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ON RECENT DISCOVERIES IN MECHANICAL CONVERSION OF MOTION.

[*Proceedings of the Royal Institution of Great Britain*, VII. (1873—75), pp. 179—198. Also *La Revue Scientifique*, 1874—75, pp. 490—498, and *Van Nostrand's Engineering Magazine* (New York) XII. (1875), pp. 313—321.]

THE speaker stated that the subject he proposed to bring under the notice of the meeting related mainly to the discovery of a perfect parallel motion,—that is to say, of a mode of producing motion in a straight line by a system of pure link-work without the aid of grooves or wheel-work, or any other means of constraint than that due to fixed centres, and joints for attaching or connecting rigid bars. This important discovery was made by M. Peaucellier, an officer of Engineers in the French army*—and first published by him, in the form of a question, in the *Annales de Mathématique* in the year 1864, and subsequently formed the subject of two communications to the *Société Philomathique* of Paris by Captain Manheim, but seems not to have received the attention it deserved from that learned body, and may be said to have passed into oblivion; so much so, that when rediscovered by a young student of the University of St Petersburg, of the name of Lipkin, several years subsequently, the discovery was attributed to Lipkin instead of to Peaucellier even in works published in the French language, and so recently as 1873 by M. Colignan, in his *Traité de Cinématique*. The eminent Professor Tchebicheff had long occupied himself with the question, but with less than his usual success in overcoming difficulties insuperable to the rest of the world. Lipkin was a student in his class, and may thus have had his attention turned to the question; at all events, Professor Tchebicheff's warm interest in the subject was displayed by his bringing Lipkin's name before the Russian Government, and securing for him a substantial reward for his

* Now Colonel Peaucellier, and in command of the fortress of Toul; at the time of his discovery lieutenant and officier d'ordonnance on the staff of the "illustrious Marshal Niel."

supposed original discovery. Before Peaucellier's time all so-called parallel motions were imperfect, and gave merely approximate rectilinear motion*; in substance they will be without exception found to be merely modifications of Watt's original construction, and to depend on the motion of a point in, or rigidly connected with, a bar joining the extremities of two other bars rotating round fixed centres, which may be described briefly as three-bar motion. Peaucellier's exact parallel motion depends on a link-work of seven

* The late lamented Professor Rankine, in his treatise on Millwork, and elsewhere, mentions a so-called "exact parallel motion," the invention of which he dubiously assigns to Mr Scott Russell. In its *exact* form this is no parallel motion at all, for it works by means of a slide, and in its modified form it ceases to be *exact*, the motion produced being no longer truly rectilinear.

Mr Kaulbach, a mechanical draughtsman, resident in London, has shown the speaker a sketch of a very ingenious *quasi*-parallel motion, which he took the first steps to patent a year or two ago, but has not thought it worth his while to proceed with further. Its principle depends upon finding a curve made to rotate about a fixed point, and enjoying the property that the tangent to each point of it, as that point passes a given vertical line, shall take up a horizontal position. A piston-rod is guided in the direction of such vertical line, and the beam, which always presses on a friction wheel attached to the rod, is so shaped in its outward contour as to satisfy the above condition; the consequence is that the reaction on the piston-rod can only take effect vertically, that is, in the direction of its motion, and no lateral pressure is produced.

