CHAPTER X.

DIFFERENTIATION, ETC., UNDER AN INTEGRATION SIGN.

354. **Differentiation of a Definite Integral with regard to a Parameter.**

A definite integral is by its nature independent of the value of the particular variable in terms of which the integration is effected, and its value depends upon any other quantities which may occur in the integrand or in the limits.

First, let us consider the differentiation with regard to c of the integral $u=\int_a^b \phi(x, c)\,dx$, where a and b are each finite and independent of c. We shall suppose also that $\phi(x, c)$ is *single-valued, finite* and *continuous*, as also its *differential coefficient with regard to c for the range of values of x from a to b.* When c changes to $c+\delta c$, suppose that the consequent change of u is to $u+\delta u$.

Then $$u+\delta u=\int_a^b \phi(x, c+\delta c)\,dx$$

and $$\delta u=\int_a^b [\phi(x, c+\delta c)-\phi(x, c)]\,dx.$$

Now $$\phi(x, c+\delta c)=\phi(x, c)+\delta c\phi'(x, c+\theta\,\delta c),$$

where the accent represents differentiation of $\phi(x, c)$ with regard to c, and θ is a positive proper fraction, $c+\theta\,\delta c$ being written for c after the differentiation is performed, *i.e.*

$$\frac{\partial u}{\partial c}=Lt_{\delta c=0}\frac{\delta u}{\delta c}=Lt_{\delta c=0}\int_a^b \phi'(x, c+\theta\,\delta c)\,dx=\int_a^b \frac{\partial\phi(x, c)}{\partial c}\,dx.$$

[See Arts. 1898, 1902, Vol. II.]

355. Next, let a and b be also functions of c.

Then $u+\delta u=\int_{a+\delta a}^{b+\delta b}\phi(x, c+\delta c)\,dx$

and $$\delta u=\int_{a+\delta a}^{b+\delta b}\phi(x, c+\delta c)\,dx-\int_a^b\phi(x, c)\,dx$$
$$=\int_b^{b+\delta b}\phi(x, c+\delta c)\,dx+\int_{a+\delta a}^{a}\phi(x, c+\delta c)\,dx$$
$$+\int_a^b[\phi(x, c+\delta c)-\phi(x, c)]\,dx.$$

Now
$$\int_b^{b+\delta b}\phi(x, c+\delta c)\,dx=\phi(b+\theta_1\delta b,\ c+\delta c)\,\delta b \quad \text{(by Art. 332)}$$
and
$$\int_{a+\delta a}^{a}\phi(x, c+\delta c)\,dx=-\phi(a+\theta_2\delta a,\ c+\delta c)\,\delta a,$$
where θ_1 and θ_2 are positive proper fractions.

Also $$Lt_{\delta c=0}\int_a^b\frac{[\phi(x, c+\delta c)-\phi(x, c)]\,dx}{\delta c}$$
has been discussed in the last article.

Hence, dividing the expression for δu by δc and taking the limit, when δc is indefinitely diminished,
$$\frac{\partial u}{\partial c}=\int_a^b\frac{\partial\phi(x, c)}{\delta c}\,dx+\phi(b, c)\frac{db}{dc}-\phi(a, c)\frac{da}{dc},$$
and the conditions under which this is true have been stated above, viz. $\phi(x, c)$ and $\dfrac{\partial\phi(x, c)}{\partial c}$ are *single-valued, finite* and *continuous* functions of x *throughout the finite range* $x=a$ *to* $x=b$, inclusive.

This is a case of the theorem on partial differentiation, *Diff. Calc.*, Art. 160, viz.
$$\frac{du}{dc}=\frac{\partial u}{\partial c}+\frac{\partial u}{\partial a}\cdot\frac{da}{dc}+\frac{\partial u}{\partial b}\cdot\frac{db}{dc}.$$

356. **Geometrical Meaning of the Process.**

We next examine the geometrical meaning of this differentiation.

Let $\alpha\beta$, $\alpha'\beta'$ be the respective graphs of
$$y=\phi(x, c),\quad y=\phi(x, c+\delta c).$$

Let the ordinates of both curves be drawn at the points

$$x=a, \quad x=b, \quad x=a+\delta a, \quad x=b+\delta b,$$

viz. $\qquad A\alpha\gamma', \quad B\beta\delta', \quad A'\gamma\alpha', \quad B'\delta\beta',$

respectively. Let NQP be any other ordinate, and draw αS, βR parallel to the x-axis. Then $\int_a^b \phi(x, c)\,dx$ is represented by the area $AB\beta\alpha$. We have to differentiate this area with

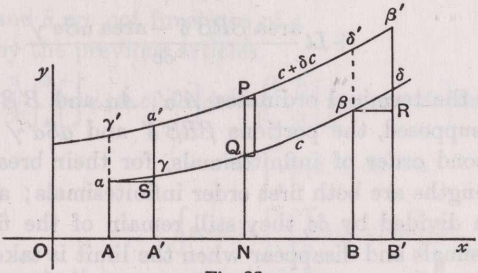

Fig. 36.

regard to c. When c is increased to $c+\delta c$, a and b being both dependent upon c, area $AB\beta\alpha$ is changed to $A'B'\beta'\alpha'$, and

$$\frac{\partial}{\partial c}\text{ area } AB\beta\alpha = Lt_{\delta c=0}\frac{\text{area } A'B'\beta'\alpha' - \text{area } AB\beta\alpha}{\delta c}$$

$$= Lt_{\delta c=0}\frac{\beta\delta'\gamma'\alpha + BB'\beta'\delta' - AA'\alpha'\gamma'}{\delta c}$$

Now

$$Lt\frac{\beta\delta'\gamma'\alpha}{\delta c} = Lt\left[\frac{\int_a^b (NP-NQ)\,dx}{\delta c}\right] = Lt\int_a^b \frac{\phi(x, c+\delta c)-\phi(x, c)}{\delta c}\,dx$$

$$= \int_a^b \frac{\partial\phi(x, c)}{\partial c}\,dx.$$

Also $\qquad Lt\dfrac{BB'\beta'\delta'}{\delta c} = Lt\dfrac{BB'R\beta + \beta R\beta'\delta'}{\delta c}$

$$= Lt\frac{\phi(b, c)\,\delta b + \beta R\beta'\delta'}{\delta c}$$

$$= \phi(b, c)\frac{db}{dc} + Lt\frac{\beta R\beta'\delta'}{\delta c};$$

and $$Lt\frac{AA'\alpha'\gamma'}{\delta c}=Lt\frac{AA'S\alpha+\alpha S\alpha'\gamma'}{\delta c}$$

$$=Lt\frac{\phi(a,c)\,\delta a+\alpha S\alpha'\gamma'}{\delta c}$$

$$=\phi(a,c)\frac{da}{dc}+Lt\frac{\alpha S\alpha'\gamma'}{\delta c};$$

$$\therefore\ \frac{d}{dc}\text{area } AB\beta\alpha=\int_a^b\frac{\partial\phi(x,c)}{dc}dx+\phi(b,c)\frac{db}{dc}-\phi(a,c)\frac{da}{dc}$$

$$+Lt\frac{\text{area }\beta R\beta'\delta'-\text{area }\alpha S\alpha'\gamma'}{\delta c}.$$

