(x-1)^2+\ldots+\frac{1}{2n+1}(x-1)^n,$$

where a_1, a_2, ... are the binomial coefficients. [St. John's, 1886.]

9. Prove that if $\quad u_n \equiv \int_0^{\frac{\pi}{4}} \sin^{2n} x\,dx,$

then $\quad u_n = \left(1-\frac{1}{2n}\right)u_{n-1} - \frac{1}{n2^{n+1}},$

and deduce

$$\int_0^{\frac{\pi}{4}} \sin^{2n} x\,dx = -\frac{1}{2^{n+1}}\left\{\frac{1}{n}+\frac{2n-1}{n(n-1)}+\frac{(2n-1)(2n-3)}{n(n-1)(n-2)}+\ldots\right\}$$
$$+\frac{(2n-1)(2n-3)\ldots 3}{2n(2n-2)\ldots 4\,.\,2}\cdot\frac{\pi}{4}.$$

[Math. Trip., 1878.]

10. Prove that

$$\int_0^1 x^{4m+1}\sqrt{\frac{1-x^2}{1+x^2}}\,dx = \frac{1\,.\,3\,.\,5\ldots(2m-1)}{2\,.\,4\,.\,6\ldots 2m}\cdot\frac{\pi}{4}-\frac{2\,.\,4\,.\,6\ldots 2m}{3\,.\,5\,.\,7\ldots 2m+1}\cdot\frac{1}{2}.$$

11. Find a reduction formula for

$$\int e^{ax}\cos^n x\,dx,$$

where n is a positive integer, and evaluate

$$\int e^{ax}\cos^4 x\,dx.$$

[Oxford, 1889.]

12. Find formulae of reduction for

$$(1)\ \int x^n \sin x\,dx,$$

$$(2)\ \int e^{ax}\sin^n x\,dx.$$

Deduce from the latter a formula of reduction for

$$\int \cos ax \sin^n x\,dx.$$

[Colleges γ, 1890.]

13. Show that

$$(m+n)(m+n-2)\int \sin^m\theta\cos^n\theta\,d\theta$$
$$=(m-1)\sin^{m+1}\theta\cos^{n-1}\theta-(n-1)\sin^{m-1}\theta\cos^{n+1}\theta$$
$$+(m-1)(n-1)\int \sin^{m-2}\theta\cos^{n-2}\theta\,d\theta.$$

[TRIN. COLL. CAMB., 1889.]

14. Show that

$$2^m\int \cos mx\cos^m x\,dx$$
$$=C+x+m\,.\,\frac{\sin 2x}{2}+\frac{m(m-1)}{1\,.\,2}\,.\,\frac{\sin 4x}{4}+\ldots+\frac{\sin 2mx}{2m},$$

where m is a positive integer. [COLLEGES α, 1885.]

15. Show that

$$\int_0^{\frac{\pi}{2}} \sin^{2m}\theta\cos^{2m-1}\theta\,d\theta=\frac{(2m-2)(2m-4)\ldots 4\,.\,2}{(4m-1)(4m-3)\ldots(2m+1)},$$

m being a positive integer. [OXFORD, 1889.]

16. Evaluate the integral $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-nx}\cos^m x\,dx$,

m being a positive integer. [COLL., 1886.]

17. Prove that if

$$I_{m,n}\equiv\int \cos^m x\sin nx\,dx,$$
$$(m+n)\,I_{m,n}=-\cos^m x\cos nx+mI_{m-1,n-1}$$

and $$\Big[I_{m,m}\Big]_0^{\frac{\pi}{2}}=\frac{1}{2^{m+1}}\left(\frac{2}{1}+\frac{2^2}{2}+\frac{2^3}{3}+\ldots+\frac{2^m}{m}\right).$$ [BERTRAND.]

18. If $$u_{m,n}\equiv\int_0^{\frac{\pi}{2}}\cos^m x\sin nx\,dx,$$

prove that $$u_{m,n}=\frac{1}{m+n}+\frac{m}{m+n}u_{m-1,n-1}.$$

Hence find the value, when m is a positive integer, of

$$\int_0^{\frac{\pi}{2}}\cos^m x\sin 2mx\,dx.$$ [γ, 1887.]

