CHAPTER XII.

QUADRATURE (I).

PLANE SURFACES, CARTESIAN AND POLAR EQUATIONS.

393. The process of finding the area bounded by any defined contour line is termed **Quadrature**, or, which amounts to the same thing, Quadrature is the investigation of the *size of a square* which shall have the same area as that of the region under consideration.

The closed contour may consist of a single curve or of a system of several arcs of different curves or straight lines.

As we shall, in most cases, have to form some rough idea of the shape of the curves under discussion so as to be able properly to assign the limits of integration, the student should be familiar with the rules of procedure adopted in the tracing of curves for the various systems of coordinates by which they may be defined, Cartesians, Polars, etc., and for such information may be referred to the author's treatise on the Differential Calculus, Chap. XII.

394. It has been already shown (Art. 11) that the area bounded by a curve whose equation is $y=\phi(x)$, any pair of ordinates, $x=a$ and $x=b$ and the x-axis, may be considered as the limit of the sum of an infinite number of inscribed rectangles; and that the expression for the area is

$$\int_a^b y\,dx, \quad \text{or} \quad \int_a^b \phi(x)\,dx\,;$$

and it was assumed that $\phi(x)$ is a finite and continuous function of x, which does not change sign between these limits. In the same way the area bounded by the curve, two given abscissae, $y=c$ and $y=d$, and the y-axis is $\int_c^d x\,dy$.

If the angle between the coordinate axes were ω instead of 90°, we should have the expressions

$$\sin\omega\int_a^b y\,dx, \quad \text{or} \quad \sin\omega\int_c^d x\,dy$$

for the area.

395. Again, if the area desired be bounded by two given curves $y=\phi(x)$ and $y=\psi(x)$, and two given ordinates $x=a$ and $x=b$, it will be clear by similar reasoning that this area

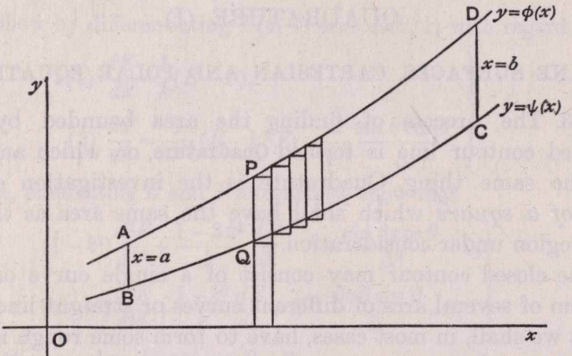

Fig. 40.

may be also considered as the limit of the sum of a series of rectangles constructed as indicated in the figure. If PQ be the portion of any of the ordinates intercepted between the curves, and δx the breadth of the elementary rectangle of which PQ is a side, the expression for the area will accordingly be

$$Lt_{\delta x=0}\sum_{x=a}^{x=b} PQ\,\delta x, \quad \text{or} \quad \int_a^b [\phi(x)-\psi(x)]\,dx,$$

where the same assumption is made as before as to $\phi(x)$ and $\psi(x)$ being finite and continuous from $x=a$ to $x=b$, and, moreover, $\phi(x)-\psi(x)$ must retain the same sign throughout the integration, *i.e.* the curves must not cross each other, and $\phi(x)$ has been assumed $>\psi(x)$ throughout.

396. **Case when the Coordinates are expressed in terms of a Parameter.**

We have regarded x as the independent variable. If this is not so the formula can be modified to suit the circumstances.

Suppose the curve defined by the equations

$$x=\phi(t), \quad y=\psi(t),$$

and that the values of t corresponding to the initial and final ordinates are t_1 and t_2.

Then $y\,\delta x=\psi(t)\,\phi'(t)\,\delta t$ to the first order, and in the limit

$$\int_a^b y\,dx=\int_{t_1}^{t_2}\psi(t)\,\phi'(t)\,dt,$$

it being supposed that the integrand remains finite and continuous throughout, and that as t changes continuously, increasing from the value t_1 to the value t_2, the point $(x, 0)$ also travels continuously along the x-axis from $(a, 0)$ to $(b, 0)$ without going over any part of its course more than once, and always in the same direction of increase of x.

397. **Case where the Arc is the Parameter.**

If the arc of the curve be the independent variable, being measured from some definite point on the curve, then at a point at which the gradient of the tangent is ψ, we have $dx=\cos\psi\,ds$, and we may write the expression $\int y\,dx$ as

$$\int y\frac{dx}{ds}ds, \quad \text{or} \quad \int y\cos\psi\,ds,$$

the limits of the integration with regard to s being the values of s corresponding to the beginning and end of the arc, and supposing that $y\cos\psi$ does not change sign.

In the same way we may write $\int x\,dy$ as

$$\int x\frac{dy}{ds}ds, \quad \text{or} \quad \int x\sin\psi\,ds.$$

398. **Area expressed by a Line Integral round the Contour.**

Let the formulae $\int y\cos\psi\,ds$, $\int x\sin\psi\,ds$ be applied to the evaluation of the area of a closed curve consisting of a single oval.

Let us suppose s measured from any point on the curve in such a direction that a person travelling along it in the direction of an increase of s has the area sought always to his left. Let ψ be the angle the tangent makes with the positive direction of the x-axis. Let $APBQ$ be the oval in question, and let

AL. BN be the tangents parallel to the y-axis. In the arc APB in the figure, ψ is changing from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$, and $\cos\psi$ is positive. In the arc BQA ψ is changing from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$, and $\cos\psi$ is negative. Integrating then $\int y\cos\psi\,ds$ from A to B, through P, we obtain the area $ALMNBPA$ taken

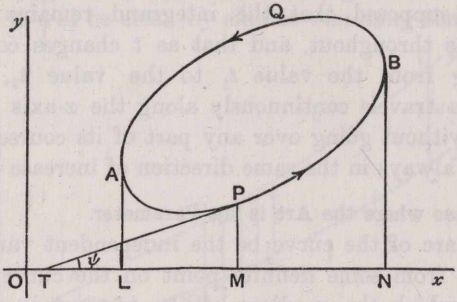

Fig. 41.

positively, whilst integration from B to A, through Q, obtains the area $BQALMNB$ taken negatively. Hence, to obtain the whole area, it is necessary to take our formula as $-\int y\cos\psi\,ds$ in integration round the whole perimeter in the counter-clockwise direction.

In the same way and under the same circumstances the area will also be given by $+\int x\sin\psi\,ds$.

This is the conventional mode of measuring s. If we measured in a clockwise direction the signs would both be reversed.

399. Precautions.

If the curve cuts itself once, having a node, as in the case of a lemniscate, it will be clear, from an inspection of the accompanying figure, that, in travelling completely round the whole curve, the directions in which the two loops are travelled round in continuously progressing in the direction of the increase of s, are one clockwise and the other counter-clockwise, and therefore, in conducting the integration completely round we get the difference of the areas of the two loops with either formula, and in the case of equality of the loops the total line-integral of $x \sin\psi$, or of $y \cos\psi$, round the complete curve will be zero. If we require the absolute area

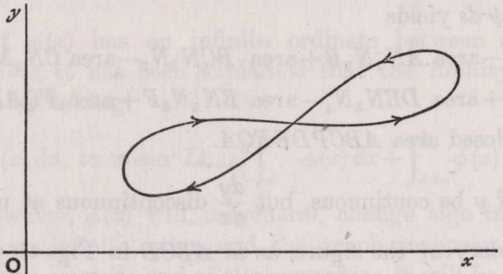

Fig. 42.

enclosed we must therefore treat each loop separately and add the positive results.

If in travelling continuously round the perimeter of the closed curve there be several nodes and several loops, we shall see in the same way that the total line-integral of $x \sin\psi$ or of $y \cos\psi$, will give the difference of the areas of the odd and even loops.

400. The student should examine the truth of the result in

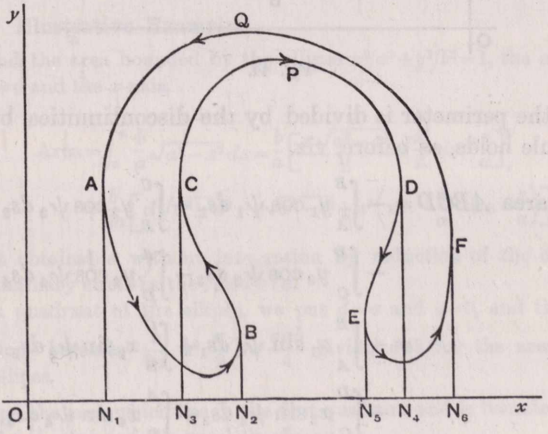

Fig. 43.

figures of other shapes—say a horseshoe-shaped closed curve, such as shown in Fig. 43.

Let $ABCDEF$ be the points at which the tangents are parallel to the y-axis, then if AN_1, BN_2, etc., be the ordinates, the integral

$-\int y \cos\psi \, ds$ yields

$$-\text{area } AN_1N_2B + \text{area } BCN_3N_2 - \text{area } CN_3N_4D$$
$$+\text{area } DEN_5N_4 - \text{area } EN_5N_6F + \text{area } FQAN_1N_6F,$$

i.e. the closed area $ABCPDEFQA$.