Now, if the terminal ordinates $A'\alpha'$, $A\alpha$ and $B'\beta'$, $B\beta$ are finite, as supposed, the portions $\beta R\beta'\delta'$ and $\alpha S\alpha'\gamma'$ are both of the second order of infinitesimals, for their breadths and greatest lengths are both first order infinitesimals; and therefore, when divided by δc, they still remain of the first order of infinitesimals and disappear when the limit is taken.

$$\therefore\ \frac{d}{dc}\int_a^b\phi(x,c)\,dx=\int_a^b\frac{\partial\phi(x,c)}{\partial c}dx+\phi(b,c)\frac{db}{dc}-\phi(a,c)\frac{da}{dc}.$$

The student will see that the truth of this theorem could not be asserted without further examination if any of the ordinates of the figure became infinite, or if either of the graphs were discontinuous, or if either graph were cut by an ordinate in more places than one for any position between the extreme ordinates of the portion considered.

When one of the limits is infinite the theorem may still be true, but special consideration is needed in each case.

357. If the integral to be differentiated with respect to c be "indefinite," *i.e.* the limits not stated, say

$$u=\int\phi(x,c)\,dx+A,$$

where A is an arbitrary constant, then

$$\frac{du}{dc}=\int\frac{\partial\phi(x,c)}{\partial c}dx+\frac{\partial A}{\partial c},$$

and A being an arbitrary constant as regards x, $\frac{\partial A}{\partial c}$ is also an arbitrary constant as regards x; and we may write the result as

$$\frac{du}{dc}=\int\frac{\partial\phi(x, c)}{\partial c}dx+A',$$

where A' is an arbitrary constant.

358. **Integration of a Definite Integral with regard to a Parameter.**

Take the integral $\quad u=\int_a^b\phi(x, c)\,dx,$

where a and b are not functions of c.

Then, by the previous articles,

$$\frac{\partial}{\partial c}\int_a^b\left[\int\phi(x, c)\,dc\right]dx=\int_a^b\frac{\partial}{\partial c}\left[\int\phi(x, c)\,dc\right]dx$$

$$=\int_a^b\phi(x, c)\,dx=u;$$

$$\therefore\ \int u\,dc=\int_a^b\left[\int\phi(x, c)\,dc\right]dx,$$

i.e. $$\int\left[\int_a^b\phi(x, c)\,dx\right]dc=\int_a^b\left[\int\phi(x, c)\,dc\right]dx.$$

359. Supposing that instead of an *indefinite* integration of u we require a *definite* integration between c_0 and c, say, regarded as independent of a and b, then we shall have in general

$$\int_{c_0}^c\left[\int_a^b\phi(x, c)\,dx\right]dc=\int_a^b\left[\int_{c_0}^c\phi(x, c)\,dc\right]dx$$

that is the order of integration is immaterial.

For putting $\quad\int_{c_0}^c\phi(x, c)\,dc=f(x, c),$ say,

then $$\int_a^b\left[\int_{c_0}^c\phi(x, c)\,dc\right]dx=\int_a^b f(x, c)\,dx,$$

and $$\frac{\partial}{\partial c}\int_a^b\left[\int_{c_0}^c\phi(x, c)\,dc\right]dx=\frac{\partial}{\partial c}\int_a^b f(x, c)\,dx$$

$$=\int_a^b\frac{\partial f(x, c)}{\partial c}dx$$

$$=\int_a^b\phi(x, c)\,dx,$$

also $$\frac{\partial}{\partial c}\int_{c_0}^c\left[\int_a^b\phi(x, c)\,dx\right]dc=\int_a^b\phi(x, c)\,dx.$$

Hence both $$\int_{c_0}^{c}\left[\int_a^b \phi(x, c)\,dx\right]dc$$

and $$\int_a^b\left[\int_{c_0}^{c} \phi(x, c)\,dc\right]dx$$

have the same differential coefficient with regard to c, and both vanish when $c=c_0$. Hence they are equal.

This theorem may be written

$$\int_{c_0}^{c}\int_a^b \phi(x, c)\,dc\,dx=\int_a^b\int_{c_0}^{c} \phi(x, c)\,dx\,dc,$$

and expresses that the order of the integrations may be changed. The theorem presupposes that the limits of integration c_0 and c are independent of the limits a and b, and also that $\phi(x, c)$ remains single-valued, finite and continuous for all values of the quantities x and c between or at their limits.

360. Notation.

The notation of this "double integration" calls for explanation. It will be noticed that we have written

$$\int_{c_0}^{c}\left[\int_a^b \phi(x, c)\,dx\right]dc \quad \text{as} \quad \int_{c_0}^{c_1}\int_a^b \phi(x, c)\,dc\,dx,$$

inverting the order of the dx and dc. The order of writing these symbols does not appear to be universally agreed upon, some authors adopting the opposite order. For the sake of clearness we may state that throughout this book the *right-hand element and the right-hand integration sign refer to the first operation*, the *left-hand element and the left-hand integration sign refer to the second.*

Thus $\int_{x_0}^{x_1}\int_{y_0}^{y_1} \phi(x, y)\,dx\,dy$ will mean that

(1) $\phi(x, y)$ is to be integrated with regard to y, keeping x constant, between limits $y=y_0$, $y=y_1$.

(2) That the result obtained is then to be integrated with regard to x between limits $x=x_0$ and $x=x_1$.

A notation which carries its own explanation, and used when there is any fear of confusion, is

$$\int_{x_0}^{x_1} dx \int_{y_0}^{y_1} dy\, \phi(x, y).$$

361. **Geometrical Interpretation.**

Writing y where we had c in $\phi(x, c)$ and dy for dc, we have to establish the theorem

$$\int_{c_0}^{c}\left[\int_a^b \phi(x, y)\, dx\right] dy = \int_a^b\left[\int_{c_0}^{c} \phi(x, y)\, dy\right] dx.$$

Imagine the rectangular space bounded by

$$x=a, \quad x=b, \quad y=c_0, \quad y=c$$

to be divided up into infinitesimal rectangles by two families of straight lines, the first set being equidistant from each other and parallel to the x-axis, and the second set being equidistant from each other and parallel to the y-axis, the distance between consecutive lines of each family being infinitesimal.

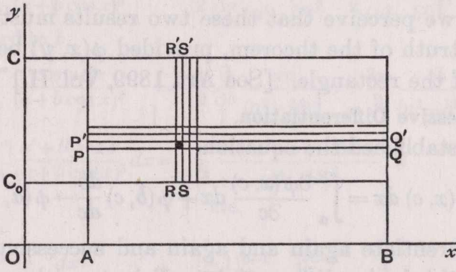

Fig. 37.