Peaucellier's invention effects the perfect conversion of circular into linear motion. An easy practical deduction from this is the conversion of spherical into plane motion, by aid of universal joints and other familiar modes of effecting free motion in space, of a shaft about a fixed point or round another shaft. The announcement of these facts has occasioned many persons unacquainted with the technical language of mechanism to suppose that the discovery of Peaucellier is connected with the quadrature of the circle or cubature of the sphere, and led to the idea that the speaker was in possession of some secret for flattening spheres and turning circles into right lines. Such a misconception was one (as indeed the wide extent of its prevalence demonstrates) quite likely to occur even to intelligent persons untrained in mathematical science. Technical names are a frequent occasion of traps to the uninitiated. A lady present at one of Mr Norman Lockyer's course of lectures on Spectral Analysis, near the close of it was overheard inquiring with some anxiety as to "when the spectres might be expected to make their appearance." Names are of course all-important to the progress of thought, and the invention of a really good name, of which the want, not previously perceived, is recognized, when supplied, as having ought to be felt, is entitled to rank on a level in importance with the discovery of a new scientific theory. Imagine *plane*, *straight*, *circle*, and you are potentially a geometer. Think the meaning of the one word *Syzygy*, and the logic of algebra has become part of your being. But, on the other hand, there are cases where over-naming does harm. The speaker has no doubt that if reading music on the piano with the fingers were taught without the intervention of learning the names of the notes, twice the velocity of execution (and quick reading is here the *sine-quâ-non* for the existence of every other kind of excellence) might be acquired in half the time required under the present system. The names of the notes of course would have to be learned at a later stage as a medium for discourse; but they should not be used as a vehicle for obtaining command of digitation, as such use amounts to throwing upon the brain the labour of going through two steps when one would suffice, and the passage of a direct nervous current from the eye to the touch in the act of reading, even at an advanced stage, becomes by force of habit interrupted and diverted into a broken channel. The new method for learning to read on the pianoforte here suggested may be distinguished as the abnominal or undenominational or tactile method. The writer is prepared to show in detail how it can be carried out in practice.

bars moving like Watt's, and the other imperfect parallel motions of the same class, round two fixed centres*.

To understand the principle of Peaucellier's link-work, it is convenient to consider previously certain properties of a linkage† (to coin a new and useful

* The perfect parallel motion of Peaucellier looks so simple and moves so easily that people who see it at work almost universally express astonishment that it waited so long to be discovered. The idea of the facility of the result by a natural mental illusion gets transferred to the process of conception, as if a healthy babe were to be accepted as proof of an easy act of parturition. No impression can be more erroneous. The speaker, on the contrary, the more he reflects upon the problem that was to be solved, and the nature of the solution (essentially a process of transformation operating on polar co-ordinates), wonders the more that it was ever found out, and can see no reason why it should have been discovered for a hundred years to come. Viewed *à priori* there was nothing to lead up to it. It bears not the remotest analogy (except in the fact of a double centring) to Watt's parallel motion or any of its progeny. In the three-bar motion the two fixed points are so to say one as good as the other, there is no distinction to be drawn between them; whereas the two fixed centres (hereafter designated as the fulcrum and pivot) in Peaucellier's seven-bar arrangement are absolutely dissimilar in position and function. Peaucellier's apparatus naturally resolves itself into a cell and a spare link; no such decomposition presents itself in the three-bar motion. Again, looking at the matter *à posteriori*, it occurs to many well-grounded mathematicians to suppose that, as the most general motion of a link-work of seven or any number of bars for each possible mode of conjunction and centring must be capable of being expressed by a general algebraical equation, the particular combination for rectilinear motion, when such motion is possible, ought to be contained therein and inferrible therefrom by studying under what conditions the characteristic of the general equation can degenerate into a power of a linear function or, as might perhaps happen (and would be sufficient if it did), into such power multiplied by a function incapable of changing its sign. But the answer to this is that *practically* there could be little or no hope of ever obtaining the general equation. In one-bar motion the general curve (that is, a circle) is of the 2nd order; in three-bar motion, as is well known, of the 6th order; very likely, therefore, in five-bar motion it would be of the 24th order at least; and in seven-bar motion, of the 120th order at least. The equation or system of equations of the 120th order, supposed to be applicable to seven-bar motion, one could hardly dream of obtaining, or of being able to manipulate if obtained. Written out at full length in a handwriting of moderate size, the area of a very large room might be insufficient to contain the whole of its terms, which would consist of 7381 groups, and might be tens or hundreds of thousands in number. No; it must either have been fallen upon in a chance or experimental way, and subsequently verified theoretically, or else hit off in some sudden glow of insight akin to but of a much intenser degree of illumination than that under which Professor Stokes was able to see that the hydrodynamical theorem of Lagrange before him, proved imperfectly by its author and others, and correctly but with great difficulty by Cauchy, was an immediate inference from the pretty nearly self-obvious fact of the complete time-derivatives of the three quantities to be proved *if ever then always zero*, being by virtue of the well-known general hydrodynamical equations, syzygetic functions of these quantities themselves. Dr Tchebicheff has informed the writer that he has succeeded in proving the non-existence of a five-bar link-work capable of producing a perfect parallel motion; he is probably therefore in possession of the actual numerical order of the general equation or system of equations applicable to this case. It is not proved, and may not be true, that Peaucellier's is the only seven-bar link-work that will solve the problem of a perfect parallel motion. Who shall say whether there may not exist some other combination of seven bars in which the same or an analogous zig-zag symmetry to that which exists in the three-bar arrangement may reappear! This is a point which should not be allowed to remain subject to doubt.

