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ON THE ROTATION OF A RIGID BODY.

[Three letters to *Nature*, Vol. I. (1870), pp. 482, 532, 582.]

The Motion of a Free rotating Body.

I SHALL feel obliged if, through the medium of your widely-circulated journal, you will allow me to point out an extraordinary mistake into which M. Radau has fallen, in a memoir inserted in the *Annales Scientifiques de l'École Normale Supérieure*, tom. VI. 1869, in which he criticises certain of my conclusions about the representation of the motion of a free rotating body contained in a paper published by me in the *Philosophical Transactions* for 1866*. In his preamble, M. Radau says, speaking of the theory of rotation in connection with the names of Poinsot, Rueb, Jacobi, and Richelot:—"Tout récemment M. Sylvester a essayé d'appliquer au même sujet des considérations nouvelles qui l'ont conduite à des résultats intéressants, à côté d'autres dont l'exactitude peut être contestée."

Later on in his memoir M. Radau points out, and accompanies with very biting (albeit toothless) criticism, the nature of his objection, which is, in short, that I suppose Poinsot's ellipsoid, under the influence of an original impulse, to roll without slipping by virtue of its friction against the plane with which it is in contact. My answer is, that of course I do. And why not? when I suppose the plane "indefinitely rough" (see p. 761 of *Philosophical Transactions*, 1866†), and have actually determined the friction and pressure at each point of the motion, so that by solving a maximum and minimum problem of one variable, the extreme value of the ratio of one of these forces to the other, or if we please to say so, the limiting angle of friction, or, in other words, the necessary degree of roughness of the plane, may be analytically determined for every given case. M. Radau falls into the school-boy blunder of making the *ratio between the friction and pressure constant throughout the motion*, confounding the actual friction with its limiting maximum value! It is, indeed, surprising that such a perversion

[* Vol. II. of this Reprint, p. 577.] [† *ibid.* p. 582.]

of the facts of the case should have found insertion in a serious journal, such as that published by the École Normale Supérieure, and I might fairly have expected from M. Radau the courtesy habitual with his adopted countrymen, of applying to me for information on anything in my paper which might have appeared to him obscure or erroneous, before rushing into print with such a *mare's nest*.

But out of evil cometh good. M. Radau says:—"Mais M. Sylvester va plus loin; il pense que le problème pourrait se résoudre par l'observation directe du mouvement d'un ellipsoide matériel tournant sur un plan fixe en même temps qu'il tournerait autour de son centre également fixe. On ne se figure pas facilement par quel artifice on fixerait le centre d'un ellipsoide matériel."

In a future number of your esteemed journal (as time at present fails me) I propose to show how, by the simplest contrivance in the world, a downright material top of ellipsoidal form may be actually made to roll, with its centre fixed, on a fixed plane and so exhibit to the eye the surprising spectacle of a motion precisely identical *in time*, as well as in its successive displacements of *position*, with that of a body, turning round a fixed centre, but otherwise absolutely unconstrained.

This mode of representation, which flashed upon my mind almost instantaneously when my eye first lighted upon M. Radau's objections, is the compensating good to the evil of being made the victim (to the temporary disturbance of my beloved tranquillity) of so hasty and futile a criticism as has been allowed insertion in the "Scientific Annals" of so great an institution as the École Normale of Paris.

The *bureau de rédaction* must surely have been nodding when they allowed such observations, so easily refuted by turning to the original memoir, to pass unchallenged. It was only within the last few days that I received M. Radau's paper.

Rotation of a Rigid Body.

My previous communication about the rotating ellipsoid to this journal, has attracted the attention of M. Radau. "One touch of *Nature* makes the whole world kin." In a note addressed to me full of true dignity, this gentleman has made much more than sufficient reparation for his previous trifling act of inadvertence, and states that to his great regret he had misunderstood my meaning, in the passage of my memoir in question, and that "sa critique n'est pas fondée." I, on my part, deeply lament the unnecessary tone of acerbity in which my reference to this criticism was couched, and wish I could recall every ungracious expression which it contains. "When I spoke that, I was ill-tempered too."

I will pass over this, to me, painful topic, to say two or three words on the mode in which the rotating ellipsoid may be supposed to roll or *wobble* on a rough plane, with its centre fixed. My solution may remind the reader of Columbus' mode of supporting an egg on its point—or, rather, of a fairer mode which Columbus might have employed, and which would not have necessitated the breaking of the shell, namely, by resting the blade of a knife or rough plate on the upper end of his egg.