Imagine that we have to find the mass of this rectangle, regarded as of variable density, such that $\phi(x, y)$ is the density at any point (x, y), and that the elementary rectangle whose corners are (x, y), $(x+\delta x, y)$, $(x+\delta x, y+\delta y)$ and $(x, y+\delta y)$, and whose area is $\delta x\, \delta y$, is so small that the density may be taken uniform all over it, or, which will amount to the same thing, that the density at any point of the small area $\delta x\, \delta y$ differs from that at x, y by an infinitesimal.

The mass of this small rectangle will be, to the second order of infinitesimals, $\phi(x, y)\, \delta x\, \delta y$. Let $PQQ'P'$ and $RSS'R'$ be the two elementary strips whose common element is $\delta x\, \delta y$. Then, in adding up all the elements of mass along the strip $PQQ'P'$, we have

$$\left[\int_a^b \phi(x, y)\, dx\right] \delta y$$

in the limit when δx is indefinitely small.

Then, if we sum the strips from $y=c_0$ to $y=c$, we have in the limit, when δy is indefinitely small,

$$\int_{c_0}^{c}\left[\int_a^b \phi(x, y)\,dx\right]dy.$$

But if we first sum the elements $\phi(x, y)\delta x\,\delta y$ along the strip $RSS'R'$, we have in the limit, when δy is indefinitely small,

$$\left[\int_{c_0}^{c}\phi(x, y)\,dy\right]\delta x.$$

And if we sum these strips from $x=a$ to $x=b$ we have in the limit, when δx is indefinitely small,

$$\int_a^b\left[\int_{c_0}^{c_1}\phi(x, y)\,dy\right]dx.$$

And as the order of addition of these elements is obviously immaterial we perceive that these two results must be equal. Hence the truth of the theorem, provided $\phi(x, y)$ be finite for all points of the rectangle. [See Art. 1899, Vol. II.]

362. **Successive Differentiation.**

Having established the equation

$$\frac{d}{dc}\int_a^b \phi(x, c)\,dx=\int_a^b \frac{\partial\phi(x, c)}{\partial c}\,dx+\phi(b, c)\frac{db}{dc}-\phi(a, c)\frac{da}{dc}$$

we can differentiate again and again and successively obtain the second, third, etc., differential coefficients with regard to c. The successive results however, in general form, rapidly get complicated. Thus, for instance, we have

$$\begin{aligned}
\frac{d^2}{dc^2}\int_a^b \phi(x, c)\,dx
&=\frac{d}{dc}\left[\int_a^b \frac{\partial\phi(x, c)}{\partial c}\,dx\right]+\frac{d}{dc}\left[\phi(b, c)\frac{db}{dc}\right]-\frac{d}{dc}\left[\phi(a, c)\frac{da}{dc}\right]\\
&=\int_a^b \frac{\partial^2\phi(x, c)}{\partial c^2}\,dx+\frac{\partial\phi(b, c)}{\partial c}\frac{db}{dc}-\frac{\partial\phi(a, c)}{\partial c}\frac{da}{dc}\\
&\quad+\frac{\partial\phi(b, c)}{\partial b}\left(\frac{db}{dc}\right)^2+\frac{\partial\phi(b, c)}{\partial c}\frac{db}{dc}+\phi(b, c)\frac{d^2b}{dc^2}\\
&\quad-\frac{\partial\phi(a, c)}{\partial a}\left(\frac{da}{dc}\right)^2-\frac{\partial\phi(a, c)}{\partial c}\frac{da}{dc}-\phi(a, c)\frac{d^2a}{dc^2},
\end{aligned}$$

which reduces to an expression with seven terms.

Similarly, the third and other differential coefficients may be found when necessary.

In particular cases there may be considerable simplification.

363. Many important results can be derived from these theorems, and new forms deduced, by differentiation or integration with regard to letters which have been regarded as constants in a previous integration.

Ex. 1. For example, taking the case

$$\int_0^\pi \frac{dx}{a+b\cos x}\,(a>b)=\frac{2}{\sqrt{a^2-b^2}}\left[\tan^{-1}\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right]_0^\pi(\text{Art. }171)=\frac{\pi}{\sqrt{a^2-b^2}},$$

we have, upon differentiation with regard to a,

$$\int_0^\pi \frac{dx}{(a+b\cos x)^2}=-\frac{\partial}{\partial a}\left(\frac{\pi}{\sqrt{a^2-b^2}}\right)=\frac{\pi a}{(a^2-b^2)^{\frac{3}{2}}}.$$

Differentiating again with regard to a,

$$\int_0^\pi \frac{dx}{(a+b\cos x)^3}=-\frac{1}{2}\frac{\partial}{\partial a}\frac{\pi a}{(a^2-b^2)^{\frac{3}{2}}}=\frac{\pi}{2}\frac{2a^2+b^2}{(a^2-b^2)^{\frac{5}{2}}},$$

or, with regard to b,

$$\int_0^\pi \frac{\cos x\,dx}{(a+b\cos x)^3}=-\frac{1}{2}\frac{\partial}{\partial b}\frac{\pi a}{(a^2-b^2)^{\frac{3}{2}}}=-\frac{3\pi}{2}\frac{ab}{(a^2-b^2)^{\frac{5}{2}}}.$$

Hence, $$\int_0^\pi \frac{a'+b'\cos x}{(a+b\cos x)^3}\,dx=\frac{\pi}{2}\frac{a'(2a^2+b^2)-3abb'}{(a^2-b^2)^{\frac{5}{2}}},$$

etc.

Generally,

$$\int_0^\pi \frac{dx}{(a+b\cos x)^n}=\frac{(-1)^{n-1}}{(n-1)\,!}\pi\frac{\partial^{n-1}}{\partial a^{n-1}}\frac{1}{\sqrt{a^2-b^2}},$$

$$\int_0^\pi \frac{\cos^{n-1}x\,dx}{(a+b\cos x)^n}=\frac{(-1)^{n-1}}{(n-1)\,!}\pi\frac{\partial^{n-1}}{\partial b^{n-1}}\frac{1}{\sqrt{a^2-b^2}}.$$

Ex. 2. Clearly $\int e^{ax}\,dx=\frac{e^{ax}}{a}$;

$$\therefore\ \int x^n e^{ax}\,dx=\left(\frac{\partial}{\partial a}\right)^n\frac{e^{ax}}{a}.$$

Also $$\int x^n e^{ax}\,dx=\frac{1}{D}e^{ax}x^n=e^{ax}\frac{1}{D+a}x^n,\quad \left(D\equiv\frac{\partial}{\partial x}\right),$$

$$=\frac{e^{ax}}{a}\left(1-\frac{D}{a}+\frac{D^2}{a^2}-\ldots\right)x^n.$$

[*Int. Calc. for Beginners*, Art. 213.]

Show that these results are identical.