19. If $$I_{m,n}\equiv\int\cos^m x\cos nx\,dx,$$

prove that $$I_{m,n}=-\frac{\cos^2 nx}{m^2-n^2}\frac{d}{dx}\left(\frac{\cos^m x}{\cos nx}\right)+\frac{m(m-1)}{m^2-n^2}I_{m-2,n},$$

and show that

$$\left[I_{m,n}\right]_0^{\frac{\pi}{2}} = \frac{m(m-1)}{m^2-n^2}\left[I_{m-2,n}\right]_0^{\frac{\pi}{2}}$$

20. Prove that $\int_0^{\frac{\pi}{2}} \cos^n x \cos nx\,dx = \frac{\pi}{2^{n+1}}$,

n being a positive integer. [Bertrand.]

21. If m and n be positive integers, and if $m+n$ be even, prove that

$$\int_0^{\frac{\pi}{2}} \cos^m\theta \cos n\theta\,d\theta = \frac{\pi}{2^{m+1}}\,\frac{m\,!}{\frac{m+n}{2}\,!\;\frac{m-n}{2}\,!}$$

[Colleges, 1882.]

22. If $\int_0^{\frac{\pi}{2}} \cos^m x \cos nx\,dx$ be denoted by $f(m, n)$, show that

$$f(m, n) = \frac{m}{m-n} f(m-1, n+1) = \frac{m}{m+n} f(m-1, n-1).$$

[Oxford, 1890.]

23. Prove that if n be a positive integer, greater than unity,

$$\int_0^{\frac{\pi}{2}} \cos^{n-2} x \sin nx\,dx = \frac{1}{n-1}.$$ [Oxford I. P., 1889.]

24. If $u_{m,n} \equiv \int x^m \operatorname{cosec}^n x\,dx$, prove that

$$(n-1)(n-2)u_{m,n} = (n-2)^2 u_{m,n-2} + m(m-1)u_{m-2,n-2}$$
$$- x^{m-1}\{m\sin x + (n-2)x\cos x\}\operatorname{cosec}^{n-1}x.$$

[Math. Trip., 1896.]

25. If $\int_0^\infty e^{-x}x^{n-1}\log x\,dx \equiv \phi(n)$,

prove that $\phi(n+2) - (2n+1)\phi(n+1) + n^2\phi(n) = 0$.

[R. P., St. John's Coll., 1881.]

26. Show that if $U_n \equiv \int_0^1 \frac{x^n e^x\,dx}{\sqrt{1-x}}$,

$$2U_{n+1} + U_n(2n-1) - 2nU_{n-1} = 0.$$ [Colleges β, 1887.]

27. Prove that if $\phi(m) \equiv \int x^m(x^3+3ax+c)^{-\frac{1}{2}}\,dx$,

then $(2m-1)\phi(m) + 3a(2m-3)\phi(m-2) + (2m-4)c\,\phi(m-3)$
$= 2x^{m-2}(x^3+3ax+c)^{\frac{1}{2}}$. [Trinity, 1886.]

28. If $u_m \equiv \int_0^{\frac{\pi}{2}} e^{-nx}\cos^m x\,dx$,

prove that $(m^2+n^2)u_m = m(m-1)u_{m-2} + n$. [Oxford I. P., 1900.]

29. Prove that if $I_m \equiv \int_0^{\frac{\pi}{2}} \frac{\sin^m x\, dx}{(1-k^2\sin^2 x)^{\frac{1}{2}}}$,

then $(m-1)k^2 I_m - (m-2)(1+k^2) I_{m-2} + (m-3) I_{m-4} = 0.$

[Trinity, 1890.]

30. Obtain a reduction formula for the integral

$$I_n \equiv \int (a\cos^2\theta + 2h\sin\theta\cos\theta + b\sin^2\theta)^{-n}\, d\theta$$

in the form

$$2(n+1)(ab-h^2) I_{n+2} - (2n+1)(a+b) I_{n+1} + 2n I_n = -\frac{1}{2n}\frac{d^2 I_n}{d\theta^2}.$$

[Math. Trip., 1898.]

31. Show that $\int_0^{\frac{\pi}{2}} \frac{d\phi}{(1-e^2\sin^2\phi)^3} = \frac{8-8e^2+3e^4}{(1-e^2)^{\frac{5}{2}}}\frac{\pi}{16}$,

e being less than unity. [St. John's Coll., 1885.]

