CHAPTER XI.

PRELIMINARY TO INTEGRATION OF $\int \frac{M}{N}\frac{dx}{\sqrt{Q}}$, WHERE Q IS A RATIONAL QUARTIC. DEFINITIONS OF ELLIPTIC FUNCTIONS. ELEMENTARY CONSIDERATIONS.

367. In many problems of both pure and applied mathematics, such as the investigation of the length of an arc of an ellipse, or of a lemniscate, or the time of a finite oscillation of an ordinary simple circular pendulum, integrals occur in which the integrand contains a square root of an algebraic function of higher degree than the second.

Now the integral $$\int \frac{dx}{\sqrt{Q}},$$
where Q is the general biquadratic function
$$a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4,$$
cannot *in general* be integrated by means of the circular, inverse circular, or inverse hyperbolic functions, though it has been seen that for particular values of the coefficients this may be possible; for no such function is known which will, on differentiation, give rise to the general expression $\frac{1}{\sqrt{Q}}$ as its differential coefficient.

Hence, in discussing such an integral as this, we are in a position similar to that which would have occurred if we had required the integral $\int \frac{dx}{\sqrt{a+bx+cx^2}}$ before the inverse circular or inverse hyperbolic functions had been discovered. The integration even of the case $\int \frac{dx}{\sqrt{1-x^2}}$ would then have presented a difficulty. And the necessity for the consideration of such an integral would have formed a suitable starting-point for the investigation of such functions as would have $\frac{1}{\sqrt{1-x^2}}$, or, mcre generally, $\frac{1}{\sqrt{a+bx+cx^2}}$ for their differential coefficients.

And the whole theory of such functions could have been built up from this starting-point.

368. For instance, let $F(x)=\int_0^x \frac{dw}{\sqrt{1-w^2}}$.

Then $F(0)=0$.

Let x and y be two variables connected by the equation

$$\frac{dx}{\sqrt{1-x^2}}+\frac{dy}{\sqrt{1-y^2}}=0,$$

i.e. $$F'(x)\,dx+F'(y)\,dy=0.$$

The integral is $F(x)+F(y)=\text{constant}=F(z)$, say, where z is the value of y when x vanishes.

But multiplying by $\sqrt{1-x^2}\,\sqrt{1-y^2}$,

$$dx\sqrt{1-y^2}+dy\sqrt{1-x^2}=0,$$

and we can integrate this by parts, viz.

$$x\sqrt{1-y^2}+\int x\frac{y}{\sqrt{1-y^2}}\,dy+y\sqrt{1-x^2}+\int y\frac{x}{\sqrt{1-x^2}}\,dx=\text{constant}=C,$$

i.e. $$x\sqrt{1-y^2}+y\sqrt{1-x^2}+\int xy\left(\frac{dx}{\sqrt{1-x^2}}+\frac{dy}{\sqrt{1-y^2}}\right)=C,$$

and the part under the integration sign vanishes.

Hence, $x\sqrt{1-y^2}+y\sqrt{1-x^2}=z$, say, where z is the value of y if x vanishes.

Hence we have the addition equation

$$F(x)+F(y)=F(x\sqrt{1-y^2}+y\sqrt{1-x^2}),$$

and if we then choose to write $\sin^{-1}$ (a supposed unknown symbol) for F, we should have

$$\sin^{-1}x+\sin^{-1}y=\sin^{-1}(x\sqrt{1-y^2}+y\sqrt{1-x^2}),$$

or writing $\sin^{-1}x=\theta$ and $\sin^{-1}y=\phi$,

$$\sin(\theta+\phi)=\sin\theta\sqrt{1-\sin^2\phi}+\sin\phi\sqrt{1-\sin^2\theta},$$

and we should thus have arrived at one of the fundamental propositions of trigonometry, and could have built up the general theory.

Such is actually our position with regard to the integration of $\int \frac{dx}{\sqrt{Q}}$, or, more generally, $\int \frac{M}{N}\frac{dx}{\sqrt{Q}}$, where M and N are rational integral algebraic functions of x, and Q is a rational integral algebraic polynomial of degree higher than the second, say the quartic

$$Q \equiv a_0x^4 + 4a_1x^3 + 6a_2x^2 + 4a_3x + a_4,$$

and the absence of knowledge of any function which, upon differentiation, would give a general result of this kind long barred the progress of geometers.

369. It was natural that after having exhausted the discussion of integrations which could be expressed algebraically or by means of logarithms, or by inverse circular functions, that is in terms of arcs of a circle, that investigators should turn their attention to such expressions as could be integrated by means of arcs of an ellipse or a hyperbola. Thus Colin Maclaurin, in his *Fluxions*, vol. ii., Art. 799, of date 1742, discusses "the fluent of $\frac{\dot{x}\sqrt{x}}{2\sqrt{xx-1}}$," or as it would now be written $\frac{1}{2}\int \frac{\sqrt{x}\,dx}{\sqrt{x^2-1}}$, *i.e.* $\frac{1}{2}\int \frac{x\,dx}{\sqrt{x(x^2-1)}}$, which he expresses as the arc of a rectangular hyperbola of semi-axis unity, viz. drawing a tangent at the vertex A of the hyperbola, centre C, and a circle with the same centre and radius x cutting the tangent at the point M, then letting the bisector of $A\hat{C}M$ cut the hyperbola at E, arc $AE = \frac{1}{2}\int \frac{x\,dx}{\sqrt{x(x^2-1)}}$, which we leave to the student to verify.

370. The real starting-point of the general theory of such integrals, which have been termed Elliptic Integrals, from their intimate connexion with that curve, may be taken to be Fagnano's discovery,* that upon every ellipse or hyperbola it is possible to assign in an infinite number of ways two

* Fagnano, *Produzioni matematiche*, tom. ii.

arcs whose difference is equal to an algebraic expression, and that the lemniscate "jouit de cette singulière propriété, que ses arcs peuvent être multipliés ou divisés algébriquement, comme les arcs de cercle, quoique chacun d'eux soit une transcendante d'un ordre supérieur."*

371. Definitions. Various mathematicians, Euler,† Lagrange,‡ Landen § and others, turned their attention to this matter, and much progress was made. But the chief advance was due to the investigations of Legendre, first in his *Mémoires sur les Transcendantes Elliptiques*, 1793, and, after a long interval, in his *Exercices de Calcul Intégral*, 1811. In this last work he treated the general reduction of the integral

$$\int \frac{P\,dx}{\sqrt{Q}},$$

where P is any rational function whatever of x, and Q is the quartic function

$$a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4,$$

showing that in all cases the integration may be made to depend upon that of three fundamental integrals, viz.

$$\left.\begin{aligned} F(\theta, k) &= \int_0^\theta \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \int_0^\theta \frac{d\theta}{\Delta}, \\ E(\theta, k) &= \int_0^\theta \sqrt{1-k^2\sin^2\theta}\,d\theta = \int_0^\theta \Delta\,d\theta, \\ \Pi(\theta, k, n) &= \int_0^\theta \frac{d\theta}{(1+n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}} \\ &= \int_0^\theta \frac{d\theta}{(1+n\sin^2\theta)\Delta}, \end{aligned}\right\} \text{where } \Delta=\sqrt{1-k^2\sin^2\theta},$$

which he calls the "Elliptic Integrals of the First, Second and Third kind respectively," k being a real constant quantity less than unity, called the *modulus*, and n any constant whatever.