Ex. 3. Starting with $\int_0^\infty e^{-ax}dx=\frac{1}{a}$,

we have $$\int_0^\infty x^n e^{-ax}\,dx=\frac{n\,!}{a^{n+1}},$$

by n differentiations with respect to a. [See Art. 1897, Vol. II.]

Ex. 4. From such integrals as

$$\int\frac{dx}{(px+q)\sqrt{ax^2+2bx+c}} \quad\text{or}\quad \int\frac{dx}{(a_1x^2+2b_1x+c_1)\sqrt{a_2x^2+2b_2x+c_2}}$$

we can deduce

$$(1)\ \int\frac{dx}{(px+q)^n\sqrt{ax^2+2bx+c}}, \quad\text{or}\quad (2)\ \int\frac{dx}{(px+q)(ax^2+2bx+c)^{\frac{3}{2}}},$$

$$\text{or}\quad (3)\ \int\frac{dx}{(a_1x^2+2b_1x+c_1)^n\sqrt{a_2x^2+2b_2x+c_2}},$$

$$\text{or}\quad (4)\ \int\frac{dx}{(a_1x^2+2b_1x+c_1)(a_2x^2+2b_2x+c_2)^{\frac{3}{2}}},$$

by respectively differentiating the first $n-1$ times with regard to q,
or once with regard to c,
or the second $n-1$ times with regard to c_1,
or once with regard to c_2,

when once the primary integral has been found (Chap. VIII.), and this will often be more convenient than the employment of a reduction formula. Differentiation with regard to other letters, p, a, b, a_1, b_1, a_2 or b_2, will give other integrals.

For example, by Art. 276 (supposing $bp>aq$, a and p positive),

$$\int\frac{dx}{(ax+b)(px+q)^{\frac{1}{2}}}=\frac{2}{\sqrt{a(bp-aq)}}\sin^{-1}\sqrt{\frac{a}{p}}\sqrt{\frac{px+q}{ax+b}}.$$

Therefore

$$\int\frac{dx}{(ax+b)^n(px+q)^{\frac{1}{2}}}=\frac{2(-1)^{n-1}}{(n-1)!}\frac{\partial^{n-1}}{\partial b^{n-1}}\left[\frac{1}{\sqrt{a(bp-aq)}}\sin^{-1}\sqrt{\frac{a}{p}}\sqrt{\frac{px+q}{ax+b}}\right]$$

and

$$\int\frac{x^{n-1}\,dx}{(ax+b)^n(px+q)^{\frac{1}{2}}}=\frac{2(-1)^n}{(n-1)!}\frac{\partial^{n-1}}{\partial a^{n-1}}\left[\frac{1}{\sqrt{a(bp-aq)}}\sin^{-1}\sqrt{\frac{a}{p}}\sqrt{\frac{px+q}{ax+b}}\right],$$

etc.

Ex. 5. If $Q=\sqrt{(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)}$, prove that

$$\int_0^\infty\left(\frac{1}{a^2+\lambda}+\frac{1}{b^2+\lambda}+\frac{1}{c^2+\lambda}-\frac{1}{\lambda}\right)\frac{\sqrt{\lambda}\,d\lambda}{Q}=0.$$

We have
$$\frac{2dQ}{Q}=\frac{d\lambda}{a^2+\lambda}+\frac{d\lambda}{b^2+\lambda}+\frac{d\lambda}{c^2+\lambda};$$

$\therefore$ the integral in question is

$$\int_0^\infty\left(\frac{2dQ}{Q\,d\lambda}-\frac{1}{\lambda}\right)\frac{\sqrt{\lambda}\,d\lambda}{Q}$$

$$=-2\int_0^\infty\frac{d}{d\lambda}\left(\frac{\sqrt{\lambda}}{Q}\right)d\lambda=-2\left[\frac{\sqrt{\lambda}}{Q}\right]_0^\infty=0.$$

Similarly

$$\int_0^\infty\left(\frac{1}{a^2+\lambda}+\frac{1}{b^2+\lambda}+\frac{1}{c^2+\lambda}\right)\frac{d\lambda}{Q}=\int_0^\infty\frac{2dQ}{Q^2}=-2\left[\frac{1}{Q}\right]_0^\infty=\frac{2}{abc}.$$

If $I=\int_0^\infty \frac{d\lambda}{Q}$, we have $\frac{\partial I}{\partial a^2}=-\frac{1}{2}\int_0^\infty \frac{1}{a^2+\lambda}\frac{d\lambda}{Q}$,

and the above equation may be written

$$\frac{\partial I}{\partial a^2}+\frac{\partial I}{\partial b^2}+\frac{\partial I}{\partial c^2}=-\frac{1}{abc}.$$

For several useful illustrations of such integrals, which occur in problems on the attraction of ellipsoidal shells, see *Analytical Statics*, by E. J. Routh, vol. II., pp. 100–101.

364. Differentiation of a Multiple Integral with regard to an Involved Constant.

It will be sufficient to take the case of a multiple integral of the second order.

Consider $$I\equiv\int_{x_0}^{x_1}dx\int_{y_0}^{y_1}dy\,\phi(x,y,c),$$

where c, x_0, x_1, y_0, y_1 are all functions of some quantity t but not involving x or y.

Let $$\int\phi(x,y,c)\,dy=F(x,y,c),$$

where x is regarded as a constant in this integration, so that

$$\frac{\partial F(x,y,c)}{\partial y}\equiv\phi(x,y,c);$$

then $$\int_{y_0}^{y_1}\phi(x,y,c)\,dy=F(x,y_1,c)-F(x,y_0,c)\equiv v, \text{ say.}$$

Then $$I=\int_{x_0}^{x_1}v\,dx.$$

Differentiating by the rule of Art. 355,

$$\frac{dI}{dt}=\int_{x_0}^{x_1}\frac{\partial v}{\partial t}dx+\frac{dx_1}{dt}v_1-\frac{dx_0}{dt}v_0,$$

where v_0 and v_1 are the values of v when x receives the values x_0 and x_1 respectively.

Also $$\frac{\partial v}{\partial t}=\int_{y_0}^{y_1}\frac{\partial\phi}{\partial t}dy+\frac{dy_1}{dt}\phi(x,y_1,c)-\frac{dy_0}{dt}\phi(x,y_0,c).$$

Thus, substituting for $\frac{\partial v}{dt}$, we have

$$\frac{dI}{dt}=\int_{x_0}^{x_1}dx\int_{y_0}^{y_1}dy\frac{\partial\phi}{\partial t}+\int_{x_0}^{x_1}\left[\frac{dy_1}{dt}\phi(x,y_1,c)-\frac{dy_0}{dt}\phi(x,y_0,c)\right]dx$$
$$+\frac{dx_1}{dt}\int_{y_0}^{y_1}\phi(x_1,y,c)dy-\frac{dx_0}{dt}\int_{y_0}^{y_1}\phi(x_0,y,c)dy.$$

This may be written in the more compact form

$$\frac{dI}{dt}=\int_{x_0}^{x_1}dx\int_{y_0}^{y_1}dy\frac{\partial\phi}{\partial t}+\int_{x_0}^{x_1}dx\left[\frac{dy}{dt}\phi(x, y, c)\right]_{y_0}^{y_1}$$
$$+\left[\frac{dx}{dt}\int_{y_0}^{y_1}\phi(x, y, c)dy\right]_{x_0}^{x_1}.$$

A similar process may be applied in cases of Multiple Integrals of a higher order. It is to be understood that all limitations with regard to the nature of ϕ, and the range of integration, which correspond to those described in Art. 355 for the case of a single variable, are supposed to be assumed.