401. If y be continuous, but $\dfrac{dy}{dx}$ discontinuous at points on the boundary of the figure, as at $ABCD$ in Fig. 44, the integration must be conducted along each of the portions into

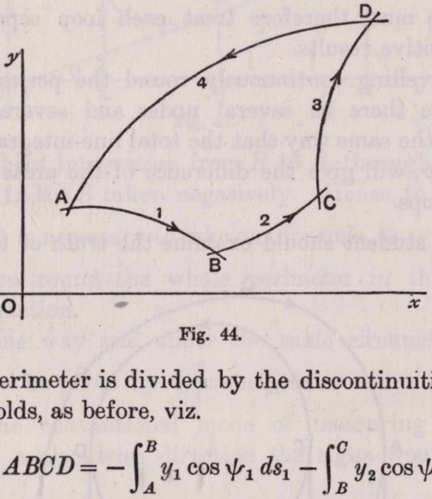

Fig. 44.

which the perimeter is divided by the discontinuities, but the same rule holds, as before, viz.

$$\text{area } ABCD = -\int_A^B y_1 \cos\psi_1 \, ds_1 - \int_B^C y_2 \cos\psi_2 \, ds_2$$
$$-\int_C^D y_3 \cos\psi_3 \, ds_3 - \int_D^A y_4 \cos\psi_4 \, ds_4,$$

or
$$= +\int_A^B x_1 \sin\psi_1 \, ds_1 + \int_B^C x_2 \sin\psi_2 \, ds_2$$
$$+\int_C^D x_3 \sin\psi_3 \, ds_3 + \int_D^A x_4 \sin\psi_4 \, ds_4,$$

suffixes denoting the several portions along which the integration is conducted, and s_1, s_2, s_3, etc., always being measured "in the same sense" along the perimeter. Here the limits of the integrals are denoted by the points $A, B, C \ldots$ of the perimeter successively arrived at in a continuous progress round it.

402. If $\phi(x)$ has an infinite ordinate between a and b, say at $x=c$, it has been explained that the infinity can be excluded by taking

$$\int_a^b \phi(x)\,dx \text{ to mean } Lt_{\epsilon=0}\left[\int_a^{c-\epsilon}\phi(x)\,dx+\int_{c+\epsilon}^b \phi(x)\,dx\right].$$

As, however, $\phi(x)$ will, in general, change sign in passing through an infinite value and the graph reappear from infinity at the opposite end of the asymptote, it will be desirable to consider the areas on opposite sides of the asymptote separately, and, after evaluation, add the *positive results* together. This is of course the same precaution we have had to take in Art. 395, in stipulating that $\phi(x)$ does not change sign between the limits, which would mean that part of the curve was above the x-axis and part below, so that carelessness in this respect would lead to a result which would represent the difference of the two portions of the area required instead of their sum.

403. **Illustrative Examples.**

1. Find the area bounded by the ellipse $x^2/a^2+y^2/b^2=1$, the ordinates $x=c$, $x=d$ and the x-axis.

Here

$$\text{Area}=\int_c^d \frac{b}{a}\sqrt{a^2-x^2}\,dx=\frac{b}{a}\left[\frac{x\sqrt{a^2-x^2}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_c^d$$

$$=\frac{b}{2a}\left[d\sqrt{a^2-d^2}-c\sqrt{a^2-c^2}+a^2\left(\sin^{-1}\frac{d}{a}-\sin^{-1}\frac{c}{a}\right)\right],$$

a result obtainable without integration by reduction of the ordinates of the auxiliary circle in the ratio $b:a$.

For a quadrant of the ellipse, we put $d=a$ and $c=0$, and the above expression becomes $\frac{b}{2a}\cdot a^2\cdot\frac{\pi}{2}$ or $\frac{\pi ab}{4}$ giving πab for the area of the whole ellipse.

2. Find the area which lies in the first quadrant and is bounded by the circle $x^2+y^2=2ax$ and the parabola $y^2=ax$.

The curves touch at the origin and cut again at (a, a).

The limits for x are therefore from $x=0$ to $x=a$.

The area sought is therefore

$$\int_0^a \{\sqrt{2ax-x^2}-\sqrt{ax}\}\,dx.$$

Putting $x=a(1-\cos\theta)$ in the first

$$\int_0^a \sqrt{2ax-x^2}\,dx=\int_0^{\frac{\pi}{2}} a^2\sin^2\theta\,d\theta=a^2\frac{1}{2}\frac{\pi}{2}=\frac{\pi a^2}{4},$$

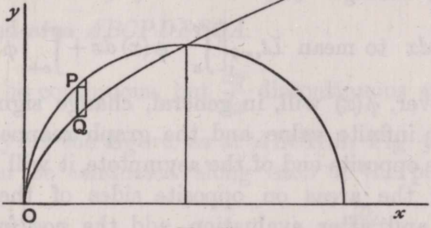

Fig. 45.

as of course might have been written down, being a quadrant of a circle of radius a; and

$$\int_0^a \sqrt{ax}\,dx=\sqrt{a}\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^a=\tfrac{2}{3}a^2.$$

Thus the area required is $\quad a^2\left(\dfrac{\pi}{4}-\dfrac{2}{3}\right)$.

3. Find the area

(1) of the loop of the curve

$$x(x^2+y^2)=a(x^2-y^2),$$

(2) of the portion bounded by the curve and its asymptote.

Here $$y^2=x^2\frac{a-x}{a+x}.$$

To trace this curve, we observe

(1) It is symmetrical about the x-axis.

(2) No real part exists for points at which $x>a$ or $<-a$.

(3) It has an asymptote $x+a=0$.

(4) It goes through the origin, and the tangents there are $y=\pm x$.

(5) It crosses the x-axis when $x=a$, and at this point $\dfrac{dy}{dx}$ is infinite.

(6) The shape of the curve is therefore that shown in the figure (Fig. 46).

Hence, for the loop the limits of integration are 0 and a, and then double the result so as to include the portion below the x-axis.

For the portion between the curve and the asymptote, the limits are $x=-a$ to $x=0$ and double as before.

For the loop we therefore have,

$$\text{Area} = 2\int_0^a x\sqrt{\frac{a-x}{a+x}}\,dx.$$

For the portion between the curve and the asymptote we have,

$$\text{Area} = -2\int_{-a}^0 x\sqrt{\frac{a-x}{a+x}}\,dx.$$

The meaning of the negative sign is this: In choosing the + sign before the radical in $y=x\sqrt{\frac{a-x}{a+x}}$, we are tracing the portion of the curve *below the x-axis* on the left of the origin and *above the x-axis* on

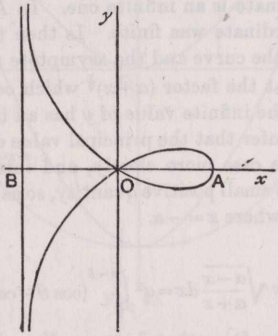

Fig. 46.

the right of the origin. Hence, y being negative between the limits $-a$ and 0, it is to be expected that we should obtain a negative result if we evaluate the expression,

$$Lt\sum_{x=-a}^{x=0} y\,dx.$$

Therefore we prefix the $-$ before the radical before integration to ensure a positive result.

To integrate $\int x\sqrt{\frac{a-x}{a+x}}\,dx$, put $x=a\cos\theta$ and $\therefore\ dx=-a\sin\theta\,d\theta$.

Thus $$\int_0^a x\sqrt{\frac{a-x}{a+x}}\,dx = -\int_{\frac{\pi}{2}}^0 a\cos\theta\sqrt{\frac{(1-\cos\theta)^2}{1-\cos^2\theta}}\,a\sin\theta\,d\theta$$

$$= a^2\int_0^{\frac{\pi}{2}}(\cos\theta-\cos^2\theta)\,d\theta$$

$$= a^2\left(1-\frac{1}{2}\frac{\pi}{2}\right)=\left(1-\frac{\pi}{4}\right)a^2.$$

And $\quad$ Area of loop $= 2a^2\left(1-\frac{\pi}{4}\right).$

Again, $$\int_{-a}^{0} x\sqrt{\frac{a-x}{a+x}}\,dx = -\int_{\pi}^{\frac{\pi}{2}} a\cos\theta\sqrt{\frac{(1-\cos\theta)^2}{1-\cos^2\theta}}\,a\sin\theta\,d\theta$$

$$= \quad a^2\int_{\frac{\pi}{2}}^{\pi}(\cos\theta-\cos^2\theta)\,d\theta$$

$$= -a^2\left(1+\frac{\pi}{4}\right),$$

and the area between the asymptote and the curve

$$= \quad 2a^2\left(1+\frac{\pi}{4}\right).$$

With regard to the latter portion of this example, it is to be observed that the greatest ordinate is an infinite one. In Arts. 11 and 394 it was assumed that every ordinate was finite. Is then the result obtained for the area bounded by the curve and the asymptote rigorously true?

It will be noted that the factor $(a+x)^{\frac{1}{2}}$ which occurs in the denominator and gives rise to the infinite value of y has an index <1 and positive. Hence (Art. 348) we infer that the principal value of the integral is finite.

Let us examine the case more closely, and integrate between $-a+\epsilon$ and 0, where ϵ is some small positive quantity, so as to exclude the infinite ordinate at the point where $x=-a$.