372. Legendre in a footnote, (pages 18, 19) of the *Exercices* suggested names for these functions, but it does not appear that the names were generally adopted, except as to the initial letter E and Π still used for the second and third. He remarks:

"Ces fonctions réunissent un si grand nombre de propriétés, que

* Legendre, *Exercices de Calcul Intégral*, 1811.

† Euler, *Novi. Com. Petrop.*, tom. vi. et vii.

‡ *Mém. de Turin*, tom. iv.

§ *Math. Memoirs*, by John Landen, 1780.

quand elles seront plus généralement connues, on jugera sans doute nécessaire de leur imposer un nom particulier, et de désigner la fonction de c et ϕ égale à $\int \frac{d\phi}{\Delta}$, comme on désigne l'arc dont le sinus est x, ou le nombre dont le logarithme est y. Il semble qu'on caractériserait assez bien la fonction F en lui donnant le nom de *Nome*, parce que cette fonction a la propriété de régler tout ce qui concerne la comparaison des fonctions elliptiques. Peut-être conviendrait-il en même temps de donner les noms d'*Epinome* et de *Paranome* aux fonctions E et Π que constituent les deux autres espèces."

373. Legendre established addition formulae for each of these functions analogous to the trigonometrical formulae for $\sin(\theta \pm \phi)$, $\cos(\theta \pm \phi)$, whence their whole theory may be deduced, as for the ordinary circular functions of trigonometry, and their numerical values calculated and tabulated for definite values of k and n. This having been done, they are available for numerical use, as in the case of the circular and inverse circular functions.

374. All three of Legendre's standard forms are comprehended in the one formula

$$H, \text{ or } \left[H\right]_0^\theta = \int_0^\theta \frac{A+B\sin^2\theta}{1+n\sin^2\theta}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}.$$

The cases are

$$\begin{aligned} &A=1,\ B=0, &&n=0, &&H\equiv F(\theta, k),\\ &A=1,\ B=-k^2, &&n=0, &&H\equiv E(\theta, k),\\ &A=1,\ B=0, && &&H\equiv \Pi(\theta, k, n). \end{aligned}$$

375. **The "Complete Values." The Real Periodicity.**

The function $\dfrac{A+B\sin^2\theta}{1+n\sin^2\theta}\cdot\dfrac{1}{\sqrt{1-k^2\sin^2\theta}}$

obviously goes through all its values four times, as θ increases from 0 to 2π, and then repeats the same cycle. The values in the second quadrant are merely repetitions of those in the first, passed through in the reverse order.

It is clear then that

$$\left[H\right]_0^{\frac{\pi}{2}} = \left[H\right]_{\frac{\pi}{2}}^{\pi} = \left[H\right]_{\pi}^{\frac{3\pi}{2}} = \left[H\right]_{\frac{3\pi}{2}}^{2\pi} = \text{etc.},$$

and that $\left[H\right]_0^{\theta} = \left[H\right]_{\pi-\theta}^{\pi}.$

We may call $\left[H\right]_0^{\frac{\pi}{2}}$ the quarter period of the integral H.

In the case of the first elliptic integral, this "complete" integral $\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$ is denoted by F_1 or K, and called the *real quarter period of* $F(\theta, k)$.

Similarly, E_1 and Π_1 are written for the "complete" integral of the second and third kinds respectively, *i.e.* when the limits are 0 and $\frac{\pi}{2}$, and E_1, Π_1 are the respective quarter periods of $E(\theta, k)$ and $\Pi(\theta, k, n)$.

$$\text{Thus} \quad \int_\theta^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} = \left(\int_0^{\frac{\pi}{2}} - \int_0^\theta\right) \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$
$$= K - F$$

$$\left[\text{analogous to } \cos^{-1}x = \frac{\pi}{2} - \sin^{-1}x\right].$$

In this respect these integrals resemble the length of the arc of an ellipse, or of any oval symmetrical about two perpendicular axes. In fact, as will be presently shown, one of them, E, represents the length of an arc of an ellipse measured from the end of the minor axis. And it was this particular fact that led Legendre to style them Elliptic functions.

It will be noticed that the "complete" values are not numerical until the values of k, n are assigned, but are functions of k and n.

376. It is not the object of the present chapter to discuss elliptic functions at length, nor to establish the mode of reduction of $\int \frac{P\,dx}{\sqrt{Q}}$ to one of the above canonical forms. These matters, as well as the addition formulae, will be postponed for later treatment. The present chapter must be regarded as an introductory description of such functions, so that the student will gradually grow accustomed to their use in cases that may appear in treating of the rectification of ellipses and other curves.

377. The Jacobian Notation.

In the integral $u=\int_0^\theta \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$ it is usual to call *the superior limit* θ the amplitude of u, and write it as

$$\theta=\operatorname{am} u,$$

and in accordance with the usual notation for inverse functions

$$u=\operatorname{am}^{-1}\theta.$$

Thus $$\operatorname{am}^{-1}\theta \equiv \int_0^\theta \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}.$$

If $x=\sin\theta$, we have $x=\sin\operatorname{am} u$, which is abbreviated into $x=\operatorname{sn} u$ and $u=\operatorname{sn}^{-1}x$.

Similarly, $\sqrt{1-x^2}=\cos\theta=\cos\operatorname{am} u$, abbreviated to $\operatorname{cn} u$;

$$\frac{x}{\sqrt{1-x^2}}=\tan\theta=\tan\operatorname{am} u, \text{ abbreviated to } \operatorname{tn} u.$$

The quantity $\sqrt{1-k^2\sin^2\theta}$, which we have called Δ, may be written $\Delta(\theta)$, (mod. k), or $\Delta(\theta, k)$ when it is necessary to put θ, k in evidence;

$$\therefore \sqrt{1-k^2\sin^2\theta}=\Delta\operatorname{am} u,$$

which is further abbreviated to $\operatorname{dn} u$.