365. Remainder after n terms of Taylor's Series expressed as a Definite Integral.

Let $f(x)$ be a function of x which is finite and continuous throughout the range of values of x, from $x=a$ to $x=a+h$, as also all its differential coefficients as far as $f^{(n)}(x)$.

Let $x=a+h-z$ be an intermediate value of x, ($z<h$).

Considering the integral $\int_0^h f'(a+h-z)\,dz$, we may

(1) integrate directly as $\Big[-f(a+h-z)\Big]_0^h=f(a+h)-f(a)$,

or (2) apply the rule of continued integration by parts (Art. 95), viz.

$$\Big[zf'(a+h-z)+\frac{z^2}{2!}f''(a+h-z)+\frac{z^3}{3!}f'''(a+h-z)+\dots$$
$$+\frac{z^{n-1}}{(n-1)!}f^{(n-1)}(a+h-z)\Big]_0^h$$
$$+\int_0^h\frac{z^{n-1}}{(n-1)!}f^{(n)}(a+h-z)\,dz,$$

i.e.
$$hf'(a)+\frac{h^2}{2!}f''(a)+\frac{h^3}{3!}f'''(a)+\dots+\frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a)$$
$$+\int_0^h\frac{z^{n-1}}{(n-1)!}f^{(n)}(a+h-z)\,dz,$$

i.e.
$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{2!}f''(a)+\frac{h^3}{3!}f'''(a)+\dots$$
$$+\frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a)+\int_0^h\frac{z^{n-1}}{(n-1)!}f^{(n)}(a+h-z)\,dz.$$

Hence the remainder after n terms is

$$R_n \equiv \frac{1}{(n-1)!}\int_0^h z^{n-1} f^{(n)}(a+h-z)\,dz.$$

By theorem IX., Art. 331, this is equal to

$$\frac{1}{(n-1)!} f^{(n)}(a+h-\zeta)\int_0^h z^{n-1}\,dz, \quad \textit{i.e.}\ \frac{h^n}{n!} f^n(a+h-\zeta)$$

for some value of ζ lying between $\zeta=0$ and $\zeta=h$, which may be written $\zeta=(1-\theta)h$ where θ is a positive proper fraction.

Hence $R_n=\dfrac{h^n}{n!}f^{(n)}(a+\theta h)$, which is Lagrange's form of remainder (see *Diff. Calc.*, Art. 130).

366. Remainders after $(n+1)$ terms in Lagrange's Theorem and in Laplace's Extension, expressed by means of a Definite Integral.

It is easy to find an expression for the remainder after $(n+1)$ terms in Laplace's extension of Lagrange's theorem (*Diff. Calc.*, Art. 518).

Lagrange's theorem states that if $z=y+x\phi(z)$ and u be any function of z, say $f(z)$, then the expansion of u in powers of x is

$$u \equiv f(z)=f(y)+x\phi(y)f'(y)+\frac{x^2}{2!}\frac{d}{dy}[\{\phi(y)\}^2 f'(y)]$$
$$+\dots+\frac{x^n}{n!}\frac{d^{n-1}}{dy^{n-1}}[\{\phi(y)\}^n f'(y)]+\dots,$$

and Laplace's extension states that if $z=F\{y+x\phi(z)\}$,

$$f(z)=f\{F(y)\}+x\phi\{F(y)\}\frac{df\{F(y)\}}{dy}+\frac{x^2}{2!}\frac{d}{dy}\left[\overline{\phi\{F(y)\}}|^2\frac{df\{F(y)\}}{dy}\right]$$
$$+\dots+\frac{x^n}{n!}\frac{d^{n-1}}{dy^{n-1}}\left[\overline{\phi\{F(y)\}}|^n\frac{df\{F(y)\}}{dy}\right]+\dots,$$

and contains the former as a particular case.

Take then $z=F\{y+x\phi(z)\}$, and consider the integral

$$I_n=\int_y^{F^{-1}(z)}[y+x\phi\{F(t)\}-t]^n f'[F(t)]F'(t)\,dt,$$

where $$f'(v)\equiv\frac{df(v)}{dv}.$$

We shall write ϕFt for $\phi\{F(t)\}$, etc., to avoid the multiplicity of brackets.

Putting $n=0$, we have

$$I_0=\int_y^{F^{-1}(z)}\frac{d}{dt}(fFt)\,dt=\Big[fFt\Big]_y^{F^{-1}(z)}=f(z)-fF(y).$$

Again, differentiating I_n with regard to y (Art. 355),

$$\frac{dI_n}{dy}=n\int_y^{F^{-1}(z)}[y+x\phi Ft-t]^{n-1}(f'Ft)(F't)\,dt$$
$$-[x\phi Fy]^n(f'Fy)(F'y)$$
$$=nI_{n-1}-x^n(\phi Fy)^n\frac{d}{dy}(fFy);$$
$$\therefore\ I_{n-1}=\frac{x^n}{n}(\phi Fy)^n\frac{d}{dy}(fFy)+\frac{1}{n}\frac{d}{dy}I_n.$$

Putting $n=1, 2, 3, \ldots$ successively in this result,

$$I_0=x\ (\phi Fy)\frac{d}{dy}(fFy)+\frac{d}{dy}I_1,$$
$$I_1=\frac{x^2}{2}(\phi Fy)^2\frac{d}{dy}(fFy)+\frac{1}{2}\frac{d}{dy}I_2,$$
$$I_2=\frac{x^3}{3}(\phi Fy)^3\frac{d}{dy}(fFy)+\frac{1}{3}\frac{d}{dy}I_3,$$
$$I_3=\frac{x^4}{4}(\phi Fy)^4\frac{d}{dy}(fFy)+\frac{1}{4}\frac{d}{dy}I_4,$$

etc.;

whence

$$f(z)-fFy=x(\phi Fy)\frac{d}{dy}(fFy)+\frac{d}{dy}I_1$$
$$=x(\phi Fy)\frac{d}{dy}(fFy)+\frac{x^2}{2!}\frac{d}{dy}\Big[\overline{\phi Fy|}^2\frac{d}{dy}\overline{fFy|}\Big]+\frac{1}{2!}\frac{d^2I_2}{dy^2}$$
$$=\text{etc.},$$

and $$f(z)=fFy+x(\phi Fy)\frac{d}{dy}(fFy)+\frac{x^2}{2!}\frac{d}{dy}\Big[\overline{\phi Fy|}^2\frac{d}{dy}\overline{fFy|}\Big]+\ldots$$
$$+\frac{x^n}{n!}\frac{d^{n-1}}{dy^{n-1}}\Big[\overline{\phi Fy|}^n\frac{d}{dy}\overline{fFy|}\Big]+\frac{1}{n!}\frac{d^n}{dy^n}I_n.$$