We have as before

$$\int_{-a+\epsilon}^{0} x\sqrt{\frac{a-x}{a+x}}\,dx = a^2\int_{\frac{\pi}{2}}^{\pi-\delta}(\cos\theta-\cos^2\theta)\,d\theta,$$

where $-a+\epsilon=a\cos(\pi-\delta)$, so that δ is a small positive angle, viz.

$$\cos^{-1}\left(1-\frac{\epsilon}{a}\right).$$

This integral is then

$$a^2\left[\sin\theta-\frac{\theta}{2}-\frac{\sin 2\theta}{4}\right]_{\frac{\pi}{2}}^{\pi-\delta} = a^2\left[(\sin\delta-1)-\left(\frac{\pi-\delta}{2}-\frac{\pi}{4}\right)+\frac{\sin 2\delta}{4}\right]$$

$$= a^2\left[-1-\frac{\pi}{4}+\frac{\delta}{2}+\sin\delta+\frac{\sin 2\delta}{4}\right],$$

and approaches indefinitely closely to the former result

$$-a^2\left(1+\frac{\pi}{4}\right),$$

when δ is made to diminish without limit to zero.

4. Prove that the whole area of the curve

$$x^4-2ax^2y+a^2(x^2+y^2)=a^4 \text{ is } \pi a^2.$$

Here, solving for y,

$$y=\frac{x^2}{a}\pm\sqrt{a^2-x^2}$$

$$=y_1\pm y_2,$$

where y_1 is the ordinate of a parabola and y_2 that of a circle of radius a.

The area of a strip parallel to the y-axis and of breadth δx is

$$[(y_1+y_2)-(y_1-y_2)]\,\delta x=2y_2\,\delta x,$$

and the total area of the curve is $2\int_{-a}^{a} y_2\,dx$, *i.e.* the same as that of the circle, $=\pi a^2$.

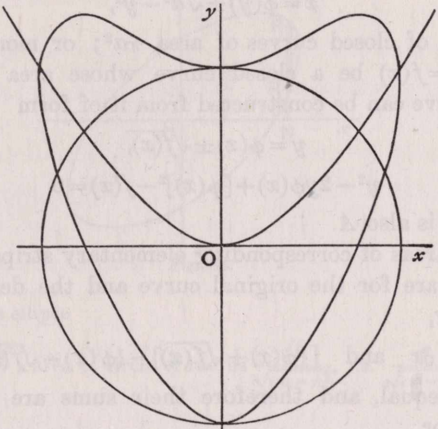

Fig. 47.

404. The last example will suggest to the student that if the curve $y=\phi(x)\pm\sqrt{a^2-x^2}$ be drawn, it may be regarded as constructed by means of two curves, viz.

$$y_1=\phi(x) \quad\text{and}\quad y_2=\sqrt{a^2-x^2},$$

the latter being a circle and the ordinates of the resultant curve being the sum or difference of y_1 and y_2, viz.

$$y=y_1\pm y_2,$$

and as in the parabola and circle of Ex. 4, the closed curve formed will be divisible into strips of length $(y_1+y_2)-(y_1-y_2)$ and breadth δx, and therefore of area $2y_2\,\delta x$.

Hence the area in any such case is $2\int_{-a}^{a} y_2\,dx=\pi a^2$, and is the same as that of the circle.

This curve, if written in rational form, is

$$x^2+y^2+[\phi(x)]^2-a^2=2y\phi(x),$$

$\phi(x)$ being supposed rational. And the areas of all such curves are $=\pi a^2$.

Similarly, for curves of form

$$x^2+y^2+[\phi(y)]^2-a^2=2x\phi(y),$$

which are clearly to be constructed as

$$x=\phi(y)\pm\sqrt{a^2-y^2},$$

and consist of closed curves of area πa^2; or more generally still, if $y^2=f(x)$ be a closed curve whose area is A, then another curve can be constructed from it of form

$$y=\phi(x)\pm\sqrt{f(x)},$$

i.e. $$y^2-2y\phi(x)+[\phi(x)]^2-f(x)=0$$

whose area is also A.

For the areas of corresponding elementary strips parallel to the y-axis are for the original curve and the derived curve respectively,

$$2\sqrt{f(x)}\,\delta x \quad \text{and} \quad [\{\phi(x)+\sqrt{f(x)}\}-\{\phi(x)-\sqrt{f(x)}\}]\,\delta x,$$

which are equal, and therefore their sums are equal also. Similarly for

$$x^2-2x\phi(y)+[\phi(y)]^2-f(y)=0.$$

405. In Art. 395 it is shown that the area between the two curves $y=\phi(x)$ and $y=\psi(x)$ and a pair of ordinates $x=a, x=b$ is

$$\int_a^b [\phi(x)-\psi(x)]\,dx.$$

It may be that $y=\phi(x)$ and $y=\psi(x)$ are different branches of *the same curve.* This is really what happens in the various cases considered in the last article.

406. Ex. Consider the case of an ellipse

$$ax^2+2hxy+by^2=1, \quad h^2<ab.$$

If y_1, y_2 are the ordinates for any abscissa x,

$$y_1+y_2=-\frac{2h}{b}x,$$

$$y_1y_2=\frac{ax^2}{b}-\frac{1}{b};$$

$\therefore$ the length of the strip is

$$y_1-y_2=2\sqrt{\frac{1}{b}-\frac{ab-h^2}{b^2}x^2}=2\,\frac{\sqrt{ab-h^2}}{b}\sqrt{\frac{b}{ab-h^2}-x^2}.$$

And the area is

$$\int_{x_1}^{x_2} (y - y_2)\, dx, \text{ between ordinates } x_1 \text{ and } x_2,$$

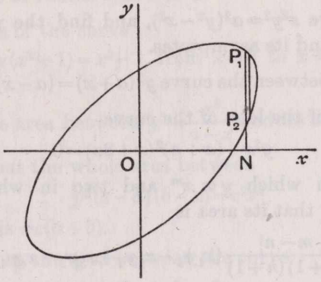

Fig. 48.

or for the whole ellipse

$$\frac{\sqrt{ab-h^2}}{b} \times \text{area of circle of radius } \frac{\sqrt{b}}{\sqrt{ab-h^2}}, \text{ i.e. } \frac{\pi}{\sqrt{ab-h^2}}.$$

Examples.

1. Obtain the area bounded by a parabola and its latus rectum. A series of ordinates are drawn between the vertex and the latus rectum, parallel to the latter, viz. $x=\left(\frac{r}{n}\right)^{\frac{2}{3}} a$, where $r=1, 2, 3, \ldots n-1$. Show that they divide the aforementioned area into n equal parts.

2. Obtain the areas bounded by the curve, the x-axis, and the specified ordinates in the following cases:

(*a*) The catenary $y=c\cosh\frac{x}{c}$, from $x=0$ to $x=h$.

(*b*) The logarithmic curve $y=e^x$, from $x=0$ to $x=h$.

(*c*) The logarithmic curve $y=\log_e x$, from $x=1$ to $x=h$ $(h>1)$

(*d*) The ellipse $y=\frac{b}{a}\sqrt{a^2-x^2}$, from $x=\sqrt{a^2-b^2}$ to $x=a$.

(*e*) The hyperbola $xy=k^2$, from $x=a$ to $x=b$, a and b both >0; first, if the hyperbola be rectangular, second, if the angle between the asymptotes be ω.

(*f*) The curve $y=xe^{x^2}$, from $x=0$ to $x=h$.

3. Obtain the area (1) bounded by the parabolas $y^2=4ax$, $x^2=4ay$;
(2) bounded by the parabolas $y^2=4ax$, $x^2=4by$.

In what ratio is this area divided by the common chord in each case?

4. Find the areas of the portions into which the ellipse $x^2/a^2+y^2/b^2=1$ is divided (1) by the straight line $y=c$;

(2) by the two straight lines $y=c$, $x=d$, supposed to cut within the ellipse.

5. Trace the curve $x^2y^2=a^2(y^2-x^2)$, and find the whole area included between the curve and its asymptotes.

6. Find the area between the curve $y^2(a+x)=(a-x)^3$ and its asymptote.

7. Find the area of the loop of the curve

$$y^2x+(x+a)^2(x+2a)=0.$$

8. Two curves in which $y \propto x^m$ and two in which $y \propto x^n$ form a quadrilateral; show that its area is

$$\frac{m \sim n}{(m+1)(n+1)}(x_1y_1-x_2y_2+x_3y_3-x_4y_4),$$

where (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4) are the coordinates of the corners taken in order. [Trinity, 1891.]

9. By means of the integral $\int y\,dx$ taken round the contour of the triangle formed by the intersecting lines,

$$y=a_1x+b_1,$$
$$y=a_2x+b_2,$$
$$y=a_3x+b_3,$$

show that they enclose the area

$$\frac{(b_1-b_3)^2}{2(a_1-a_3)}+\frac{(b_2-b_1)^2}{2(a_2-a_1)}+\frac{(b_3-b_2)^2}{2(a_3-a_2)}.$$

[Smith's Prize, 1876.]

10. A four-sided figure is formed by the three parabolas,

$$y^2-9ax+81a^2=0,$$
$$y^2-4ax+16a^2=0,$$
$$y^2-\ ax+\ \ a^2=0,$$

and the axis of x. Prove that its area is $12a^2$, and is equal to the area enclosed by the chords of the area. [Colleges α, 1886.]