Thus $$\operatorname{dn} u \equiv \Delta\operatorname{am} u \equiv \Delta\theta \equiv \sqrt{1-k^2\sin^2\theta}.$$

The names of these expressions, $\operatorname{sn} u$, $\operatorname{cn} u$, $\operatorname{dn} u$, are spoken as spelt, *i.e.* each letter read off.

378. Differentiation.

From the integral itself $\frac{du}{d\theta}=\frac{1}{\sqrt{1-k^2\sin^2\theta}}=\frac{1}{\operatorname{dn} u}$

Hence we can differentiate each of these functions.

Thus

$$\frac{d}{du}\operatorname{sn} u=\frac{d}{du}\sin\theta=\cos\theta\frac{d\theta}{du}=\operatorname{cn} u\operatorname{dn} u,$$

$$\frac{d}{du}\operatorname{cn} u=\frac{d}{du}\cos\theta=-\sin\theta\frac{d\theta}{du}=-\operatorname{sn} u\operatorname{dn} u,$$

$$\frac{d}{du}\operatorname{dn} u=\frac{d}{du}\sqrt{1-k^2\sin^2\theta}=-\frac{k^2\sin\theta\cos\theta}{\sqrt{1-k^2\sin^2\theta}}\frac{d\theta}{du}=-k^2\operatorname{sn} u\operatorname{cn} u.$$

It follows that any expression involving such functions may be differentiated by the ordinary rules of differentiation.

379. Integration.

Conversely, we can integrate various forms involving such functions.

Thus $$\int \operatorname{cn} u \operatorname{dn} u \, du = \operatorname{sn} u,$$

$$\int \operatorname{sn} u \operatorname{dn} u \, du = -\operatorname{cn} u,$$

$$\int \operatorname{sn} u \operatorname{cn} u \, du = -\frac{1}{k^2} \operatorname{dn} u.$$

380. The **elementary transformations** are merely those of ordinary trigonometry for single angles.

Thus $$\operatorname{cn}^2 u = \cos^2\theta = 1 - \sin^2\theta = 1 - \operatorname{sn}^2 u,$$

$$\operatorname{sn}^2 u = \sin^2\theta = 1 - \cos^2\theta = 1 - \operatorname{cn}^2 u,$$

$$\operatorname{dn}^2 u = 1 - k^2 \sin^2\theta \qquad = 1 - k^2 \operatorname{sn}^2 u,$$

$$\operatorname{tn} u = \frac{\operatorname{sn} u}{\operatorname{cn} u}, \quad \operatorname{ctn} u \equiv \cot \operatorname{am} u = \frac{\operatorname{cn} u}{\operatorname{sn} u} = \frac{1}{\operatorname{tn} u},$$

$$\operatorname{sn}^2 u + \operatorname{cn}^2 u = 1,$$

$$\operatorname{dn}^2 u + k^2 \operatorname{sn}^2 u = 1,$$

etc.

381. If $x = \sin\theta$, $F = \int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$,

which exhibits the quartic nature of the radical.

The equation $u = \int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$ may then be written as

$$x = \operatorname{sn} u, (\text{mod. } k); \quad \text{or as } \operatorname{sn}(u, k);$$

and $$u = \operatorname{sn}^{-1} x, (\text{mod. } k); \quad \text{or} \quad \operatorname{sn}^{-1}(x, k).$$

382. The earlier authors treating of this subject, Legendre, Euler and others, regarded the direct integral u as the function to be studied, and θ as its inverse.

The course followed by all later writers, Abel, Clifford, Ferrers, Cayley, Greenhill and others, is to regard θ as the direct function and u as its inverse.

383. The inverse nature of u is expressed in calling it $\mathrm{am}^{-1}\theta$, and this is in conformity with the simple case where $k=0$, viz.

$$u_1 \equiv \int_0^x \frac{dx}{\sqrt{1-x^2}} = \sin^{-1}x,$$

whilst
$$u_2 \equiv \int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \mathrm{sn}^{-1}(x, k).$$

384. **Complementary Modulus.**

It is desirable to introduce a new quantity k' such that

$$k^2+k'^2=1;$$

k' is called the complementary modulus

385. **Transformations.**

Each of the functions, $\mathrm{sn}\,u$, $\mathrm{cn}\,u$, $\mathrm{dn}\,u$, $\mathrm{tn}\,u$, can be expressed in terms of the others.

If $\mathrm{sn}\,u=x$,
$$\left.\begin{aligned} \mathrm{cn}\,u &= \sqrt{1-x^2} = \sqrt{1-\mathrm{sn}^2 u} \\ \mathrm{dn}\,u &= \sqrt{1-k^2x^2} = \sqrt{1-k^2\,\mathrm{sn}^2 u} \\ \mathrm{tn}\,u &= \frac{x}{\sqrt{1-x^2}} = \frac{\mathrm{sn}\,u}{\mathrm{cn}\,u} = \frac{\mathrm{sn}\,u}{\sqrt{1-\mathrm{sn}^2 u}} \end{aligned}\right\}$$

If $\mathrm{cn}\,u=x$,
$$\begin{aligned} \mathrm{sn}\,u &= \sqrt{1-x^2} = \sqrt{1-\mathrm{cn}^2 u}, \\ \mathrm{dn}\,u &= \sqrt{1-k^2(1-x^2)} = \sqrt{k'^2+k^2\,\mathrm{cn}^2 u}, \\ \mathrm{tn}\,u &= \frac{\sqrt{1-x^2}}{x} = \frac{\sqrt{1-\mathrm{cn}^2 u}}{\mathrm{cn}\,u}. \end{aligned}$$

If $\mathrm{dn}\,u=x$,
$$\begin{aligned} \mathrm{sn}\,u &= \frac{\sqrt{1-x^2}}{k} = \frac{\sqrt{1-\mathrm{dn}^2 u}}{k}, \\ \mathrm{cn}\,u &= \frac{\sqrt{k^2-1+x^2}}{k} = \frac{\sqrt{\mathrm{dn}^2 u-k'^2}}{k}, \\ \mathrm{tn}\,u &= \frac{\sqrt{1-x^2}}{\sqrt{x^2-k'^2}} = \sqrt{\frac{1-\mathrm{dn}^2 u}{\mathrm{dn}^2 u-k'^2}}. \end{aligned}$$