32. If $I_n \equiv \int \frac{\sin nx}{\sin x} dx,$

prove that $(n-1)(I_n - I_{n-2}) = 2\sin(n-1)x,$

and hence that

$$\int_0^{\frac{\pi}{2}} \frac{\sin nx}{\sin x} = \frac{\pi}{2}, \text{ if } n \text{ be odd},$$

$$= 2\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots + (-1)^{\frac{n}{2}+1}\frac{1}{n-1}\right), \text{ if } n \text{ be even}.$$

33. If $X \equiv a + bx^n + cx^{2n}$ and $I_{m,p} \equiv \int x^m X^p\, dx$,

prove the existence of reduction formulae of the nature of

(i) $x^{m+1}X^{p+1} = A_1 I_{m,p} + B_1 I_{m+n,p} + C_1 I_{m+2n,p}$;

(ii) $x^{m-2n+1}X^{p+1} = A_2 I_{m,p} + B_2 I_{m-n,p} + C_2 I_{m-2n,p}$;

(iii) $x^{m+1}X^p = A_3 I_{m,p} + B_3 I_{m+n,p-1} + C_3 I_{m+2n,p-1}$;

(iv) $x^{m+1}X^p = A_4 I_{m,p} + B_4 I_{m,p-1} + C_4 I_{m+n,p-1}$;

(v) $x^{m-n+1}X^{p+1} = A_5 I_{m,p} + B_5 I_{m-n,p} + C_5 I_{m-n,p+1}$;

and find the values of the fifteen constants.

34. Show that $\int \frac{dx}{(a+bx^2+cx^4)^p}$

can be reduced to the integration of

$$\int \frac{dx}{a+bx^2+cx^4} \text{ and } \int \frac{x^2 dx}{a+bx^2+cx^4} \quad (b>0,\ b^2>4ac),$$

and integrate these expressions; p being integral. [Bertrand.]

35. Show that, if $\qquad u_{m,n} \equiv \int \frac{x^m}{(\log x)^n}\,dx,$

$$(n-1)\,u_{m,n} = -\frac{x^{m+1}}{(\log x)^{n-1}} + (m+1)\,u_{m,n-1}.$$

[OXFORD, I. P., 1889.]

36. Find reduction formulae for

$$(\alpha)\ \int \tanh^n x\,dx\,;$$

$$(\beta)\ \int \frac{x}{\sin^n x}\,dx\,;$$

$$(\gamma)\ \int \frac{dx}{(a+be^x+ce^{-x})^n}.$$

37. If $I_m \equiv \int \frac{x^m\,dx}{\sqrt{X}}$, where $X \equiv ax^2+2bx+c$, show that

$$amI_m + (2m-1)\,bI_{m-1} + (m-1)\,cI_{m-2} = x^{m-1}\sqrt{X}.$$ [β, 1891.]

38. Establish a reduction formula for

$$I_n \equiv \int \frac{dx}{(Ax^2+B)^n\sqrt{X}},$$

where $X \equiv ax^4+bx^2+c$, in the form

$$\frac{x\sqrt{X}}{(Ax^2+B)^{n-1}} = \lambda I_n + \mu I_{n-1} + \nu I_{n-2} + \xi I_{n-3},$$

showing that

$$\left.\begin{aligned}
\lambda &= \quad (2n-2)\,\frac{B}{A^2}\,(A^2c - \ \ ABb + \ \ B^2a),\\
\mu &= -(2n-3)\,\frac{1}{A^2}\,(A^2c - 2ABb + 3B^2a),\\
\nu &= -(2n-4)\,\frac{1}{A^2}\,(Ab - 3Ba),\\
\xi &= -(2n-5)\,\frac{1}{A^2}\,a.
\end{aligned}\right\}$$

39. Show that, if

$$I_{m,n} \equiv \int_0^\pi \sin^m\theta\cos n\theta\,d\theta, \quad J_{m,n} \equiv \int_0^\pi \sin^m\theta\sin n\theta\,d\theta,$$

then $\qquad J_{m-1,n+1} = \left(1-\frac{n}{m}\right) I_{m,n}, \quad I_{m-1,n+} = -\left(1-\frac{n}{m}\right) J_{m,n},$

where m is a positive integer; and point out how these results can be used to find the values of $I_{m,n}$ and $J_{m,n}$. [C. S., 1896.]

40. If T be a function of x such that

$$\left(\frac{dT}{dx}\right)^2 = A + 3BT + 3CT^2 + DT^3,$$

prove that

$$\frac{d}{dx}\left(\frac{1}{T^{n-1}}\frac{dT}{dx}\right) = -\frac{(n-1)A}{T^n} - \frac{3}{2}(2n-3)\frac{B}{T^{n-1}} - 3\frac{(n-2)C}{T^{n-2}} - \frac{(2n-5)D}{2T^{n-3}}.$$

Apply the result to investigate a reduction formula for

$$\int \frac{dx}{T^n}.$$

By a consideration of the case where $C = 0$, $D = 0$ (or in any other way), obtain a reduction formula for

$$\int \frac{dx}{(a + 2bx + cx^2)^n}.$$

[I. C. S., 1897.]