11. Find the curvilinear area enclosed between the parabola $y^2=4ax$ and its evolute. [Oxf. I. P., 1889.]

12. Show that the area cut off from a semi-cubical parabola by a tangent is divided by the tangent at the cusp in the ratio 64 : 17.

[Oxford II. P., 1889.]

13. (i) Find the area of a loop of the curve

$$ay^2=x^2(a-x).$$ [I. C. S., 1882.]

(ii) Find the whole area of the curve

$$a^2y^2=a^2x^2-x^4.$$ [I. C. S., 1881.]

14. Trace the curve $a^2x^2=y^3(2a-y)$, and prove that its area is equal to that of the circle whose radius is a. [I. C. S., 1887 AND 1890.]

15. Trace the curve $a^4y^2=x^5(2a-x)$, and prove that its area is to that of the circle of radius a as 5 to 4.

16. Find the area of the curve
$$y(x^2+1)=x^3-1 \text{ from } x=0 \text{ to } x=1.$$
[ST. JOHN'S, 1881.]

17. (i) Find the area between $y^2=\dfrac{x^3}{a-x}$ and its asymptote.

(ii) Show that the whole area between
$$y^2(x-a)(b-x)=c^2x^2$$
and its asymptote is $\pi c(a+b)$. [OX. II. P., 1903.]

(iii) Show that the area between the curve
$$y^2x=a^3-a^2x$$
and its asymptote is that of a circle of radius a. [ST. JOHN'S, 1889.]

18. Find the area between the axis of x, the hyperbola $x^2/a^2-y^2/b^2=1$, and the line $y=x\tan\alpha$, where
$$\frac{b}{a}>\tan\alpha>0.$$
[OX. I. P., 1901.]

If A be the vertex, O the centre, and P any point on the hyperbola $x^2/a^2-y^2/b^2=1$, prove that
$$x=a\cosh\frac{2S}{ab},\quad y=b\sinh\frac{2S}{ab},$$
where S is the sectorial area AOP [MATH. TRIPOS, 1885.]

19. Find by integration the area lying on the same side of the axis of x as the positive part of the axis of y, and which is contained by the lines
$$y^2=4ax,$$
$${}^2+y^2=2ax,$$
$$x=y+2a.$$

Express the area both when x is the independent variable and when y is the independent variable. [COLLEGES, 1882.]

20. Prove that the area of the loop of
$$a(x-y)(x-2y)=y^3 \text{ is } \frac{a^2}{60}.$$
[COLL. β, 1891.]

21. Find the areas of the two regions of space bounded by the straight line $y=c$, and the curves whose equations are
$$x^2+y^2=c^2,$$
$$y^2+4x^2=4c^2.$$
[I. C. S., 1891.]

22. Prove that the area contained between the curve
$$(x+3a)(x^2+y^2)=4a^3$$
and its asymptote is $3a^2\sqrt{3}$. [OXF. I. P., 1901.]

23. Prove that the area of the curve

$$x^4 - 3ax^3 + a^2(2x^2 + y^2) = 0$$

is $\frac{3}{8}\pi a^2$. [MATH. TRIP., 1893.]

24. Find the area of one loop of the curve

$$y^4 - y^2 + x^2 = 0.$$ [COLLEGES α, 1885.]

25. Through the cusp of the evolute of a parabola, a line is drawn perpendicular to the axis. Show that it divides the area between the parabola and the evolute in the ratio 17 : 5. [C. S., 1896.]

26. Show that the ordinate $x=a$ divides the area between $y^2(2a-x)=x^3$ and its asymptote into two parts in the ratio

$$3\pi - 8 : 3\pi + 8.$$ [MATH. TRIP. I., 1912.]

407. Sectorial Areas. Polar Coordinates.

When the area to be found is bounded by a curve $r=f(\theta)$ and two radii vectores drawn from the origin in given directions, we may divide the area into elementary sectors with the same small angle $\delta\theta$, as shown in the figure. Let the

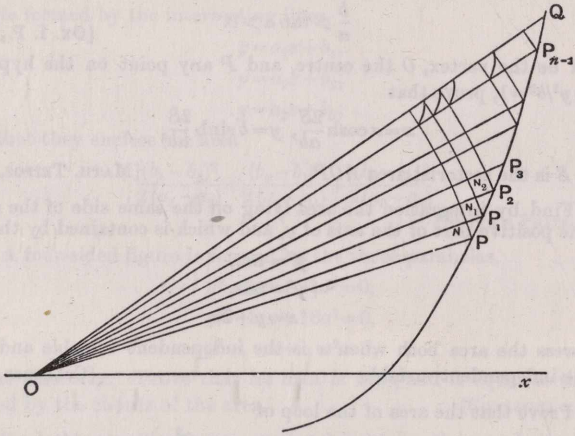

Fig. 49.

area to be found be bounded by the arc PQ and the radii vectores OP, OQ. Draw radii vectores $OP_1, OP_2, \ldots OP_{n-1}$ at equal angular intervals, so that

$$P\hat{O}P_1 = P_1\hat{O}P_2 = \ldots = P_{n-1}\hat{O}Q = \delta\theta.$$

Then by drawing with centre O the successive circular arcs PN, P_1N_1, P_2N_2, etc., it may be at once seen that the limit of the sum of the circular sectors OPN, OP_1N_1, OP_2N_2, etc.

is the area required. For the remaining elements PNP_1, $P_1N_1P_2$, $P_2N_2P_3$, etc., may be made rotate about O so as to occupy new positions on the greatest sector, say $OP_{n-1}Q$, as indicated in the figure Their sum is plainly less than this sector; and in the limit when the angle of this sector is indefinitely diminished its area also diminishes without limit, provided the radius vector OQ is finite.

Now the area of a circular sector is

$$\tfrac{1}{2}(\text{radius})^2 \times \text{circular measure of angle of sector.}$$

Thus the area required $= \frac{1}{2}\,\mathrm{Lt}\,\Sigma\, r^2\,\delta\theta$, the summation being conducted for such values of θ as lie between $\theta = x\hat{O}P$ and $\theta = x\hat{O}P_{n-1}$ *i.e.*, $x\hat{O}Q$ in the limit, Ox being the initial line.

In the notation of the integral calculus, if $x\hat{O}P = \alpha$ and $x\hat{O}Q = \beta$, this will be expressed as

$$\tfrac{1}{2}\int_\alpha^\beta r^2\,d\theta \quad \text{or} \quad \tfrac{1}{2}\int_\alpha^\beta [f(\theta)]^2\,d\theta.$$

It is assumed that $f(\theta)$ is finite and continuous from $\theta = \alpha$ to $\theta = \beta$ inclusive.

408. If the curve consist of a closed oval and the origin be within it, the limits of integration to find the whole area are 0 and 2π, viz. the extent to which a radius vector must rotate about O to sweep out the whole area (Fig. 50).

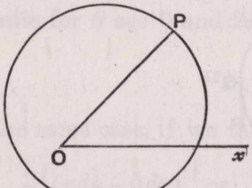

Fig. 50.

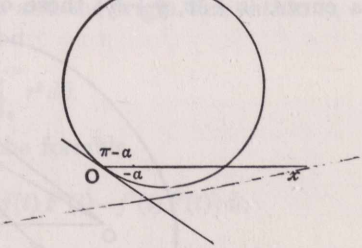

Fig. 51.

If the origin be on the perimeter of the oval, and if it be not a singular point, the limits will be from $-\alpha$ to $+\pi-\alpha$ if the tangent at the origin makes an angle $-\alpha$ with the x-axis as shown in Fig. 51.

In this case, if the initial line be an axis of symmetry, it is sufficient to integrate from 0 to $\frac{\pi}{2}$ and double the result (Fig. 52)

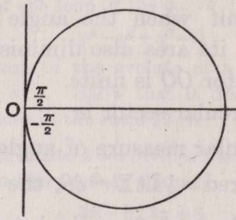

Fig. 52.

If there be a loop and the origin be a singular point on the curve at which the tangents make an angle 2α with each

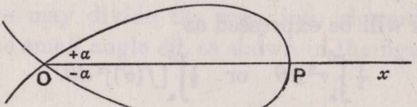

Fig. 53.

other, and if the initial line be an axis of symmetry, the limits for the area of the loop will be 0 and α and double the result (Fig. 53).

409. Another Expression for an Area.

Let (x, y) be the Cartesian coordinates of any point P on a curve, $(x+\delta x,\ y+\delta y)$ those of an adjacent point Q. Let

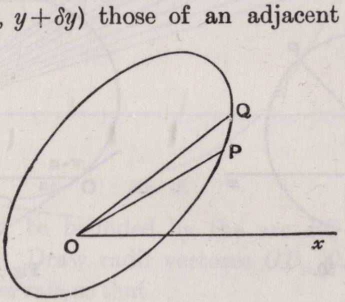

Fig. 54.

(r, θ), $(r+\delta r,\ \theta+\delta\theta)$ be the corresponding polar coordinates. Also, we shall suppose that, in travelling along the curve from P to Q on an infinitesimal arc PQ, the direction of rotation of the radius vector OP is counter-clockwise, and that the area to be considered is on the *left hand to a person travelling in this direction* (Fig. 54).