If $\mathrm{tn}\,u=x$,
$$\begin{aligned} \mathrm{sn}\,u &= \frac{x}{\sqrt{1+x^2}} = \frac{\mathrm{tn}\,u}{\sqrt{1+\mathrm{tn}^2 u}}, \\ \mathrm{cn}\,u &= \frac{1}{\sqrt{1+x^2}} = \frac{1}{\sqrt{1+\mathrm{tn}^2 u}}, \\ \mathrm{dn}\,u &= \frac{\sqrt{1+k'^2x^2}}{\sqrt{1+x^2}} = \sqrt{\frac{1+k'^2\,\mathrm{tn}^2 u}{1+\mathrm{tn}^2 u}}. \end{aligned}$$

386. Inverse Notation.

With the inverse notation the same formulae would be written

$$\mathrm{sn}^{-1}x=\mathrm{cn}^{-1}\sqrt{1-x^2}=\mathrm{dn}^{-1}\sqrt{1-k^2x^2}=\mathrm{tn}^{-1}\frac{x}{\sqrt{1-x^2}},$$

$$\mathrm{cn}^{-1}x=\mathrm{sn}^{-1}\sqrt{1-x^2}=\mathrm{dn}^{-1}\sqrt{k'^2+k^2x^2}=\mathrm{tn}^{-1}\frac{\sqrt{1-x^2}}{x},$$

$$\mathrm{dn}^{-1}x=\mathrm{sn}^{-1}\frac{\sqrt{1-x^2}}{k}=\mathrm{cn}^{-1}\frac{\sqrt{x^2-k'^2}}{k}=\mathrm{tn}^{-1}\frac{\sqrt{1-x^2}}{\sqrt{x^2-k'^2}},$$

$$\mathrm{tn}^{-1}x=\mathrm{sn}^{-1}\frac{x}{\sqrt{1+x^2}}=\mathrm{cn}^{-1}\frac{1}{\sqrt{1+x^2}}=\mathrm{dn}^{-1}\frac{\sqrt{1+k'^2x^2}}{\sqrt{1+x^2}}.$$

387. Ex. $\mathrm{cn}^{-1}\left(\sqrt{\frac{\cos 2\theta+\cos 2\beta}{1+\cos 2\beta}},\ \sqrt{\frac{1+\cos 2\beta}{2}}\right)$

$$=\mathrm{sn}^{-1}\left(\sqrt{1-\frac{\cos 2\theta+\cos 2\beta}{1+\cos 2\beta}},\ \cos\beta\right)$$

$$=\mathrm{sn}^{-1}\left(\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\beta}},\ \cos\beta\right)$$

$$=\mathrm{sn}^{-1}\left(\frac{\sin\theta}{\cos\beta},\ \cos\beta\right).$$

Similarly

$$\mathrm{cn}^{-1}\left(\sqrt{\frac{\cos 2\theta-\cos 2\beta}{1-\cos 2\beta}},\ \sqrt{\frac{1-\cos 2\beta}{2}}\right)=\mathrm{sn}^{-1}\left(\frac{\sin\theta}{\sin\beta},\ \sin\beta\right).$$

388. Illustrative Examples of Reduction to the Legendrian Form.

1. Consider $\quad I\equiv\int_0^x\frac{dx}{\sqrt{(a^2-x^2)(b^2-x^2)}}\quad (x<b<a).$

Let $x=b\sin\theta$,

$$I=\int_0^\theta\frac{b\cos\theta\,d\theta}{\sqrt{(a^2-b^2\sin^2\theta)\,b^2\cos^2\theta}}$$

$$=\frac{1}{a}\int_0^\theta\frac{d\theta}{\sqrt{1-\frac{b^2}{a^2}\sin^2\theta}}=\frac{1}{a}F\left(\theta,\ \frac{b}{a}\right).$$

$$\theta=\mathrm{am}\,(aI),$$

$$x=b\sin\theta=b\,\mathrm{sn}\,(aI)\ ;\quad \text{mod. } \frac{b}{a},$$

$$I=\frac{1}{a}\ \mathrm{sn}^{-1}\left(\frac{x}{b},\ \frac{b}{a}\right).$$

2. Consider the case $I=\int_x^1\frac{dx}{\sqrt{1-x^4}}$.

Put $x=\cos\theta$,

$$I=\int_\theta^0 \frac{-\sin\theta\,d\theta}{\sin\theta\sqrt{1+\cos^2\theta}}$$

$$=\int_0^\theta \frac{d\theta}{\sqrt{2-\sin^2\theta}}=\frac{1}{\sqrt{2}}\int_0^\theta \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}}$$

$$=\frac{1}{\sqrt{2}}F\left(\theta,\ \frac{1}{\sqrt{2}}\right);$$

$$\theta=\operatorname{am}(I\sqrt{2});\ \text{mod.}\ \frac{1}{\sqrt{2}},$$

$$x=\operatorname{cn} I\sqrt{2},$$

$$I=\frac{1}{\sqrt{2}}\operatorname{cn}^{-1}\left(x,\ \frac{1}{\sqrt{2}}\right).$$

3. Consider

$$I\equiv\int_x^\infty \frac{dx}{\sqrt{4(x-a)(x-b)(x-c)}},\quad \text{where } a<b<c<x.$$

Let $x-a=(c-a)\operatorname{cosec}^2\theta$.

Then $$I=\int_\theta^0 \frac{-(c-a)\,2\operatorname{cosec}^2\theta\cot\theta\,d\theta}{\sqrt{4(c-a)\operatorname{cosec}^2\theta\{(c-a)\operatorname{cosec}^2\theta-(b-a)\}\{(c-a)\cot^2\theta\}}}$$

$$=\int_0^\theta \frac{d\theta}{\sqrt{(c-a)-(b-a)\sin^2\theta}}$$

$$=\frac{1}{\sqrt{c-a}}\int_0^\theta \frac{d\theta}{\sqrt{1-\frac{b-a}{c-a}\sin^2\theta}}$$

$$=\frac{1}{\sqrt{c-a}}F\left(\theta,\ \sqrt{\frac{b-a}{c-a}}\right),$$

$$\theta=\operatorname{am}(\sqrt{c-a}\,I);\ \text{mod.}\ \sqrt{\frac{b-a}{c-a}},$$

$$\sqrt{\frac{c-a}{x-a}}=\sin\theta=\operatorname{sn}(\sqrt{c-a}\,I);$$

$$\therefore\ I=\frac{1}{\sqrt{c-a}}\operatorname{sn}^{-1}\left(\sqrt{\frac{c-a}{x-a}},\ \sqrt{\frac{b-a}{c-a}}\right).$$

4. Consider the case

$$I\equiv\int_x^1 \frac{dx}{\sqrt{(1-x^2)(x+\lambda)}},\quad (\lambda<1).$$

Put $x+\lambda=(1+\lambda)\cos^2\phi$.