† A link-work consists of an odd number of bars, a linkage of an even number. A linkage may be converted into a link-work *additively* by fixing one point of it as a fulcrum and attaching

word of general application), consisting of an arrangement of six links, obtained in the following manner:—first conceive a rhomb or diamond formed by four equal links joined to one another; and now suppose a pair of equal links to be joined on to two opposite angles of such figure and to each other. All six links are supposed to lie (and to be constrained by the nature of their attachments to remain) in the same plane. The point of junction of the last-named pair of links (which it will be found convenient to call the fulcrum), according as they are greater or smaller than the sides of the diamond, will lie outside or inside the diamond. The linkage consisting of the six links may be termed a positive *cell* in the one case and a negative *cell* in the other*. It is easily seen, as a geometrical necessity, that the fulcrum,

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a second point disconnected from the first by a new link to another fulcrum, or *ablatively* by fixing two ends of a link, which may then be removed. When one point only of a linkage is fixed, any other point may be made to describe an arbitrary curve, but then the path of every other point becomes prescribed. In order for a combination of links to fulfil this so to say fatalistic condition, and to entitle it to the name of a linkage in the speaker's sense, which when greater precision is required may be distinguished as a *perfect* linkage, equivalent to the French *système de tiges à liaison complète*, a numerical relation must be satisfied between the number of links and the number of joints, namely, three times the number of links must be four greater than twice the number of joints. In applying this rule it must be understood that, if three links are jointed together, the junction counts for two joints; if four are jointed together, for three joints; and so on. A compass or a pair of scissors is the simplest kind of linkage; a set of lazy-tongs is another; a Peaucellier cell, subsequently described in the text, a third. If no three joints lie on the same link, the above numerical relation between joints and links may be stated in another form, namely, twice the number of joints is four greater than the number of links. But in applying the rule in this form all joints count alike as units, and for a simple compass the ends must be reckoned as joints.

* Mr Penrose, the eminent architect and surveyor to St Paul's Cathedral, the scientific expositor and elucidator in succession to Mr Pennethorne of the surprising law of curvilinearity in the temples of the Greeks, has put up a house-pump worked by a negative Peaucellier cell, to the great wonderment of the plumber employed, who could hardly believe his senses when he saw the sling attached to the piston-rod moving in a true vertical line, instead of wobbling as usual from side to side. There seems to be no reason why the perfect parallel motion should not be employed with equal advantage in the construction of ordinary water-closets. The author has been admitted to see the geometrical pump at work in Mr Penrose's kitchen at Wimbledon. A sister pump of the ordinary construction stands beside it. The former, although quite as compact as its neighbour, throws up a considerably larger head of water with the same sweep of the handle. Its elegance, and the frictionless ease with which it can be worked (beauty as usual the stamp and seal of perfection) have made it the pet of the household. Some circular steps outside St Paul's Cathedral very lately requiring repair, Mr Penrose employed a circulo-circularly-adjusted Peaucellier cell to cut out templets in zinc for the purpose. The radius of the steps is about 40 feet, but to the great comfort and delectation of his clerk of the works, they were able to operate with a radius of not more than 6 or 7 feet in length. General Sir H. James, R.E., lately gave a lecture on the subject at Southampton, and informs the writer that this has been the means of inducing a gentleman of fortune residing there, well known in the yachting world, to fit up a marine engine with a Peaucellier parallel motion to use on board a steam yacht.