Then, to the first order of infinitesimals,

$$\tfrac{1}{2}r^2\delta\theta = \text{sectorial area } OPQ$$

$$= \tfrac{1}{2}\begin{vmatrix} x, & y, & 1 \\ x+\delta x, & y+\delta y, & 1 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \tfrac{1}{2}(x\,\delta y - y\,\delta x).$$

Hence, another expression for the area of a sectorial portion of a curve bounded by a definite portion of an arc is

$$\frac{1}{2}\int(x\,dy - y\,dx) \quad \text{or} \quad \frac{1}{2}\int\left(x\frac{dy}{ds} - y\frac{dx}{ds}\right)ds,$$

the limits being the initial and final values of s, corresponding to the portion of the sectorial area to be found.

Obviously we might take any other independent variable, say t, and supposing the curve expressed as

$$x = f(t), \quad y = F(t),$$

and that the values of t, corresponding to the beginning and end of the arc, are t_1 and t_2 respectively,

$$\text{sectorial area} = \frac{1}{2}\int_{t_1}^{t_2}\{f(t)\,F'(t) - f'(t)\,F(t)\}\,dt.$$

If the curve be a closed curve and the origin lies within it, the limits for θ are 0 and 2π, and

$$\text{area} = \frac{1}{2}\int_0^{2\pi} r^2\,d\theta.$$

In the same case, if we take the formula

$$\frac{1}{2}\int(x\,dy - y\,dx) \quad \text{or} \quad \frac{1}{2}\int[f(t)\,F'(t) - f'(t)\,F(t)]\,dt,$$

the limits for t must be such that the point (x, y) travels once, and once only, completely round the curve.

410. If the origin lies outside the curve, as the current point P travels round the curve, we obtain sectorial elements such as OP_1Q_1 (Fig. 55), including portions of space such as OP_2Q_2,

shown in the figure, which lie outside the curve. These portions are, however, ultimately removed from the whole integral

$$\frac{1}{2}\int(x\,dy-y\,dx),$$

when the point P travels over the element P_2Q_2, for the

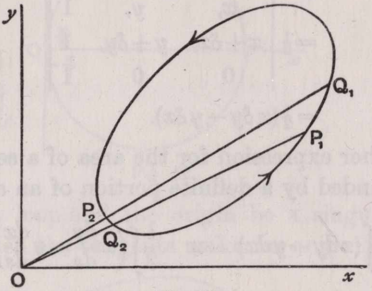

Fig. 55.

sectorial element OP_2Q_2 is reckoned negatively as θ is decreasing and $\delta\theta$ is negative.

411. Precautions.

If the curve cross itself as in Fig. 56, the expression

$$\frac{1}{2}\int(x\,dy-y\,dx),$$

taken round the whole perimeter, no longer represents the *sum* of the areas of the several regions. For draw two contiguous radii vectores OP_1, OQ_1, cutting the curve again at Q_2, P_3, Q_4 and P_2, Q_3, P_4 respectively. Then, in travelling round the curve continuously through the complete perimeter, we obtain positive elements such as OP_1Q_1 and OP_3Q_3, and negative elements such as OP_2Q_2 and OP_4Q_4.

Now, taking all these elements positively,

$$OP_1Q_1-OP_2Q_2+OP_3Q_3-OP_4Q_4$$
$$=\text{quadrilateral } P_1Q_1P_4Q_4-\text{quadrilateral } P_2Q_2P_3Q_3,$$

and in integrating for the whole curve we therefore obtain the difference of the two regions instead of their sum.

Similarly, if the curve cuts itself more than once, the integral $\frac{1}{2}\int(x\,dy-y\,dx)$ gives the difference of the sum of the odd regions and the sum of the even regions. Thus, to obtain the absolute area bounded by such a curve, we must take *our limits for each area separately* and obtain the *absolute area of each region,* and then *add together the results.* It is

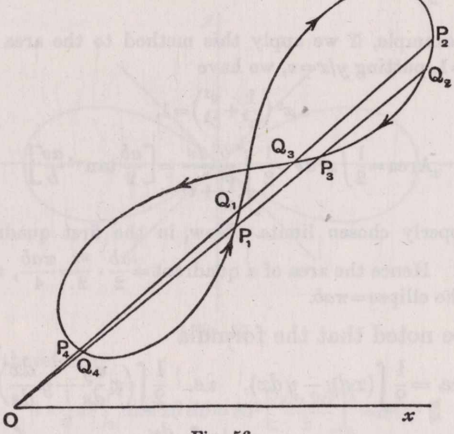

Fig. 56.

obvious that in curves consisting of several equal regions, or loops, it will be sufficient to ascertain the area of any one, and then to multiply that area by the number of the loops.

412. Another Form.

If we write $\dfrac{y}{x}=v$, we have

$$x\,dy-y\,dx=x^2dv,$$

and accordingly we may transform the formula

$$\frac{1}{2}\int(x\,dy-y\,dx) \text{ into } \frac{1}{2}\int x^2dv.$$

This is equivalent to a choice of new coordinates, of which one is the Cartesian abscissa and the other, viz. v, is the tangent of the polar angle θ.

In using the formula, x is to be expressed in terms of v, and the limits of the integration so chosen that the current point (x, y) travels from the beginning to the end of the arc, *i.e.* if α, β be the limits for θ, $\tan\alpha$ and $\tan\beta$ will be the limits for v.

In using this formula, however, *care must be taken not to integrate through an infinite value of v.* It must be remembered that $v=\tan\theta$ and becomes infinite when $\theta=\frac{\pi}{2}$, or any odd multiple of $\frac{\pi}{2}$.

413. For example, if we apply this method to the area of an ellipse $x^2/a^2+y^2/b^2=1$, putting $y/x=v$, we have

$$x^2\left(\frac{1}{a^2}+\frac{v^2}{b^2}\right)=1,$$

and
$$\text{Area}=\frac{1}{2}\int x^2\,dv=\frac{1}{2}\int\frac{b^2\,dv}{\frac{b^2}{a^2}+v^2}=\left[\frac{ab}{2}\tan^{-1}\frac{av}{b}\right]$$

between properly chosen limits. Now, in the first quadrant v varies from 0 to ∞. Hence the area of a quadrant $=\frac{ab}{2}\cdot\frac{\pi}{2}=\frac{\pi ab}{4}$, and therefore the area of the ellipse $=\pi ab$.

It will be noted that the formula

$$\text{Area}=\frac{1}{2}\int(x\,dy-y\,dx),\quad \textit{i.e.}\quad \frac{1}{2}\int\left(x\frac{dy}{ds}-y\frac{dx}{ds}\right)ds,$$

is equivalent to half the sum of $\int x\frac{dy}{ds}\,ds$ and $-\int y\frac{dx}{ds}\,ds$, each of which has been shown to represent the area when the integration follows the complete perimeter.

414. It may also be worth the student's notice to remark that the problem of finding the area bounded by $y=\phi(x)$, the x-axis, and a pair of ordinates $x=a$, $x=b$, viz. $A=\int_a^b\phi(x)\,dx$, is manifestly the same as that of finding the mass of a rod of small section but of line density $\phi(x)$, of length $b-a$, and of any shape if x be measured along the rod. For the mass of a length δx of the rod is $\phi(x)\,\delta x$, the limit of the sum of such expressions being required, when δx is indefinitely diminished, between limits $x=a$ and $x=b$, that is $\int_a^b\phi(x)\,dx$.

415. **Illustrative Examples.**

1. Obtain the area of the semicircle bounded by $r=a\cos\theta$ and the initial line.

Here the radius vector sweeps over the angular interval from

$$\theta=0\quad\text{to}\quad\theta=\frac{\pi}{2}.$$

Hence the area is

$$\frac{1}{2}\int_0^{\frac{\pi}{2}} a^2\cos^2\theta\, d\theta = \frac{a^2}{2}\,\frac{1}{2}\,.\,\frac{\pi}{2} = \frac{\pi a^2}{8}, \quad i.e.\ \frac{1}{2}\pi\,(\text{radius})^2.$$

2. Find the area of the lemniscate $r^2 = a^2\cos 2\theta$.

Here the axis is a line of symmetry ; the tangents at the origin are $\theta = \pm\frac{\pi}{4}$.

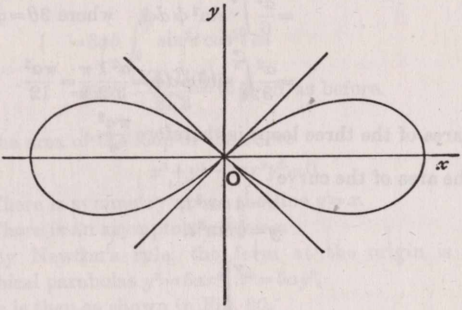

Fig. 57.