Thus,

$$I=\int_\phi^0 \frac{-(1+\lambda)\,2\cos\phi\sin\phi\,d\phi}{\sqrt{\{1-\lambda+(1+\lambda)\cos^2\phi\}\{(1+\lambda)-(1+\lambda)\cos^2\phi\}(1+\lambda)\cos^2\phi}}$$

$$=2\int_0^\phi \frac{d\phi}{\sqrt{2-(1+\lambda)\sin^2\phi}}$$

$$=\sqrt{2}\int_0^\phi \frac{d\phi}{\sqrt{1-\frac{1+\lambda}{2}\sin^2\phi}}=\sqrt{2}\,F\left(\phi,\ \sqrt{\frac{1+\lambda}{2}}\right);$$

$$\therefore \ \phi = \operatorname{am}\left(\frac{I}{\sqrt{2}}\right), \quad \cos\phi = \operatorname{cn}\left(\frac{I}{\sqrt{2}}\right),$$

$$I = \sqrt{2}\,\operatorname{cn}^{-1}\left(\sqrt{\frac{x+\lambda}{1+\lambda}},\ \sqrt{\frac{1+\lambda}{2}}\right).$$

If $x = \cos 2\theta$ and $\lambda = \cos 2\beta$,

$$I = \sqrt{2}\,\operatorname{sn}^{-1}\left(\frac{\sin\theta}{\cos\beta},\ \cos\beta\right)\!\Big). \quad \text{(Art. 387.)}$$

Similarly,

$$\int_x^1 \frac{dx}{\sqrt{(1-x^2)(x-\lambda)}} = \sqrt{2}\,\operatorname{cn}^{-1}\left(\sqrt{\frac{x-\lambda}{1-\lambda}},\ \sqrt{\frac{1-\lambda}{2}}\right)$$

$$= \sqrt{2}\,\operatorname{sn}^{-1}\left(\frac{\sin\theta}{\sin\beta},\ \sin\beta\right).$$

These integrals are useful in the rectification of a Cassinian oval.

5. Consider the integration

$$I \equiv \int_0^x \sqrt{\frac{a^2-x^2}{c^2-x^2}}\,dx, \quad x < c < a.$$

Putting $x = c\sin\theta$,

$$I = \int_0^\theta \sqrt{a^2 - c^2\sin^2\theta}\,d\theta = a\int_0^\theta \sqrt{1 - \frac{c^2}{a^2}\sin^2\theta}\,d\theta$$

$$= aE\left(\theta, \frac{c}{a}\right).$$

6. Consider the integration

$$I \equiv \int_c^x \sqrt{\frac{x^2-a^2}{x^2-c^2}}\,dx, \quad \text{where } x > c > a.$$

Here we may put

$$\frac{x^2-c^2}{x^2-a^2} = \sin^2\omega.$$

Then
$$x = \frac{\sqrt{c^2-a^2\sin^2\omega}}{\cos\omega},$$

and
$$\frac{dx}{d\omega} = \frac{(c^2-a^2)\sin\omega}{\cos^2\omega\sqrt{c^2-a^2\sin^2\omega}};$$

$$\therefore \ I = \int_0^\omega \sec^2\omega\,\frac{c^2-a^2}{\sqrt{c^2-a^2\sin^2\omega}}\,d\omega$$

$$= \int_0^\omega \sec^2\omega\,\frac{c^2-a^2\sin^2\omega-a^2\cos^2\omega}{\sqrt{c^2-a^2\sin^2\omega}}\,d\omega$$

$$= \int_0^\omega \sec^2\omega\sqrt{c^2-a^2\sin^2\omega}\,d\omega - \int_0^\omega \frac{a^2\,d\omega}{\sqrt{c^2-a^2\sin^2\omega}}$$

$$= \tan\omega\sqrt{c^2-a^2\sin^2\omega} + \int_0^\omega \frac{a^2\sin^2\omega - a^2}{\sqrt{c^2-a^2\sin^2\omega}}\,d\omega$$

$$= \tan\omega\sqrt{c^2-a^2\sin^2\omega} + \int_0^\omega \frac{(c^2-a^2)-(c^2-a^2\sin^2\omega)}{\sqrt{c^2-a^2\sin^2\omega}}\,d\omega$$

$$= \tan \omega \sqrt{c^2 - a^2 \sin^2 \omega} + \frac{(c^2 - a^2)}{c} \int_0^{\omega} \frac{d\omega}{\sqrt{1 - \frac{a^2}{c^2} \sin^2 \omega}}$$

$$- c \int_0^{\omega} \sqrt{1 - \frac{a^2}{c^2} \sin^2 \omega}\, d\omega$$

$$= \tan \omega \sqrt{c^2 - a^2 \sin^2 \omega} + \frac{c^2 - a^2}{c} F\left(\omega, \frac{a}{c}\right) - cE\left(\omega, \frac{a}{c}\right),$$

the integration needed in the rectification of a hyperbola.

7. Reduce the integral

$$I \equiv \int_{\theta}^{\frac{\pi}{2}} \sqrt{\frac{a^2 \cos^2 \theta + b^2 \sin^2 \theta}{a^2 \cos^2 \theta + b^2 \sin^2 \theta + c^2}}\, d\theta,$$

to Legendrian form, taking $a > b$.

Write $b \tan \theta = a \cot \chi$.

Then $$d\theta = -\frac{a}{b} \frac{\operatorname{cosec}^2 \chi\, d\chi}{1 + \frac{a^2}{b^2} \cot^2 \chi} = -ab \frac{d\chi}{b^2 \sin^2 \chi + a^2 \cos^2 \chi}.$$