The remainder sought is therefore

$$R_{n+1}\equiv\frac{1}{n!}\Big(\frac{d}{dy}\Big)^n\int_y^{F^{-1}(z)}[y+x\phi\{F(t)\}-t]^n f'\{F(t)\}F'(t)\,dt.$$

This includes, as a particular case, the remainder after $(n+1)$ terms in Lagrange's theorem, when $z=y+x\phi(z)$, viz.

$$R_{n+1}\equiv\frac{1}{n!}\left(\frac{d}{dy}\right)^n\int_y^z [y+x\phi(t)-t]^n f'(t)\,dt,$$

cited by Professor Williamson (*Encyclopaedia Britannica*, "Infinitesimal Calculus," § 151) as due to M. Popoff (*Comptes Rendus*, 1861), the demonstration of which by M. Zolotareff, quoted in the *Encyclopaedia Britannica*, is similar to the above.

GENERAL EXAMPLES.

1. Prove that

$$\frac{d}{da}\int_{a^q}^{a^p} a^m x^n\,dx = a^{m-1}\left[\left(\frac{m}{n+1}+p\right)a^{(n+1)p}-\left(\frac{m}{n+1}+q\right)a^{(n+1)q}\right],$$

and verify the result by performing the integration first.

2. If A be the area bounded by a parabola and its latus rectum $(4a)$, prove

(1) by differentiating the integral $4\int_0^a \sqrt{ax}\,dx$ with regard to a,

(2) by first integrating and then differentiating with regard to a, that

$$\frac{dA}{da}=\frac{16a}{3}.$$

3. Apply the method of Art. 355 to prove that

$$\frac{d}{dc}\int_{\frac{c}{2}}^{\frac{c\sqrt{3}}{2}}\sqrt{c^2-x^2}\,dx=\tfrac{1}{6}\pi c,$$

and explain geometrically each step of the process.

Obtain the same result by first integrating and then differentiating the result with regard to c; and also geometrically.

4. Show that if $$u=\int_0^\infty e^{-ax^3-bx^2}\,dx,$$

then $$3ab\frac{\partial^2 u}{\partial b^2}-3a\frac{\partial u}{\partial b}-2b^2\frac{\partial u}{\partial a}=1,$$

provided a be positive. [TRINITY, 1888.]

5. Show that $\dfrac{d^n}{dc^n}\displaystyle\int_{-c}^{c} f(x+c)\,dx=2^n f^{(n-1)}(2c)$. [$\alpha$, 1883.]

6. If $f(x+c)=f(x)$ for all values of x, show that

$$\int_0^c f(y+az)\,dy$$

is independent of z. [α, 1887.]

7. Prove that

$$\int_0^{\frac{\pi}{2}} \frac{dx}{(a^2\cos^2 x + \beta^2 \sin^2 x)^{n+1}} = \frac{\pi}{2^{n+1}} \frac{(-1)^n}{\lfloor n} \left(\frac{1}{a}\frac{\partial}{\partial a} + \frac{1}{\beta}\frac{\partial}{\partial \beta}\right)^n (a\beta)^{-1}.$$

8. Prove that if $u = (a^n + b^n)\int_\beta^\alpha F\{(a-b)x\}\,dx$,

where F denotes any function, β and α being independent of a and b, and n being a positive integer, then

$$\left\{(a^n+b^n)\left(\frac{\partial}{\partial a} + \frac{\partial}{\partial b}\right)^n - 2\lfloor n\right\} u = 0.$$

[OXFORD, 1886.]

9. If
$$u = \int_0^c \phi(x, t)\,dt,$$

where c is a function of u and x, prove that

$$\frac{du}{dx} = \frac{\displaystyle\int_0^c \frac{\partial \phi}{\partial x} dt + \frac{\partial c}{\partial x}\phi(x, c)}{1 - \dfrac{\partial c}{\partial u}\phi(x, c)}.$$

[δ, 1885.]

10. If
$$u = \int_\alpha^\beta \phi(x, y)\,dy,$$

where α and β are functions of x and u, prove that

$$\frac{du}{dx} = \frac{\displaystyle\int_\alpha^\beta \frac{\partial \phi}{\partial x} dy + \phi(x, \beta)\frac{\partial \beta}{\partial x} - \phi(x, \alpha)\frac{\partial \alpha}{\partial x}}{1 - \phi(x, \beta)\dfrac{\partial \beta}{\partial u} + \phi(x, \alpha)\dfrac{\partial \alpha}{\partial u}}.$$

11. Comment upon the application of the rule of Art. 355 to the case

$$\frac{d}{da}\int_{-a}^{a} \frac{\phi(x)\,dx}{\sqrt{a^2 - x^2}}.$$

Prove that in this case the true result is

$$\frac{1}{a}\int_{-a}^{a} \frac{x\phi'(x)\,dx}{\sqrt{a^2-x^2}}.$$

12. If
$$u = \int_{f(\alpha)}^{F(\alpha)} \phi(\theta, \alpha)\,d\theta,$$

we have $\displaystyle \frac{du}{d\alpha} = \int_{f(\alpha)}^{F(\alpha)} \frac{\partial \phi(\theta, \alpha)}{\partial \alpha} d\theta + \phi\{F(\alpha), \alpha\}\frac{dF}{d\alpha} - \phi\{f(\alpha), \alpha\}\frac{df}{d\alpha}.$

Do you consider that this formula fails in the case in which

$$F(\alpha) = \alpha, \quad f(\alpha) = 0 \quad \text{and} \quad \phi(\theta, \alpha) = \frac{1}{\sqrt{\cos\theta - \cos\alpha}}\,?$$

If so, to what extent and in what respect?

Prove that in this case

$$\frac{du}{d\alpha}=\frac{\sin\alpha}{2\sqrt{2}}\int_0^{\frac{\pi}{2}}\frac{\sin^2\theta\,d\theta}{\left(1-\sin^2\frac{\alpha}{2}\sin^2\theta\right)^{\frac{3}{2}}}.$$

Make any remarks that occur to you as to the reasons for the peculiar form which the general formula assumes in this case.

[ϵ, 1884.]

13. Show that the equation

$$\frac{d}{dx}\int_0^1 x^3 e^{x^2\left(1-\frac{1}{t}\right)}\frac{dt}{t^2}=\int_0^1\frac{\partial}{\partial x}\left[x^3 e^{x^2\left(1-\frac{1}{t}\right)}\frac{1}{t^2}\right]dt$$

ceases to hold for $x=0$. [MATH. TRIPOS, 1897.]