The area is therefore

$$4\times\frac{1}{2}a^2\int_0^{\frac{\pi}{4}}\cos 2\theta\, d\theta = 2a^2\left[\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}} = a^2.$$

3. Find the area of the pedal of an ellipse with regard to the centre
With the usual axes and notation, the equation of the pedal is

$$r^2 = a^2\cos^2\theta + b^2\sin^2\theta,$$

and
$$\text{Area} = 4\times\frac{1}{2}\int_0^{\frac{\pi}{2}}(a^2\cos^2\theta + b^2\sin^2\theta)\, d\theta = \pi\frac{a^2+b^2}{2}.$$

4. Find the area of one loop of the curve $r = a\sin 3\theta$.

The curve consists of three equal loops, as indicated in the figure

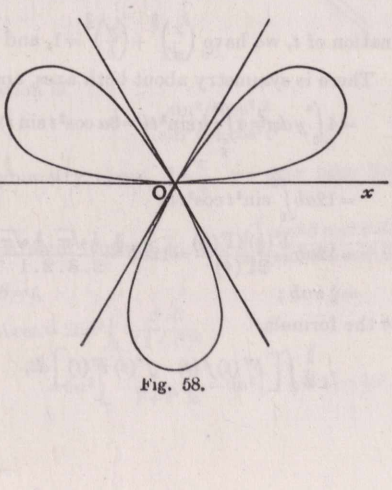

Fig. 58.

The proper limits for the integration extending over the first loop are $\theta=0$ and $\theta=\frac{\pi}{3}$, for these are two successive values of θ for which r vanishes:

$$\therefore \text{ Area of loop} = \frac{1}{2}\int_0^{\frac{\pi}{3}} a^2 \sin^2 3\theta \, d\theta$$

$$= \frac{a^2}{6}\int_0^{\pi} \sin^2\phi \, d\phi, \quad \text{where } 3\theta=\phi,$$

$$= \frac{a^2}{3}\int_0^{\frac{\pi}{2}} \sin^2\phi \, d\phi = \frac{a^2}{3}\frac{1}{2}\frac{\pi}{2} = \frac{\pi a^2}{12}.$$

The total area of the three loops is therefore $\frac{\pi a^2}{4}$.

5. Find the area of the curve

$$x = a\cos^3 t,$$
$$y = b\sin^3 t.$$

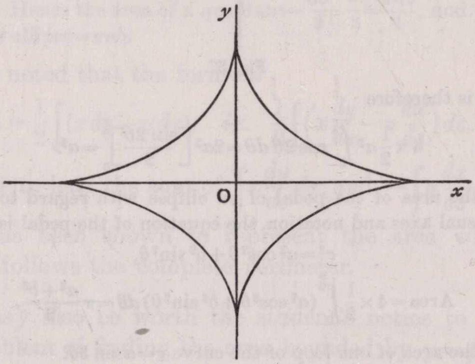

Fig. 59.

Upon elimination of t, we have $\left(\frac{x}{a}\right)^{\frac{2}{3}} + \left(\frac{y}{b}\right)^{\frac{2}{3}} = 1$, and the shape is shown in the figure. There is symmetry about both axes, and the area

$$= 4\int_0^a y\,dx = 4\int_{\frac{\pi}{2}}^0 b\sin^3 t(-3a\cos^2 t\sin t)\,dt$$

$$= 12ab\int_0^{\frac{\pi}{2}} \sin^4 t\cos^2 t\,dt$$

$$= 12ab\,\frac{\Gamma(\frac{5}{2})\,(\Gamma(\frac{3}{2})}{2\Gamma(4)} = 12ab\,\frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}\cdot\frac{1}{2}\sqrt{\pi}}{2\,.\,3\,.\,2\,.\,1}$$

$$= \tfrac{3}{8}\pi ab;$$

or we may use the formula

$$\frac{1}{2}\int\Big[F'(t)f(t) - f'(t)F(t)\Big]\,dt,$$

which gives

$$4\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}}(a\cos^4 t\,.\,3b\sin^2 t+b\sin^4 t\,.\,3a\cos^2 t)\,dt$$

$$=6ab\int_0^{\frac{\pi}{2}}(\cos^4 t\sin^2 t+\sin^4 t\cos^2 t)\,dt$$

$$=6ab\int_0^{\frac{\pi}{2}}\sin^2 t\cos^2 t\,dt$$

$$=6ab\,\frac{\Gamma(\frac{3}{2})\,\Gamma(\frac{3}{2})}{2\Gamma 3}=\tfrac{3}{8}\pi ab, \text{ as before.}$$

6. Find the area of the loop of the curve

$$x^5+y^5-5ax^2y^2=0.$$

(1) There is symmetry about the line $y=x$.

(2) There is an asymptote $x+y=a$.

(3) By Newton's rule, the form at the origin is that of two semicubical parabolas $y^3=5ax^2$, $x^3=5ay^2$.

The shape is then as shown in Fig. 60.

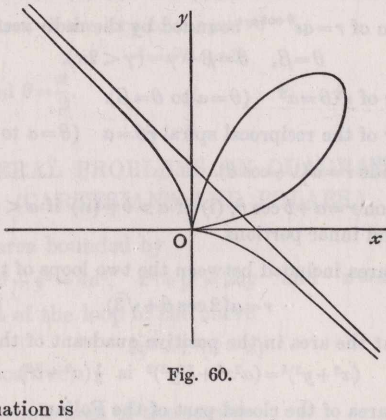

Fig. 60.

The polar equation is

$$r=5a\frac{\sin^2\theta\cos^2\theta}{\sin^5\theta+\cos^5\theta}.$$

As there is symmetry about $\theta=\frac{\pi}{4}$, we may take limits 0 to $\frac{\pi}{4}$ and double.

$$\text{Area of loop}=2\,.\,\tfrac{1}{2}\,.\,25a^2\int_0^{\frac{\pi}{4}}\frac{\sin^4\theta\cos^4\theta\,d\theta}{(\sin^5\theta+\cos^5\theta)^2},$$

or, putting $\tan\theta=t$,

$$\text{Area}=25a^2\int_0^1\frac{t^4\,dt}{(1+t^5)^2}$$

$$=5a^2\left[-\frac{1}{1+t^5}\right]_0^1=5a^2\left[-\tfrac{1}{2}+1\right]=\tfrac{5}{2}a^2.$$

Otherwise; this curve is unicursal; and we may write (putting $y=tx$)

$$x=\frac{5at^2}{1+t^5},\quad y=\frac{5at^3}{1+t^5},$$

and integrate $\frac{1}{2}\int\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)dt$, with limits 0 and ∞, which gives

$$\frac{(5a)^2}{2}\int_0^\infty \frac{t^2(3t^2-2t^7)-t^3(2t-3t^6)}{(1+t^5)^3}dt=\frac{25a^2}{2}\int_0^\infty\frac{t^4}{(1+t^5)^2}dt=\frac{5a^2}{2},$$

as before.

EXAMPLES.

Find the areas bounded by

1. $r^2=a^2\cos^2\theta-b^2\sin^2\theta$, the central pedal of a hyperbola.

2. One loop of $r=a\sin 4\theta$. Also state the total area.

3. One loop of $r=a\sin 5\theta$. Also state the total area.

4. One loop of $r=a\sin n\theta$.
Give the total area in the cases, (i) n even; (ii) n odd.

5. The portion of $r=ae^{\theta\cot\alpha}$ bounded by the radii vectores

$$\theta=\beta,\quad \theta=\beta+\gamma\quad (\gamma<2\pi).$$

6. Any sector of $r^{\frac{1}{2}}\theta=a^{\frac{1}{2}}\quad (\theta=\alpha \text{ to } \theta=\beta)$.

7. Any sector of the reciprocal spiral $r\theta=a\quad (\theta=\alpha \text{ to } \theta=\beta)$.

8. The cardioide $r=a(1-\cos\theta)$.

9. The Limaçon $r=a+b\cos\theta$, (i) if $a>b$; (ii) if $a<b$ obtain the two areas of outer and inner portions.

10. Find the area included between the two loops of the curve

$$r=a(2\cos\theta+\sqrt{3}).$$ [Oxf. I. P., 1889.]

11. Prove that the area in the positive quadrant of the curve

$$(x^2+y^2)^5=(a^2x^3+b^2y^3)^2 \text{ is } \tfrac{1}{3}(a^2+b^2).$$ [γ, 1899.]

12. Find the area of the closed part of the Folium

$$r=\frac{3a\sin\theta\cos\theta}{\sin^3\theta+\cos^3\theta}.$$ [I. C. S., 1884.]

13. Show that the area of a loop of the curve

$$ax^{2n+1}-b^2x^ny^n+cy^{2n+1}=0$$

is $\dfrac{1}{2(2n+1)}\dfrac{b^4}{ac}$, a and c being positive. [Colleges, 1881.]

14. Trace the curve whose equation is

$$r^4=a^4\sec\theta\tan\theta,$$

and find the area between the curve and any pair of radii vectores drawn from the pole. [Trinity, 1882.]

15. Trace the lemniscate $r^2=a^2\cos 2\theta$ and its first positive pedal, and show that the area of a loop of the latter is double the area of a loop of the former.

Find the areas of each of the two small lozenge-shaped portions common to the two loops of the pedal.

16. Show that the area contained between the curve

$$r=a\cos 5\theta$$

and the circle $r=a$ is three-fourths of the area of the circle.

[OXF. I. P., 1888.]

17. Find the area between the curve $r=a(\sec\theta+\cos\theta)$ and its asymptote. [ST. JOHN'S, 1881.]

18. Prove that the area of the curve

$$r^2(2c^2\cos^2\theta-2ac\sin\theta\cos\theta+a^2\sin^2\theta)=a^2c^2$$

is equal to πac. [I. C. S., 1879.]

19. Find the area of the curve

$$r=3a\cos\theta+a\cos 3\theta.$$ [MATH. TRIP., 1882.]