A very good idea of the form and operation of a negative cell may be gained by putting together the fore-fingers and ring-fingers of the two hands, and placing one middle finger a little over the other so as to keep all six fingers in the same plane. The first Peaucellier cell constructed in this country was a positive one, made by the speaker's friend, the eminent musician in whatever way the linkage is moved about, will always lie in a straight line with the two free angles of the diamond, which may be called its poles, and the distances of these poles from the fulcrum, or the ideal lines which represent those distances, may be called the arms of the cell. It is upon the geometrical relation between these arms that the remarkable mechanical properties of Peaucellier's cell depend. The cell may be made to change its

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and inventor of the laryngoscope, Mr Manuel Garcia, Ph.D., who happened to visit him shortly after his memorable interview with Dr Tchebicheff, in which that great mathematician announced in answer to his inquiries after the progress of the disproof of the impossibility of the exact conversion of circular into rectilinear motion, which had so long occupied the attention of his illustrious guest, that it, the thing itself, not the proof of its impossibility, had been actually effected in France, and subsequently in Russia, by a freshman student in his own class. He showed Mr Garcia the drawing of the cell and mounting left by Tchebicheff, and the next day was gratified by receiving from him a model constructed with a few pieces of wood, fastened together with nails as pivots, which, rough as it was, worked perfectly, and drew forth the most lively expressions of admiration from some of the most distinguished members of the Philosophical Club of the Royal Society (not mathematicians, but naturalists, geologists, chemists, and physicists), when it was brought in with the dessert, to be seen by them after dinner, as is the laudable custom among the members of that eminent body in making known to each other the latest scientific novelties. Presently after the speaker exhibited the same model in the hall of the Athenæum Club to his brilliant friend Sir William Thomson, of Glasgow, who nursed it as if it had been his own child, and when a motion was made to relieve him of it, replied, "No! I have not had nearly enough of it—it is the most beautiful thing I have ever seen in my life." This rude but invaluable model ought to be preserved in some physical laboratory as a historical relic. It served as an instrument by which the speaker in every case where it was seen gained immediate converts to the belief of the importance of Peaucellier's great discovery, whereas a mere geometrical diagram would have been as little regarded as a figure of the celebrated asses' bridge in Euclid at last, so great is the difference of the impression produced on the practical English mind by the *esse* and the *posse*—being told how a thing ought to act, and seeing it actually going. Considering the extraordinary conversions worked with Mr Garcia's model, it would not be unsuitable to write in letters of gold on the board attached to it which gives support to the two frail centres, the famous motto of Constantine—"In hoc signo vinces."

Apropos of the mistaken impressions of great men. Did not Newton live and die in the belief of the incurability of chromatic dispersion; Cayley affirm the infinitude of the number of the asyzygetic invariants of binary quantics beyond the sixth order, thereby arresting for many years the progress of the triumphal car which he had played a principal part in setting in motion; Pontecoulant the possibility of the existence of a rotating fluid ellipsoid of equilibrium for other than forms of revolution?

And as regards the speaker himself, twenty years ago he emitted* in the *Philosophical Magazine* a conjectural criterion for distinguishing *à priori*, geometrical propositions capable only of indirect demonstration from those susceptible of direct, when, lo and behold! but a few days ago came over a seemingly incontrovertible refutation of his supposed law, addressed to the Vice-Chancellor of our University of Cambridge (as a sort of Patriarch of the West, and recognized Official Defender of the Faith (as it is in Euclid) for the British isles), by Miss Chart, of Oakland, California, U.S., which it is to be hoped will speedily appear in the same journal where the erroneous hypothetical dogma first saw the light. His sin, after so long a delay, and travelling half round the world in the interim, has found him out. It ought to be added that Miss Chart does not claim for herself the merit of the refutation, but represents herself as having received it some years ago from a gentleman bearing the, to geometrical ears, auspicious-sounding name of Hesse.

[* Vol. I. of this Reprint, p. 395.]