Hence

$$I = \int_{\theta}^{\frac{\pi}{2}} \frac{\sqrt{a^2 + b^2 \tan^2 \theta}}{\sqrt{(a^2 + c^2) + (b^2 + c^2) \tan^2 \theta}}\, d\theta$$

$$= -\int_{\chi}^{0} \frac{a \operatorname{cosec} \chi}{\sqrt{(a^2 + c^2) + (b^2 + c^2) \frac{a^2}{b^2} \cot^2 \chi}} \frac{ab\, d\chi}{b^2 \sin^2 \chi + a^2 \cos^2 \chi}$$

$$= \int_0^{\chi} \frac{a^2 b^2\, d\chi}{[a^2 - (a^2 - b^2) \sin^2 \chi] \sqrt{(a^2 + c^2) b^2 \sin^2 \chi + (b^2 + c^2) a^2 \cos^2 \chi}}$$

$$= \int_0^{\chi} \frac{a^2 b^2\, d\chi}{[a^2 - (a^2 - b^2) \sin^2 \chi] \sqrt{(b^2 + c^2) a^2 - (a^2 - b^2) c^2 \sin^2 \chi}}$$

$$= \frac{a^2 b^2}{a^3 \sqrt{(b^2 + c^2)}} \int_0^{\chi} \frac{d\chi}{\left(1 - \frac{a^2 - b^2}{a^2} \sin^2 \chi\right) \sqrt{1 - \frac{a^2 - b^2}{b^2 + c^2} \frac{c^2}{a^2} \sin^2 \chi}};$$

$$\therefore\ I = \frac{b^2}{a \sqrt{b^2 + c^2}} \Pi\left(\chi,\ \frac{c}{a} \sqrt{\frac{a^2 - b^2}{b^2 + c^2}},\ -\frac{a^2 - b^2}{a^2}\right),$$

an integral of the Third Species.

This integral is needed in the rectification and quadrature of a spheroconic.

389. **The Simple Pendulum. Dynamical illustration of the real periodicity of F.**

Consider the *finite oscillation* of a simple circular pendulum. Let θ be the angular displacement of the rod from the vertical at time t, α the extreme value of θ, m the mass of the bob, a the length of the rod. The line of zero-velocity in this case cuts the circle described by the bob at two points A, A' between which the bob oscillates.

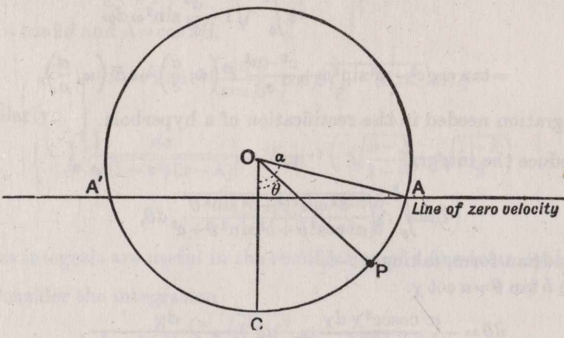

Fig. 38.

The energy equation is

$$\tfrac{1}{2} ma^2\dot{\theta}^2 = mg(a\cos\theta - a\cos\alpha)$$

$$= 2mga\left(\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}\right)$$

giving
$$t = \tfrac{1}{2}\sqrt{\frac{a}{g}}\int_0^\theta \frac{d\theta}{\sqrt{\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}}}$$

t being measured from the instant at which the bob passes through its lowest position.

Let
$$\sin\frac{\theta}{2} = \sin\frac{\alpha}{2}\sin\phi;$$

$$\therefore\ d\theta = \frac{2\sin\frac{\alpha}{2}\cos\phi\, d\phi}{\sqrt{1 - \sin^2\frac{\alpha}{2}\sin^2\phi}};$$

$$\therefore\ t = \sqrt{\frac{a}{g}}\int_0^\phi \frac{d\phi}{\sqrt{1 - \sin^2\frac{\alpha}{2}\sin^2\phi}};$$

$$\therefore\ t = \sqrt{\frac{a}{g}}\,\mathrm{am}^{-1}\phi;\ \left(\mathrm{mod.}\ \sin\frac{\alpha}{2}\right),$$

$$\textit{i.e.}\ \sin\frac{\theta}{2} = \sin\frac{\alpha}{2}\,\mathrm{sn}\left(\sqrt{\frac{g}{a}}\,t\right).$$

When $\theta=\alpha$, and $\therefore\ \phi=\frac{\pi}{2}$, $\dot{\theta}=0$, and the time to this point, viz. T, is given by

$$T=\sqrt{\frac{a}{g}}\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-\sin^2\frac{\alpha}{2}\sin^2\phi}}=\sqrt{\frac{a}{g}}\,F_1(\phi);\ \left(\text{mod.}\sin\frac{\alpha}{2}\right),$$

and is the quarter period of the whole time of a complete oscillation. Writing K for F_1 it appears that the function

$$\int_0^{\phi}\frac{d\phi}{\sqrt{1-\sin^2\frac{\alpha}{2}\sin^2\phi}}$$

is periodic and has a real period $4K$. Thus F_1 or K is called the "quarter period of the integral F," viz.,

$$K=\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}},\quad\text{where } k=\sin\frac{\alpha}{2}.$$

For an indefinitely small oscillation α is infinitesimal and $T=\frac{\pi}{2}\sqrt{\frac{a}{g}}$, the ordinary formula for a small oscillation.

390. **Complete Revolutions.**

Case of the pendulum making complete revolutions.

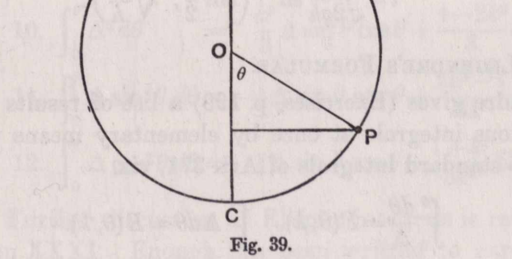

Fig. 39.

In the case when the line of zero velocity is at a height h ($>2a$) above the lowest point and does not cut the circle described by the bob of the pendulum, the velocity of the bob is not exhausted when it arrives at the highest point of its path. The rod then makes complete revolutions and does not oscillate. In this case the energy equation is

$$\tfrac{1}{2} ma^2 \dot{\theta}^2 = mg\,[h - a\overline{1-\cos\theta}];$$

$$\therefore\ \dot{\theta}^2 = \frac{2g}{a}\left(\frac{h}{a} - 2\sin^2\frac{\theta}{2}\right)$$

$$= \frac{2gh}{a^2}\left(1 - \frac{2a}{h}\sin^2\frac{\theta}{2}\right),$$

and
$$\frac{\sqrt{2gh}}{a}\,.\,t = \int_0^\theta \frac{d\theta}{\sqrt{1 - \frac{2a}{h}\sin^2\frac{\theta}{2}}}.$$