20. Find the area of the loop of the curve

$$r^2=a^2\theta\cos\theta$$

between $\theta=0$ and $\theta=\frac{\pi}{2}$.

GENERAL PROBLEMS ON QUADRATURE. (CARTESIANS AND POLARS.)

1. Find the area bounded by

$$x^2+y^2=4a^2,\quad x^2+y^2=2ay\quad\text{and}\quad x=a.$$ [H. C. S.]

Also the area of the loop of the curve

$$by^2=x^2(a-x)$$

(a and b both positive). [I. C. S., 1882.]

2. Find the whole area of the curve

$$y^2=x^2\frac{a^2-x^2}{a^2+x^2}.$$ [I. C. S., 1885; COLLEGES, 1892.]

3. A parabola $y^2=ax$ cuts the hyperbola $x^2-y^2=2a^2$ at the points P, Q; and the tangent at P to the hyperbola cuts the parabola again at R. Find the area of the curvilinear triangle PQR.

4. Find the area included between one of the branches of the curve $x^2y^2=a^2(x^2+y^2)$ and its asymptotes.

Find the whole area of the curve

$$x^4+y^4=a^2(x^2+y^2).$$ [COLLEGES a, 1887.]

5. Trace the curve $a^2y^2 = x^3(2a - x)$, and prove that its area is equal to that of the circle whose radius is a. [J. C. S., 1887.]

6. Prove that the whole area of

$$(x^2 + a^2) y^2 + 3a^3y + 2a^4 = 0$$

is $$(3 - 2\sqrt{2})\pi a^2.$$ [COLLEGES β, 1891.]

7. Find the area of the loops of the curve

$$y^4 - x^4 - a^2y^2 + b^2x^2 = 0 \text{ when } b^2 > a^2.$$

[OXFORD I. P., 1902.]

8. Find the area bounded by the cycloid

$$x = a(\theta + \sin\theta),$$
$$y = a(1 - \cos\theta),$$

and the straight line joining two consecutive cusps.

9. Show that the coordinates of a point P on the Folium of Descartes $x^3 + y^3 = axy$ can be expressed as

$$x = \frac{at}{1+t^3}, \quad y = \frac{at^2}{1+t^3}.$$

Show that as t varies from 0 to ∞ P traces out a closed loop, and that its area is $\dfrac{a^2}{6}$. [COLLEGES, 1896.]

10. Prove that the area of either loop of the curve

$$x^5 + y^5 - 5a^2x^2y = 0$$

is $$\frac{2\pi a^2}{\sqrt{10 + 2\sqrt{5}}}.$$ [γ, 1893.]

11. Show that in that part of the curve $(x + y - 3c)xy + c^3 = 0$ for which x is positive, the area between the curve, the axis of x, and the ordinate which touches the curve is $\frac{1}{2}c^2$. [ST. JOHN'S, 1886.]

12. Trace the curve $\quad y^4 + x^3y = a^2x^2,$
and show that the area of the segment which lies between the axis of y and the straight line whose equation is $y = x$ is $\dfrac{1}{6}a^2 \log 2$.

[COLLEGES ϵ, 1883.]

13. Pairs of ordinates of the hyperbola $xy = a^2$ are determined by the condition that the area included by any pair, the curve, and the x-axis is constant; show that the lengths of any such pair are in a constant ratio. [OXFORD I. P., 1888.]

14. Show that the area between the curve

$$x(x^2+y^2-a^2)+\frac{2}{9}a^3\sqrt{3}=0$$

and its asymptote is πa^2. [ST. JOHN'S, 1892.]

15. Show that the area between the inner branch of the curve

$$(x^2+y^2-a^2)^2=\tfrac{1}{4}x^2(x^2+y^2)$$

and the positive parts of the two axes is $\pi a^2/3\sqrt{3}$. [ST. JOHN'S, 1888.]

16. Prove that the whole area of the epicycloid generated by a point on a circle of radius $\frac{a}{4}$ rolling on a fixed circle of radius a is to the area of the fixed circle in the ratio of 15 to 8.

17. Find the whole area of the curve whose equation is $(x^2+y^2)(x+y+a)(x+y-a)+x^2y^2=0$. [COLLEGES, 1886.]

18. Find the area of a loop of the curve

$$x^4+y^4=2a^2xy.$$ [OXFORD I. P., 1888.]

19. Find the area cut off from an ellipse by a focal chord. [COLLEGES α, 1883.]

20. Prove that the areas cut off by the equiangular spiral $r=ae^{\theta\cot\alpha}$ from the space bounded by any two fixed lines through the pole are in geometrical progression. [OXFORD I. P., 1900.]

21. Find the area of the curve $r=a\theta e^{b\theta}$ enclosed between two given radii vectores and two successive branches of the curve. [TRINITY, 1881.]

22. Find the area of the loop of the curve $r=a\theta\cos\theta$ between $\theta=0$ and $\theta=\frac{\pi}{2}$. [OXFORD II. P., 1890.]

23. Find the area of the curve

$$(r-a\cos\theta)^2=a^2\cos 2\theta.$$ [COLLEGES α, 1887.]

24. Show that the area of the loop of the folium $x^3+y^3=3axy$ is divided by the parabola $y^2=ax$ in the ratio $5:4$.

In what ratio does the line $x+y=2a$ cut the loop in the above folium. [OXFORD I. P., 1889.]

25. Find the area included between the axis of y and the curve

$$y^2+2y-2x(y+1)=x^4-3x^3+3,$$

the curve being supposed to stop at the node. [ST. JOHN'S, 1884.]

26. Determine by integration the area of the ellipse

$$x^2+xy+y^2=1.$$

27. (i) Find the whole area enclosed by the hypocycloid

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}.$$ [OXFORD I. P., 1888.]

(ii) Prove that the area of the locus of intersection of pairs of tangents at right angles for this curve is $\frac{1}{4}\pi a^2$. [MATH. TRIPOS, 1888.]

28. Prove that the locus of the points of bisection of the intercepts on the normals of a cycloid between the cycloid and its base divides the area between the cycloid and its base into two parts in the ratio 7 : 5. [OXFORD II. P., 1886.]

29. Trace the curve $x^{2n+1}+y^{2n+1}=(2n+1)ax^ny^n$, when n is even, and when n is odd, n being a positive integer; and prove that the area of the loop is $(2n+1)\dfrac{a^2}{2}$. Prove that this is also the area between the infinite branches of the curve and the asymptote. [ST. JOHN'S, 1882.]

30. Find the whole area contained between the curve

$$x^2(x^2+y^2)=a^2(y^2-x^2)$$

and its asymptotes. [OXFORD I. P., 1887.]

31. Find the area bounded by the circle $x=a\cos\theta$, $y=a\sin\theta$ and the hyperbola $x=b\cosh u$, $y=b\sinh u$; that area being taken which lies within the circle and on the convex side of the hyperbola, and b being less than a. [TRINITY, 1888.]

32. (a) Show that in the Archimedean Spiral $r=a\theta$, if A_1, A_2, A_3, A_4, ... be the areas of the inner loop and the successive heart-shaped figures formed by the convolutions of the curve

$$A_1=\frac{\pi^2a^2}{4},\quad A_{n+1}=2n\pi^2a^2.$$

(b) In the Reciprocal Spiral $r\theta=a$, if A_1, A_2, A_3 ... be the areas of the successive closed loops,

$$A_n=\frac{4a^2}{\pi}\,\frac{1}{4n^2-1}.$$

33. Find the area of the loop of the curve

$$(x+y)(x^2+y^2)=2axy.$$ [OXFORD I. P., 1890.]

34. At all points of the first negative pedal of the curve $r=\cosh(m\theta\cot\alpha)$ lines are drawn making a constant angle α with

the tangent. Show that the area bounded by any pair of such lines, the curve enveloped and the first negative pedal is

$$A\{1+(m^2-1)\cos^2\alpha\},$$

where A is the area of the corresponding portion of the first negative pedal bounded by radii vectores from the pole.

[COLLEGES α, 1891.]

35. Find the area of that portion of the loop of the curve

$$r^2 = p\cos\theta + q\sin\theta,$$

which is not enclosed by the curve

$$r^2 = b + a\cos\theta.$$

If a family of such curves be taken (by varying p and q), such that this area is constant, show that the envelope of the system is a curve whose equation is

$$r^2 = c + a\cos\theta.$$

[COLLEGES β, 1889.]

36. Show that the whole area enclosed by the outer line of the curve $r^{\frac{2}{3}} = a^{\frac{2}{3}}\cos\frac{2}{3}\theta$ is $\frac{9}{8}a^2\sqrt{3}$. [COLLEGES, 1876.]

37. In a hyperbola, C is the centre, A the end of the transverse axis and P any point (x, y) on the same branch of the curve as A; prove that twice the area of the sector CAP is

$$ab\log\left(\frac{x}{a}+\frac{y}{b}\right).$$

38. Show that the area contained between a hyperbola, any tangent and a line parallel to the asymptote which bisects the part of the tangent intercepted between the curve and the asymptote

$$=\frac{ab}{2}(\log 2 - \tfrac{5}{8}),$$

and is constant. [TRINITY, 1886.]