14. Find a curve in which the abscissa of the centroid of the area of that portion bounded by the curve, the coordinate axes and an ordinate is proportional to the abscissa of the bounding ordinate.

[COLLEGES, 1878.]

15. A vessel in the form of a right circular cylinder with vertical axis and a flat horizontal base is filled to varying depths with liquid of varying density. If the depth of the centre of gravity of the liquid be always $\frac{1}{n}$ of the immersed portion of the axis, show that the density varies as $(\text{depth})^{\frac{2-n}{n-1}}$.

16. Find the general equation of all solids of revolution for which the distance from the vertex of the centroid of a segment made by a plane perpendicular to the axis, is proportional to the height of the segment. [TODHUNTER, *Integral Calculus*, p. 198.]

17. Find the form of the curve for which the area bounded by the curve, the coordinate axes and an ordinate is such that the moments of inertia of this area about the coordinate axes are in a constant ratio.

18. A body moves from rest at a distance a towards a centre of attraction varying inversely as the distance. Show that the time of describing the space between βa and $\beta^n a$ will be a maximum when

$$\beta n^{\frac{1}{2(n-1)}}=1$$

$$\left[\text{It may be assumed that }\left(\frac{dx}{dt}\right)^2\propto\log\frac{a}{x}.\right]$$

[TAIT AND STEELE, *Dynamics of a Particle*.]

19. Find the density of a parabolic plate as a function of the abscissa in order that the distance of the centroid from the vertex may vary as the square root of the length of the plate. [α, 1881.]

20. Find the equation of the curve such that the area included by the ordinate at any point, the axis of x and the curve is in a constant ratio to the area included by the ordinate, the axis of x and the tangent. [MATH. TRIPOS, 1882.]

21. Prove that

$$\frac{d}{da}\int_0^a \frac{F(x)\,dx}{\sqrt{a-x}} = \int_0^a \frac{F(x) + 2x\dfrac{dF(x)}{dx}}{2a\sqrt{a-x}}\,dx.$$

Under what circumstances will $\displaystyle\int_0^a \frac{F(x)\,dx}{\sqrt{a-x}}$ be independent of a?

[TODHUNTER, *Int. Calc.*]

22. If $\qquad \tan\frac{\pi}{8}\sin\phi = x\sqrt{\frac{1-x^2}{1+x^2}} = \sin\psi,$

verify that

$$\int_0^x \frac{dx}{\sqrt{1-x^8}} = \frac{1}{2\sqrt{2}}\int_0^\phi \frac{d\phi}{\sqrt{1-\tan^2\frac{\pi}{8}\sin^2\phi}} + \sin^2\frac{\pi}{8}\int_0^\psi \frac{d\psi}{\sqrt{1-\tan^2\frac{\pi}{8}\sin^2\psi}}.$$

[MATH. TRIPOS, 1896.]

23. Prove that

$$b^2\int_0^\infty \frac{\cos bx}{a^2+x^2}\,dx + 6\int_0^\infty \frac{\cos bx}{(a^2+x^2)^2}\,dx - 8a^2\int_0^\infty \frac{\cos bx}{(a^2+x^2)^3}\,dx = 0.$$

[ϵ, 1884.]

24. Verify that

$$y = A\int_0^1 v^{\beta-1}(1-v)^{\gamma-\beta-1}(1-xv)^{-\alpha}\,dv$$

satisfies the differential equation of the hypergeometric series, viz.

$$x(1-x)\frac{d^2y}{dx^2} + \{\gamma - (\alpha+\beta+1)x\}\frac{dy}{dx} - \alpha\beta y = 0,$$

when $\beta > 0$ and $\gamma > \beta$.

25. If $\qquad u = \int_0^\pi e^{qx\cos\theta}\{A + B\log(x\sin^2\theta)\}\,d\theta,$

verify that $\qquad x\frac{d^2u}{dx^2} + \frac{du}{dx} - q^2xu = 0.$

26. Prove that $y=\frac{1}{\pi}\int_0^\pi (x+\cos\phi\sqrt{x^2-1})^n\,d\phi$ satisfies the equation

$$\frac{d}{dx}\left[(x^2-1)\frac{dy}{dx}\right]=n(n+1)y.$$

[ST. JOHN'S, 1892.]

27. Show that the differential equation

$$\frac{d^2u}{dx^2}+u=\frac{a}{x}+\frac{b}{x^2}$$

is satisfied by

$$u=\int_0^\infty \frac{a\sin z+b\cos z}{x+z}\,dz.$$

Write down the complete solution. [ST. JOHN'S, 1883.]

28. If $y=\int_0^\pi \sqrt{x}e^{nx\cos\theta}\cos\{\sqrt{7}\log(\sqrt{x}\sin\theta)+\alpha\}d\theta$,

prove that

$$\frac{d^2y}{dx^2}=\left(n^2-\frac{2}{x^2}\right)y.$$

[ST. JOHN'S, 1889.]

29. Prove that

$$\int_0^\pi (\cosh x-\sinh x\cos\phi)^{\frac{2n-1}{2}}\,d\phi=\int_0^\pi \frac{d\phi}{(\cosh x-\sinh x\cos\phi)^{\frac{2n+1}{2}}}.$$

Prove also that if

$$P=\int_0^\pi (\cosh x-\sinh x\cos\phi)^{\frac{1}{2}}\,d\phi,$$

$$\frac{dP}{dx}=\frac{1}{2}\int_0^\pi \frac{\cos\phi\,d\phi}{(\cosh x-\sinh x\cos\phi)^{\frac{3}{2}}}.$$

[α, 1886.]

30. If $\frac{y}{x}=\int_0^{\frac{\pi}{2}}\cos(mx^n\sin\phi)\cos^{\frac{1}{n}}\phi\,d\phi$, prove that y satisfies the equation

$$\frac{d^2y}{dx^2}+m^2n^2x^{2n-2}y=0.$$

[α, 1886.]

31. Verify that

$$y=\int e^{ux}V\left[A+B\log\{U_1(a_2+b_2x)\}\right]du,$$

where $U_1=b_2u^2+b_1u+b_0$, $\log VU_1=\int\frac{a_2u^2+a_1u+a_0}{U_1}\,du$,

and the limits are given by $e^{ux}VU_1=0$, satisfies the equation

$$(a_2+b_2x)\frac{d^2y}{dx^2}+(a_1+b_1x)\frac{dy}{dx}+(a_0+b_0x)\,y=0,$$

provided $a_1b_2-a_2b_1=b_2^2$. [SPITZER, *Crelle*, vol. liv.]