Let $\theta = 2\phi$,

$$\frac{\sqrt{2gh}}{2a}t = \int_0^\phi \frac{d\phi}{\sqrt{1 - \frac{2a}{h}\sin^2\phi}},\quad \left(\frac{2a}{h} < 1\right)$$

$$= F\left(\phi, \sqrt{\frac{2a}{h}}\right).$$

The time of a half revolution is given by $\phi = \frac{\pi}{2}$,

and
$$T = \frac{2a}{\sqrt{2gh}} F_1;\ \text{mod.}\ \sqrt{\frac{2a}{h}},$$

$$\frac{\theta}{2} = \phi = \text{am}\,\frac{\sqrt{2gh}}{2a}t,\quad \sin\frac{\theta}{2} = \text{sn}\,\frac{\sqrt{2gh}}{2a}t,$$

$$t = \frac{2a}{\sqrt{2gh}}\,\text{sn}^{-1}\left(\sin\frac{\theta}{2},\ \sqrt{\frac{2a}{h}}\right).$$

391. Legendre's Formulae.

Legendre gives (Exercises, p. 199) a list of results connecting various integrals at once by elementary means with the first two standard integrals of Art. 371, viz.

$$\int_0^\theta \frac{d\theta}{\Delta} = F(\theta, k),\quad \int_0^\theta \Delta\, d\theta = E(\theta, k).$$

These we may usefully reproduce for reference, and they will furnish a useful set of examples for the student to verify.

EXAMPLES (LEGENDRE).

Prove the following twelve results:

1. $\int_0^\theta \frac{d\theta}{\Delta^3} = \frac{1}{k'^2} E(\theta, k) - \frac{k^2}{k'^2} \frac{\sin\theta\cos\theta}{\Delta}$.

[Putting $P = \frac{\sin\theta\cos\theta}{\Delta}$ and differentiating, we obtain, after a little reduction, $k^2 \frac{dP}{d\theta} = \Delta - \frac{k'^2}{\Delta^3}$, then integrating we obtain the result stated.]

2. $\int_0^\theta \frac{\sin^2\theta\, d\theta}{\Delta} = \frac{1}{k^2}(F - E)$.

3. $\int_0^\theta \frac{\cos^2\theta\, d\theta}{\Delta} = \frac{1}{k^2}(E - k'^2 F)$.

4. $\int_0^\theta \frac{\sec^2\theta\, d\theta}{\Delta} = \frac{1}{k'^2}(\Delta\tan\theta + k'^2 F - E)$.

5. $\int_0^\theta \frac{\tan^2\theta\, d\theta}{\Delta} = \frac{1}{k'^2}(\Delta\tan\theta - E)$.

6. $\int_0^\theta \frac{\tan^2\frac{\theta}{2}\, d\theta}{\Delta} = 2\Delta\tan\frac{\theta}{2} + F - 2E$.

7. $\int_0^\theta \frac{\sec^2\frac{\theta}{2}\, d\theta}{\Delta} = 2\Delta\tan\frac{\theta}{2} + 2F - 2E$.

8. $\int_0^\theta \Delta\sec^2\theta\, d\theta = \Delta\tan\theta + F - E$.

9. $\int_0^\theta \Delta\tan^2\theta\, d\theta = \Delta\tan\theta + F - 2E$.

10. $\int_0^\theta \Delta^3 d\theta = \frac{k^2}{3}\Delta\sin\theta\cos\theta + \frac{4-2k^2}{3}E - \frac{k'^2}{3}F$

11. $\int_0^\theta \Delta\sin^2\theta\, d\theta = -\tfrac{1}{3}\Delta\sin\theta\cos\theta + \frac{2k^2-1}{3k^2}E + \frac{k'^2}{3k^2}F$.

12. $\int_0^\theta \Delta\cos^2\theta\, d\theta = \tfrac{1}{3}\Delta\sin\theta\cos\theta + \frac{1+k^2}{3k^2}E - \frac{k'^2}{3k^2}F$.

392. Further discussion of Elliptic integrals is reserved till Chapter XXXI. Enough has been written to explain their nature, and the student will be able to employ the notation when wanted in the intervening chapters.

EXAMPLES.

1. By putting $x=\dfrac{1-\sin\theta}{1+\sin\theta}$, shew that

$$u=\int_x^1 \frac{dx}{\sqrt{x(1+6x+x^2)}}=\frac{1}{\sqrt{2}}\int_0^\theta \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}}=\frac{1}{\sqrt{2}}F\left(\theta, \frac{1}{\sqrt{2}}\right);$$

and that

$$x=\frac{1-\operatorname{sn}(u\sqrt{2})}{1+\operatorname{sn}(u\sqrt{2})}, \left(\text{mod. } \frac{1}{\sqrt{2}}\right); \quad \textit{i.e. } u=\frac{1}{\sqrt{2}}\operatorname{sn}^{-1}\left(\frac{1-x}{1+x}, \frac{1}{\sqrt{2}}\right).$$

2. Prove that

$$F_1(\theta, k)=\frac{\pi}{2}\left(1+\frac{1^2}{2^2}k^2+\frac{1^2 . 3^2}{2^2 . 4^2}k^4+\frac{1^2 . 3^2 . 5^2}{2^2 . 4^2 . 6^2}k^6+\dots\right);$$

and that $\qquad F_1(\theta, \tfrac{1}{10})=1{\cdot}574745$ very nearly.

3. Prove that

$$E_1(\theta, k)=\frac{\pi}{2}\left(1-\frac{1}{2^2}k^2-\frac{1^2 . 3}{2^2 . 4^2}k^4-\frac{1^2 . 3^2 . 5}{2^2 . 4^2 . 6^2}k^6-\dots\right).$$

4. Prove that

$$\Pi_1(\theta, k, n)=\frac{\pi}{2}\left[1+\left(\frac{1}{2}k^2-n\right)\frac{1}{2}+\left(\frac{1 . 3}{2 . 4}k^4-\frac{1}{2}k^2n+n^2\right)\frac{1 . 3}{2 . 4}\right.$$
$$+\left(\frac{1 . 3 . 5}{2 . 4 . 6}k^6-\frac{1 . 3}{2 . 4}k^4n+\frac{1}{2}k^2n^2-n^3\right)\frac{1 . 3 . 5}{2 . 4 . 6}$$
$$\left.+\dots\right] \text{ if } n \text{ be } < 1.$$

5. Establish the truth of

$$(a)\ \left(\operatorname{sn}u+\frac{1}{\operatorname{cn}u}\right)^2+\left(\operatorname{cn}u+\frac{1}{\operatorname{sn}u}\right)^2=\left(1+\frac{1}{\operatorname{sn}u\operatorname{cn}u}\right)^2.$$

$$(b)\ \frac{\dfrac{\operatorname{cn}u}{\operatorname{sn}u}-\dfrac{\operatorname{sn}u}{\operatorname{cn}u}}{\operatorname{cn}u+\operatorname{sn}u}=\frac{1}{\operatorname{sn}u}-\frac{1}{\operatorname{cn}u}$$

$$(c)\ \left(\frac{1}{\operatorname{sn}u}-\operatorname{sn}u\right)\left(\frac{1}{\operatorname{cn}u}-\operatorname{cn}u\right)\left(\frac{\operatorname{sn}u}{\operatorname{cn}u}+\frac{\operatorname{cn}u}{\operatorname{sn}u}\right)=1.$$

$$(d)\ \frac{1-\operatorname{cn}u}{1+\operatorname{cn}u}=\left(\frac{1}{\operatorname{sn}u}-\frac{\operatorname{cn}u}{\operatorname{sn}u}\right)^2.$$