So, to make an ellipsoidal or spheroidal top roll, with its centre fixed—say, upon a rough horizontal plane—imagine a second horizontal plane in contact with the upper portion of its surface; then the line joining the two points of contact will pass through the centre of the top. We may conceive a slight perforation in either or each plane at its initial point of contact with the top, and a screw wire introduced through this, and inserted into a female screw in the body to be set rolling (a mode of spinning which Sir C. Wheatstone recommends as the most elegant in any case, and in this case evidently the most eligible). On withdrawing the wire with a jerk, the top may be set in motion about its centre, in such a direction as to remain in contact with the two planes, and if these be sufficiently rough the motion will eventually be reduced to one of pure rolling between them, the axis (that is, the line joining the two points of contact), continually shifting, but the centre remaining absolutely stationary: for, vertical motion this point cannot have, so long as the top continues to touch both planes, and any slight horizontal motion (if it should chance to take on such at the outset) would be checked and ultimately destroyed by the friction, which would also keep the two points of contact stationary (like the single point of contact of a wheel rolling on a rail), in each successive atom of time. Thus the motion upon the lower plane would in the end be precisely the same as if the upper plane were withdrawn, and the centre of the top kept fixed by some mechanical adjustment. If the spin were not sufficiently vigorous, after a time the rolling top might quit the upper plane, and of course sooner or later by the diminution of the *vis-viva* due to adhesion, resistance of the air, imperfection or deformation of the surfaces, and other disturbing causes, this would take place, but abstracting from these circumstances the principal axes of the spheroidal or ellipsoidal top would move precisely in place and time like the "axes of spontaneous rotation" of any free body of which the top was the "Kinematic Exponent."

I do not pretend to offer an opinion what materials for the planes and rolling body (ground glass and ebony or roughened ebonite have been suggested to me) it would be best to employ, or whether the "wobbling top" could easily be made to exhibit its evolutions. It is enough for a non-effective, unpractical man (as unfortunately I must confess to being) to have shown that there is no intrinsic impossibility in the execution of the conception.

With regard to the friction and pressure: if W be the weight of the body, F and P the friction and pressure in the case of a single plane (the values of which are set out in my memoir, pp. 764—766, *Philosophical Transactions*, 1866*), it may easily be proved that eventually the friction at each point of contact will be $\frac{F}{2}$, the pressure upwards at the lower point $\frac{P+W}{2}$, and downwards at the upper one $\frac{P-W}{2}$, so that if P should become equal to W the top would quit the upper plane and the experiment come to an end. At p. 766 of my memoir the factor $\sqrt{M\Lambda}$ has accidently dropped out of the expression for P which I mention here, in case any one should feel inclined to consult the memoir in consequence of this note. Mr Ferrers has taken up my investigations, and given more compendious expressions than mine for F and P; with the aid of these it would probably be not difficult to determine the maximum value of $\frac{F}{P}$ so as to assign the necessary degree of roughness of the confining planes, and also to ascertain under what circumstances $P-W$ would become zero, but I do not feel sufficient interest in the question, nor have I the courage to undertake these calculations with the complicated forms of P and F contained in my memoir. Mr Ferrers' results are contained in a memoir ordered to be printed in the *Philosophical Transactions*, and will shortly appear.

In my memoir will be found an exact kinematical method of reckoning the time of rotation by Poinsot's ellipsoid when the lower surface is made to roll on one fixed plane at the same time that its upper surface is sharpened off in a particular way (therein described) so as to roll upon a parallel plane which turns round a fixed axis; this upper plane is compelled to turn by the friction, and acts the part of a moveable dial in marking the time of the free body imaginarily associated with the ellipsoid. I have also shown there that the motion of any free body about a fixed centre may be regarded as compounded of a uniform motion of rotation and the motion of a disc, or, if one pleases, a pair of mutually bisecting cross-wires left to turn freely about their centre. But I fear that *Nature*, used to a more succulent diet, has had as much as it can bear upon so dry a topic, and, although having more to say, deem it wiser to bring these remarks to an end.

An after-dinner experiment.