32. Verify that if x be positive

$$u = C_1 \int_{-q}^{q} e^{xt}(t^2 - q^2)^{\frac{a}{2}-1}\, dt + C_2 \int_{-\infty}^{-q} e^{xt}(t^2 - q^2)^{\frac{a}{2}-1}\, dt,$$

and if x be negative

$$u = C_1 \int_{-q}^{q} e^{xt}(t^2 - q^2)^{\frac{a}{2}-1}\, dt + C_2 \int_{q}^{\infty} e^{xt}(t^2 - q^2)^{\frac{a}{2}-1}\, dt$$

solves the differential equation

$$x\frac{d^2u}{dx^2} + a\frac{du}{dx} - q^2 xu = 0.$$

[PETZVAL.]

33. Prove that

$$\int_{\frac{\pi}{2}-a}^{\frac{\pi}{2}} \sin\theta \operatorname{arc\,cos}\frac{\cos a}{\sin\theta}\, d\theta = \frac{\pi}{2}(1 - \cos a).$$

[TRINITY, 1886.]

34. Prove that

$$\int_0^1 \log\frac{1+ax}{1-ax}\frac{dx}{x\sqrt{1-x^2}} = \pi \sin^{-1} a.$$

[OXFORD, 1888.]

35. Establish the known result

$$\int_0^\infty \frac{\log x}{(a+x)^2}\, dx = \frac{\log a}{a},$$

and hence prove that when n is a positive integer

$$(\alpha)\ \int_0^\infty \frac{\log x}{(a+x)^{n+2}}\, dx = \frac{1}{(n+1)a^{n+1}}\left\{\log a - \frac{1}{1} - \frac{1}{2} - \frac{1}{3} - \dots - \frac{1}{n}\right\},$$

$$(\beta)\ \int_0^\infty \frac{(\log x)^2}{(a+x)^{n+2}}\, dx = \frac{1}{(n+1)a^{n+1}}\left\{\left(\frac{\pi^2}{3} - \frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \dots - \frac{1}{n^2}\right) + \left(\log a - \frac{1}{1} - \frac{1}{2} - \frac{1}{3} - \dots - \frac{1}{n}\right)^2\right\}.$$

[MATH. TRIPOS, 1883.]

36. If the operator Δ, applied to a function of a, has the effect of changing a to $a+1$, and subtracting the original function, show that

$$\Delta \int_a^b \phi(x, a)\, dx = \int_a^b \Delta\phi(x, a)\, dx,$$

where a and b are independent of a.

Prove that

$$\int_0^\infty e^{-ax}(e^{-x} - 1)^n\, dx = \frac{(-1)^n}{a(a+1)\dots(a+n)}.$$

[BERTRAND, *C.I.*, p. 183.]

37. Given $u=\int_0^\infty \frac{\cos ax}{1+x^2}dx$, differentiating twice we have

$$\frac{d^2u}{da^2}=-\int_0^\infty \frac{x^2\cos ax}{1+x^2}\,dx.$$

But this is indeterminate when x is infinite. Discuss the validity of the differentiation. [BERTRAND, *Cal. Int.*, p. 181.]

38. Is it true that

$$\int_0^1\left[\int_0^1 \frac{a^2-x^2}{(a^2+x^2)^2}\,dx\right]da=\int_0^1\left[\int_0^1 \frac{a^2-x^2}{(a^2+x^2)^2}\,da\right]dx\ ?$$

If not, why not? [See Art. 1899, Vol. II.]

Evaluate each side separately and compare the results.

[BERTRAND, *Cal. Int.*, p. 187.]

39. If $P+\iota Q=\phi(x+\iota y)$, show that in general

$$\int_a^b\int_\alpha^\beta \frac{\partial Q}{\partial y}\,dx\,dy=\int_\alpha^\beta\int_a^b \frac{\partial P}{\partial x}\,dy\,dx$$

and
$$\int_a^b\int_\alpha^\beta \frac{\partial P}{\partial y}\,dx\,dy=-\int_\alpha^\beta\int_a^b \frac{\partial Q}{\partial x}\,dy\,dx.$$

Examine the case $\phi(x+\iota y)=e^{-(x+\iota y)^2}$, taking $a=0$, $\alpha=0$ and $b=\infty$.

[BERTRAND.]

40. If $f(x)=(2-x)^{\frac{1}{2}}\int_0^{\frac{\pi}{2}}(1-x\sin^2\theta)^{-\frac{1}{2}}\,d\theta$, show that

$$\frac{df(x)}{dx}=\frac{1}{2}(2-x)^{-\frac{1}{2}}\int_0^{\frac{\pi}{4}}\cos 2\theta\left[(1-x\cos^2\theta)^{-\frac{3}{2}}-(1-x\sin^2\theta)^{-\frac{3}{2}}\right]d\theta.$$

Hence show that as x increases from 0 to 1, $f(x)$ increases from $\frac{\pi}{\sqrt{2}}$ to ∞. [C. S., 1898.]

41. Prove that

$$\int_0^u du\int_0^u du\int_0^u du\ldots\int_0^u du\,f(u)=\frac{1}{(n-1)!}\int_0^u (u-z)^{n-1}f(z)\,dz,$$

there being n integration signs in the left member of the equality.

[R. P.]

42. Show that

$$\frac{d^2}{dc^2}\left\{\int_0^c\int_0^c \phi(x+y+c)\,dx\,dy\right\}=9\phi(3c)-8\phi(2c)+\phi(c).$$

[OXF. II. P., 1890.]

43. Show that the quartic function

$$Q \equiv ax^4 + 4bx^3 + 6cx^2 + 4dx + e$$

can, in general, be expressed in three different ways as the sum of two squares $P^2 + R^2$, where

$$P \equiv a^{-\frac{3}{2}}[(ax+b)^2 + 3(ac-b^2) - 2\lambda]$$

and $$R \equiv a^{-\frac{3}{2}}\lambda^{-\frac{1}{2}}[2(ax+b)\lambda + a^2d - 3abc + 2b^3],$$

λ having any one of three determinate values λ_1, λ_2, λ_3.

Verify the evaluation of the integral $\int \frac{dx}{Q}$ in the form

$$\frac{a^2}{4}\left[(\lambda_2-\lambda_3)\lambda_1^{\frac{1}{2}}\tan^{-1}\frac{R_1}{P_1} + (\lambda_3-\lambda_1)\lambda_2^{\frac{1}{2}}\tan^{-1}\frac{R_2}{P_2}\right.$$
$$\left.+ (\lambda_1-\lambda_2)\lambda_3^{\frac{1}{2}}\tan^{-1}\frac{R_3}{P_3}\right]\Big/\Lambda,$$

where $$\Lambda = (\lambda_2-\lambda_3)(\lambda_3-\lambda_1)(\lambda_1-\lambda_2).$$

[MATH. TRIP., 1897.]

44. Show that

$$\int_0^a x^{n-1}(1-a+x)^{-n-r}dx = \frac{(n-1)!}{(n+r-1)!}\left(\frac{d}{da}\right)^{r-1}\frac{a^{n+r-1}}{1-a}$$

[I. C. S., 1892.]