41. Prove that

$$\int_0^\infty e^{-x^2} x^{2n}\, dx = \sqrt{\pi}\,\frac{1 . 3 . 5 \ldots (2n-1)}{2 . 4 . 6 \ldots 2n}\int_0^\infty e^{-x^2} x^{2n+1}\, dx,$$

where n is a positive integer. [Colleges α, 1890.]

42. If $$u_n \equiv \int_a^b x^n \{(b-x)(x-a)\}^{-\frac{1}{2}} dx,$$

show that $2nu_n = (2n-1)(a+b)u_{n-1} - 2(n-1)abu_{n-2}$.

[Oxford I. Pub., 1912.]

43. By applying the substitution $x = a\cos^2\theta + b\sin^2\theta$ (or otherwise), prove that the definite integral

$$\int_a^b \frac{x^n\, dx}{\sqrt{(x-a)(b-x)}}$$

is a rational integral function of a and b when n is an integer; and evaluate it when $n = 3$. [Oxf. I. P., 1913.]

44. If $$u_n \equiv \int \frac{\cos 2nx}{\sin^2 x}\, dx,$$

obtain a formula of reduction connecting u_n and u_{n-1}.

Hence, or otherwise, evaluate

$$\int_x^{\frac{\pi}{2}} \frac{\cos 2nx}{\sin^2 x}\, dx,$$

where n is a positive integer and $\frac{\pi}{2} > x > 0$.

Consider the case when the lower limit is negative.

[Oxf. I. P., 1915.]

45. By multiplying the inequality $1 \geqslant 2\sin x - \sin^2 x$ by $\sin^{2n-1}x$ and by $\sin^{2n}x$, and integrating between 0 and $\frac{1}{2}\pi$, show that

$$\left\{\frac{(4n+3)(2n+1)}{4n+4}\cdot\frac{\pi}{2}\right\}^{\frac{1}{2}} > \frac{2.4\ldots 2n}{1.3\ldots(2n-1)} > \left\{\frac{2n(2n+1)}{4n+1}\cdot\pi\right\}^{\frac{1}{2}}.$$

[MATH. TRIP. I., 1915.]

46. The expression

$$\frac{1-a}{(1-a\sin^2\theta)^{\frac{3}{2}}} - (1-a\sin^2\theta)^{\frac{1}{2}},$$

where $1 > a > 0$, is expanded in ascending powers of a, and the coefficient of a^n is denoted by u_n. Prove that

$$\int_0^{\frac{\pi}{2}} u_n d\theta = 0.$$

[MATH. TRIP. I., 1916.]

47. If $\quad s_n \equiv \int \frac{\sin(2n-1)x}{\sin x}dx, \quad v_n \equiv \int \frac{\sin^2 nx}{\sin^2 x}dx,$

prove the reduction formulae

$$n(s_{n+1} - s_n) = \sin 2nx, \quad v_{n+1} - v_n = s_{n+1};$$

and show that if v_n be taken between the limits 0 and $\frac{1}{2}\pi$, its value is $\frac{1}{2}n\pi$, where n is an integer. [MATH. TRIP. I., 1914.]

48. If $\quad \lambda^2 \equiv \cos^2\phi/a^2 + \sin^2\phi/b^2, \quad$ find $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\phi}{\lambda^2}$,

and prove that $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\phi}{\lambda^6} = \pi ab\{3(a^4+b^4) + 2a^2b^2\}/16,$

and that $\displaystyle\int_0^1 \frac{3\mu^2-1}{2}\,\frac{d\mu}{(1-e^2\mu^2)^{\frac{5}{2}}} = \frac{e^2}{3(1-e^2)^{\frac{5}{2}}}.$ [ϵ, 1883.]

49. If $U_n \equiv \int \sin^m x(a+b\cos x)^{-n}dx$, prove that U_n can be calculated by means of a reduction formula of the nature

$$AU_n + BU_{n-1} + CU_{n-2} = \sin^{m+1}x(a+b\cos x)^{-n+1};$$

and determine the constants A, B, C.

50. Prove that $\displaystyle\int_c^\infty \frac{dx}{(a^2-2cx+x^2)^{n+1}} = \frac{2n!}{n!\,n!}\,\frac{\pi}{\lambda^{n+\frac{1}{2}}},$

where λ denotes $4(a^2-c^2)$ and is supposed positive. [TRIN., 1887.]