Suppose in the experiment of an ellipsoid or spheroid, referred to in my last letter, rolling between two parallel horizontal planes, we were to scratch on the rolling body the two equal similar and opposite closed curves (the *polhods* so-called), traced upon it by the successive axes of instantaneous rotation; and suppose, further, that we were to cut away the two extreme

[* Vol. II., above, pp. 585, 587.]

segments marked off by those tracings, retaining only the barrel or middle portion, and were then to make this barrel roll under the action of friction upon its bounding curved edges between the two fixed planes as before, or more generally, imagine a body of any form whatever bounded by and rolling under the action of friction upon these two edges between two parallel fixed planes; it is easy to see that, provided the centre of gravity and direction of the principal axis be not displaced, the law of the motion will depend only on the relative values of the principal moments of inertia of the body so rolling, in comparison with the relative values of the axes of the ellipsoid or spheroid to which the *polhods* or rolling edges appertain; and consequently, that, when a certain condition is satisfied between these two sets of ratios, the motion will be similar in all respects to that of a free body about its centre of gravity.

That condition (as shown in my memoir in the *Philosophical Transactions**) is, that the nine-membered determinant formed by the principal moments of inertia of the rolling body, the inverse squares and the inverse fourth powers of the axes of the ellipsoid or spheroid shall be equal to zero—a condition manifestly satisfied in the case of the spheroid, provided that two out of the three principal moments of inertia of the rolling solid are equal to one another.

My friend Mr Froude, the well-known hydraulic engineer, with his wonted sagacity, lately drew my attention to the familiar experiment of making a wine-glass spin round and round on a table or table-cloth upon its base in a circle without slipping, believing that this phenomenon must have some connection with the motion referred to in my preceding letter to *Nature*: an intuitive anticipation perfectly well-founded on fact; for we need only to prevent the initial tendency of the centre of gravity to rise by pressing with a second fixed plane (say a rough plate or book-cover) on the top of the wine-glass, and we shall have an excellent representation of the free motion about their centre of gravity of that class of solids which have, so to say, a natural momental axis, that is (in the language of the schools) two of their principal moments of inertia equal. For greater brevity let me call solids of this class uniaxal solids. I suppose that the centre of gravity of the glass is midway between the top and bottom, and that the periphery of the base and of the rims are circles of equal radius. These circles will then correspond to *polhods* of a spheroid, conditioned by the angular magnitude and dip of the spinning glass; to determine from which two elements the ratio of the axes of the originally supposed but now superseded representative spheroid is a simple problem in conic sections; this being ascertained, the proportional values of the moments of inertia of the represented solid may be immediately inferred. The wine-glass

[* Vol. II., above, p. 583.]

itself belonging to the class of uniaxal bodies, the condition that ought to connect its moments of inertia with the axis of the representative spheroid (in order that the motion may proceed *pari passu* with that of a free body) is necessarily satisfied.

The conclusion which I draw from what precedes is briefly this—that a wine-glass equally wide at top and bottom, and with its centre of gravity midway down, spinning round upon its base and rim in an inclined position between two rough but level fixed horizontal surfaces, yields, so long as its *vis-viva* remains sensibly unaffected by disturbing causes, a perfect representation, both in space and time, of the motion of a free uniaxal solid, as for example, a prolate or oblate spheroid, or a square or equilateral prism or pyramid about its centre of gravity, and conversely that every possible free motion about its centre of gravity of every such solid admits of being so represented.

To revert for an instant to the general question of the representative rolling ellipsoid, I think it must be admitted that the addition of the time element to the theory and the substitution of a second fixed plane in lieu of a fixed centre, considerably enhance the value and give an unexpected roundness and completeness to Poinsot's image of the free motion of rotation of a rigid body, of which so much and not altogether undeservedly has been made. From an idea or shadow Poinsot's representation has now become a corporeal fact and reality, as if, so to say, Ixion's cloud, in a moment of fruition, had substantified into a living Juno. I heard the late Professor Donkin, of revered and ever-to-be-cherished memory, state that when as a referee of the Royal Society he first took in hand my paper on rotation, he did so with a conviction that all had already been said that could be said on the subject, and that it was a closed question; but that when he laid down the memoir he saw reason to change his opinion. I owe my thanks to M. Radau and the editors of the Annals of the École Normale Supérieure for having been at the pains to disentomb the little-known conclusions therein contained from their honourable place of sepulture in the *Philosophical Transactions*.