39. Prove that the area of the curve

$$x = \frac{ap}{(1+p^2)^2}, \quad y = \frac{1}{2}\,\frac{ap^2(1-p^2)}{(1+p^2)^2}$$

is $\frac{1}{32}\pi a^2$. [MATH. TRIPOS, 1882.]

40. Show that the area cut off from the ellipse

$$ax^2 + 2hxy + by^2 = 1$$

by the line $lx + my = 1$ is

$$\alpha\beta(\theta - \sin\theta\cos\theta),$$

where α, β are the semiaxes of the ellipse and

$$\cos\theta = \frac{\sqrt{ab-h^2}}{\sqrt{am^2+bl^2-2hlm}}.$$

[COLLEGES, 1892.]

41. Trace the curve whose equation is

$$x(x^{2n}+y^{2n})=a^2y^{2n-1},$$

and prove that the area between the curve, the axis of x and a tangent parallel to the axis of y is

$$\frac{a^2}{4n}(2n-1-\log 2n).$$

[ST. JOHN'S, 1885.]

42. Show that in the curve

$$r^2=\sec 2\theta \log(2\cos^2\theta)$$

the area between the curve and the lines $\theta=\pm\frac{1}{4}\pi$ is $(\frac{1}{4}\pi)^2$.

[ST. JOHN'S, 1886.]

43. Find in integral form, and completely, the area enclosed between two confocal conics and two given radii from the centre.

[TRINITY, 1881.]

44. Prove that the area of each of the two equal and similar pieces of the ellipse $x^2/a^2+y^2/b^2=1$ which are cut off by the hyperbola $x^2/\alpha^2-y^2/\beta^2=1$ $(\alpha<a)$ is

$$ab\sin^{-1}\frac{\beta(a^2-\alpha^2)^{\frac{1}{2}}}{(a^2\beta^2+\alpha^2b^2)^{\frac{1}{2}}}-\alpha\beta\sinh^{-1}\frac{b(a^2-\alpha^2)^{\frac{1}{2}}}{(a^2\beta^2+\alpha^2b^2)^{\frac{1}{2}}}.$$

[ST. JOHN'S, 1887.]

45. Prove that the areas of the two loops of the curve

$$r^2-2ar\cos\theta-8ar+9a^2=0$$

are $\quad (32\pi+24\sqrt{3})a^2 \quad$ and $\quad (16\pi-24\sqrt{3})a^2.$

[MATH. TRIPOS, 1875.]

46. The area between two tangents to the same convolution of an equiangular spiral at right angles to one another, and the curve, is

$$pp'+\tfrac{1}{2}(p^2-p'^2)\cot 2\gamma,$$

where p, p' are the perpendiculars from the pole on the tangents and γ is the angle of the spiral.

[COLLEGES, 1882.]

47. A circle with centre at the origin cuts the loop of the Folium $x^3+y^3-3axy=0$. If the angle subtended at the origin by the common chord equals

$$2\tan^{-1}\frac{2^{\frac{1}{3}}-1}{2^{\frac{1}{3}}+1},$$

prove that the area between the loop and the circle is

$$\frac{a^2}{2}\left[1-(2^{\frac{2}{3}}+2^{\frac{4}{3}})\tan^{-1}\frac{2^{\frac{2}{3}}-1}{2^{\frac{4}{3}}}\right].$$

[COLLEGES, 1885.]

48. The centre of a circle of constant radius a moves along a fixed straight line AB in its plane, and from A a fixed point in the line a tangent AP is drawn to the circle. Show that the area included between the locus of P and its asymptotes is πa^2.

[MATH. TRIPOS, 1882.]

49. Show that the curve

$$r = a\left(\frac{1}{2}\sqrt{3} + \cos\frac{\theta}{2}\right)$$

has three loops, whose areas are

$$a^2(\tfrac{5}{4}\pi + 2\sqrt{3}), \quad a^2(\tfrac{5}{6}\pi - \tfrac{5}{4}\sqrt{3}), \quad a^2(\tfrac{5}{12}\pi - \tfrac{3}{4}\sqrt{3})$$

respectively. [COLLEGES, 1892.]

50. Show that the area of the Cassinian

$$r^4 - 2a^2r^2\cos 2\theta + a^4 = b^4$$

is $$2\int_0^{\frac{\pi}{2}} \sqrt{b^4 - a^4\sin^2\phi}\, d\phi, \quad \text{provided } b > a,$$

but is $$2\int_0^{\frac{\pi}{2}} \frac{b^4\cos^2\phi\, d\phi}{\sqrt{a^4 - b^4\sin^2\phi}}, \quad \text{when } a > b.$$

[MATH. TRIPOS, 1883.]

51. Prove that the area of the first negative pedal of an ellipse with respect to the focus is

$$\frac{\pi a^2(2-3e^2)}{2\sqrt{1-e^2}} \quad (e < \tfrac{1}{2}),$$

where a and e are the semi major axis and the eccentricity of the ellipse. [COLLEGES, 1892.]

How do you interpret this result if $e < \frac{1}{2}$?

52. Find the area of the curve whose Cartesian equation is

$$a^2(y-x)^2 = (a+x)^3(a-x).$$

[MATH. TRIPOS, 1896.]

53. Find the value of $\int_0^1 v_x\, dx$, v_x being the real root of the cubic

$$v_x{}^3 + v_x{}^2 v_1 + v_x v_1{}^2 - \frac{c}{x} = 0.$$

[COLLEGES, 1872; R. P.]

54. Find the area in the first quadrant bounded by the axes of coordinates and the curve

$$\sinh^{-1}\frac{x}{a} + \sinh^{-1}\frac{y}{b} = c,$$

taking a, b, c all positive. [I. C. S., 1897.]

55. Trace the whole curve

$$x^2y^2 = c^2(a-x)(x-b),$$

where $0 < b < a$, and find its whole area. [I. C. S., 1898.]

56. It is given that the abscissa ON and ordinate NP of a point on any arch of a cycloidal arc are $a(\theta - \sin\theta)$ and $a(1-\cos\theta)$. NP is produced to K so that $NK = 2a$, and the rectangle $ONKA$ is completed. Prove that the area included by ON, NP and the arc OP never differs from three-fourths of $ONKA$ by more than $\frac{3a^2}{8}\sqrt{3}$; and find for what positions of P the difference vanishes.

[I. C. S., 1912.]

57. Trace on squared centimetre paper the curves

$$x^4 + y^4 = 4a^2xy,$$
$$x^4 + y^4 = 4ax^2y,$$

taking $a = 10$ cm., and estimate the area of a loop of each curve.

Prove that $$\int_0^\infty \frac{t^2}{(1+t^4)^2}\,dt = \frac{1}{4}\int_0^\infty \frac{t^2}{1+t^4}\,dt = \frac{\pi}{8\sqrt{2}},$$

and hence calculate the area of a loop of the second curve. Find also the area of a loop of the first curve. Give each area to the nearest square centimetre when a is 10 centimetres.

[C. S., 1913.]

58. Obtain the area contained between the two curves

$$r^2\cos 2\theta = 4a^2\cos^4\theta \quad \text{and} \quad r^2\cos 2\theta = a^2.$$

[Oxf. I. P., 1912.]

59. Show that the area of the loop of the curve

$$x^7 + y^7 = ax^3y^3$$

is equal to $a^2/14$. [Oxf. I. P., 1914.]

60. Prove by any method that the area of the ellipse

$$\{a(x-2)+3y\}^2 + 4(x+1)(x-2) = 0$$

is independent of a, and find the area.

Prove also that the straight line $y = x$ divides the ellipse $x^2 + 3y^2 = 6y$ into two areas which are in the ratio

$$4\pi - 3\sqrt{3} : 8\pi + 3\sqrt{3}.$$ [Oxf. I. P., 1916.]

61. Trace the curve

$$r\cos\theta = a\sin 3\theta,$$

and show that the area of a loop is

$$\tfrac{1}{8}a^2(9\sqrt{3} - 4\pi).$$ [Math. Trip. I., 1919.]

62. Show that the curve $r = a(2\cos\theta + \cos 3\theta)$ has three loops, the area of the larger loop being $\dfrac{10\pi + 9\sqrt{3}}{12}a^2$, and the areas of the two smaller loops being $\dfrac{5\pi - 9\sqrt{3}}{24}a^2$. [MATH. TRIP. I., 1916.]

63. Show that the coordinates of any point on the curve

$$y^2(a+x) = x^2(3a-x)$$

may be taken as

$$x = a\sin 3\theta/\sin\theta, \quad y = a\sin 3\theta/\cos\theta,$$

and prove that the area of the loop and the area between the curve and its asymptote are both equal to $3\sqrt{3}a^2$. [MATH. TRIP. I., 1915.]

64. Show that the area of the loop of the curve

$$(x^2+y^2)^2 - 4axy^2 = 0$$

in the positive quadrant is $\frac{1}{4}\pi a^2$. [MATH. TRIP. I., 1920.]

65. Having established Simpson's Rule, that if

$$y = y(x) \equiv a_0 + a_1x + a_2x^2 + a_3x^3,$$

then $$\int_0^1 y\,dx = \tfrac{1}{6}\{y(0) + y(1) + 4y(\tfrac{1}{2})\},$$

prove that if $y(x)$ also contains a term a_4x^4 the error in still using Simpson's Rule is

$$\frac{1}{120}a_4.$$ [MATH. TRIP. I., 1920.]