6. Prove that

$$(1)\ \operatorname{dn}^2u-k^2\operatorname{cn}^2u=k'^2,$$

$$(2)\ \frac{1}{\operatorname{cn}^2u}=1+\operatorname{tn}^2u,$$

$$(3)\ \frac{1}{\operatorname{sn}^2u}=1+\frac{1}{\operatorname{tn}^2u}.$$

7. Prove that

(1) $\frac{d}{du}\operatorname{sn}^2 u = 2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u,$

(2) $\int \operatorname{sn}^p u \operatorname{cn} u \operatorname{dn} u \, du = \frac{\operatorname{sn}^{p+1} u}{p+1},$

(3) $\int \frac{\operatorname{dn} u}{a + b \operatorname{cn} u} du = \frac{1}{\sqrt{a^2 - b^2}} \cos^{-1}\left(\frac{a \operatorname{cn} u + b}{a + b \operatorname{cn} u}\right), \quad a > b.$

8. Prove

$$2 \operatorname{sn} u \operatorname{cn} v = \sin(\operatorname{am} u + \operatorname{am} v) + \sin(\operatorname{am} u - \operatorname{am} v),$$
$$2 \operatorname{cn} u \operatorname{cn} v = \cos(\operatorname{am} u + \operatorname{am} v) + \cos(\operatorname{am} u - \operatorname{am} v).$$

9. By putting $x = a \cos\theta$, show that

$$\int_x^a \frac{dx}{\sqrt{a^4 - x^4}} = \frac{1}{a\sqrt{2}} \operatorname{cn}^{-1}\left(\frac{x}{a}, \frac{1}{\sqrt{2}}\right).$$

10. Prove $\int_a^x \frac{dx}{\sqrt{x^4 - a^4}} = \frac{1}{a\sqrt{2}} \operatorname{cn}^{-1}\left(\frac{a}{x}, \frac{1}{\sqrt{2}}\right).$

11. By putting $x = a\sqrt{\frac{1+z}{1-z}}$, show that

$$\int_x^\infty \frac{dx}{\sqrt{a^4 + x^4}} = \frac{1}{2a} \operatorname{cn}^{-1}\left(\frac{x^2 - a^2}{x^2 + a^2}, \frac{1}{\sqrt{2}}\right).$$

12. Prove that

$$\int_x^\infty \frac{dx}{\sqrt{a^4 + 2a^2x^2\cos 2\alpha + x^4}} = \frac{1}{2a} \operatorname{cn}^{-1}\left(\frac{x^2 - a^2}{x^2 + a^2}, \sin\alpha\right).$$

13. Prove that

$$\operatorname{sn} K = 1, \ \operatorname{cn} K = 0, \ \operatorname{dn} K = k', \ \operatorname{tn} K = \infty.$$

14. Prove that

(1) $\frac{d}{du}(\operatorname{sn} u + \operatorname{cn} u)^n = n(\operatorname{sn} u + \operatorname{cn} u)^{n-1}(\operatorname{cn} u - \operatorname{sn} u)\operatorname{dn} u,$

(2) $\int (k^2 \operatorname{sn} u + \operatorname{cn} u)^n (k^2 \operatorname{cn} u - \operatorname{sn} u) \operatorname{dn} u \, du = \frac{(k^2 \operatorname{sn} u + \operatorname{cn} u)^{n+1}}{(n+1)}.$

15. Draw graphs of $y = \Delta\theta$ and $y = \frac{1}{\Delta\theta}$, showing that the former consists of an undulating curve lying entirely below the line $y = 1$ and the other of an undulating line lying entirely above the line $y = 1$. Take the cases $k^2 = \frac{1}{2}$ and $k^2 = \frac{1}{4}$.

Show that the areas bounded by these curves, the x-axis, the y-axis and any ordinate at a point whose abscissa is θ represent $E(\theta)$ and $F(\theta)$ completely. Examine what happens in the limiting cases $k = 0$ and $k = 1$.

16. Show that the complete elliptic integrals of the First and Second Species may be expressed as

$$F_1 = \frac{\pi}{2} f(\tfrac{1}{2}, \tfrac{1}{2}, 1, k^2),$$

$$E_1 = \frac{\pi}{2} f(-\tfrac{1}{2}, \tfrac{1}{2}, 1, k^2),$$

where $f(a, b, c, x)$ is the hypergeometric series

$$1 + \frac{a \,.\, b}{1 \,.\, c} x + \frac{a\,\overline{a+1}\,b\,\overline{b+1}}{1 \,.\, 2\, c\,\overline{c+1}} x^2 + \dots.$$

17. Show by differentiating $F(\theta, k)$ and $E(\theta, k)$ with regard to k

$$\left.\begin{aligned} &(1)\quad \frac{dE}{dk} = \frac{1}{k}(E - F), \\ &(2)\quad \frac{dF}{dk} = \frac{1}{kk'^2}(E - k'^2 F) - \frac{k}{k'^2}\frac{\sin\theta\cos\theta}{\Delta}. \end{aligned}\right\}$$

Hence, eliminating E and F alternately, show that

$$\left.\begin{aligned} &(1-k^2)\frac{d^2F}{dk^2} + \frac{1-3k^2}{k}\frac{dF}{dk} - F + \frac{\sin\theta\cos\theta}{\Delta^3} = 0, \\ &(1-k^2)\frac{d^2E}{dk^2} + \frac{1-k^2}{k}\frac{dE}{dk} + E - \frac{\sin\theta\cos\theta}{\Delta} = 0, \end{aligned}\right\}$$

and for the complete functions F_1, E_1

$$\left.\begin{aligned} &(1-k^2)\frac{d^2F_1}{dk^2} + \frac{1-3k^2}{k}\frac{dF_1}{dk} - F_1 = 0, \\ &(1-k^2)\frac{d^2E_1}{dk^2} + \frac{1-k^2}{k}\frac{dE_1}{dk} + E_1 = 0. \end{aligned}\right\}$$