form like a set of lazy-tongs or any other kind of linkage, by closing or opening the diamond: as this is done evidently the lengths of the arms alter; but it will be found, and is capable of easy geometrical proof, that they remain subject to a very simple condition, namely, one increases just as much as the other decreases, so that their product remains invariable; this product is equal to the difference between the square of either of the links (called the connectors) proceeding to the fulcrum and the square of any side of the diamond, to which we may give the name of the modulus of the cell. The speaker illustrated this property experimentally, using a negative cell for the purpose. When the fulcrum was midway between the two poles each arm was 12 inches in length. When one arm was made 18 inches the other was found to be 8; when again it was stretched to the length of 24 inches the other was 6, and so on, the product of the two remaining always 144; or, reckoning in feet, to the lengths 1, $\frac{3}{2}$, 2, 3 of one arm corresponded the lengths 1, $\frac{2}{3}$, $\frac{1}{2}$, $\frac{1}{3}$ of the other; showing that the length of one arm was so governed by the length of the other as that the numbers denoting the two were always inverse or reciprocal to each other when the modulus was taken as unity. Hence a Peaucellier's cell may be conveniently termed a Reciprocator or Inverter. If we were to suppose the connectors at their free ends, instead of being attached to the side angles of the diamond, to be joined on to two adjoining sides in such a manner as to become parallel to the other pair of sides, this parallelism would continue to subsist for all positions of the linkage, and the arms or distances of the fulcrum from the opposite angles or poles of the diamond would still remain in the same right line, but the relation between them would now be one of direct instead of inverse proportion. Conceive the fulcrum in such an arrangement to become fixed. Since we can not only alter the angles of the diamond, but make the whole arrangement turn round the fixed point, we can make either pole describe any plane curve whatever: the other pole will then describe a curve precisely similar in shape, but drawn on a different scale, as in any ordinary pantigraph*.

But if we revert to the Peaucellier cell or Reciprocator, whether of the positive or negative form, and treat it in the same manner as the supposed pantigraphic arrangement, fixing the fulcrum, and making one of the poles—that is, an extremity of one of the arms—describe any plane curve, the other pole will no longer describe a similar curve, but what in the language of geometry

* Sometime, according to the authority of a questionist in the *Educational Times* for the current month, called a Pentegram. Theoretically two Peaucellier cells are equivalent to a Pentegram (for we may change r into $\frac{K}{r}$ by one, and that into Kr by the other), but whilst combinations of the former are adequate to the transformation of r into any algebraic function of r, the latter are absolutely sterile, leading only to the one single sort of transformation (if it may be called so), r into Kr. It seems then going too far to say (as does the writer alluded to above) that the germ of Peaucellier's invention is contained in the Pentegram.

is termed an inverse of the curve in question, the fulcrum being the origin of the inversion.

Suppose now one of the poles is made to describe a circle, the other will describe the inverse of a circle, which geometricians are well aware will in general be another circle, subject to the exception that if the arc described by one pole is part of a circle passing through the fulcrum, which is here the origin of the inversion, the path of the second pole will be no longer a circle, but a perfect straight line, which, under a mathematical point of view, may be regarded as a circle with an infinite radius. If then, in addition to fixing the fulcrum, we still further constrain the motion of the Peaucellier cell by attaching one of the poles to a centre (which for the sake of distinction from the other fixed point above defined we may term the *pivot*) round which it can revolve, situated at an equal distance from that pole and the fulcrum, the other pole will describe a perfect straight line perpendicular to the line joining the fulcrum and the pivot. We have thus a combination of seven radiating bars attached to two fixed centres, one point of which describes a true rectilinear path, and thus the long-sought-for problem of a perfect parallel motion meets for the first time its complete solution*.

* The centre above spoken of may be taken in the line itself, which joins the poles and the fulcrum. If it be taken not too far out of this position of symmetry it will in the course of the motion be brought into such position; but if it be taken at starting (as it may be), at a sufficiently great distance from the cell, the position of symmetry may never be attained throughout the whole possible course of the motion. This circumstance has been generally overlooked, and accordingly too narrow a rule has been given for the construction of a Peaucellier parallel motion, namely, it is laid down that the pivot is to be taken midway between the fulcrum and one of the poles for some certain position of the instrument. The position of the fulcrum relative to the two poles gives rise to the distinction between a negative and positive cell; but the preceding remark shows that there is a further subdivision of Peaucellier parallel motions depending on the length of the mounting radius, and that positive and negative mounted cells each of them embrace two radically different forms or genera, which may be distinguished as the symmetrical and non-symmetrical respectively; in the one form there exists a position where the *first* lies in the line containing the *fulcrum* and the two poles, in the other, no such position can be found. In the ordinary rule given for the construction of a P.P.M. only the former of these two genera is included which, as machines, differ between themselves as much as do the ellipse and hyperbola as curves.

It ought to be added that the motion of the *parallel-point* is always perpendicular to the line of centres, and in every position makes, with the line containing the fulcrum and the poles, an angle equal to the angle contained in the segment of the circle (of which one pole describes an arc), which lies between it and the fulcrum. If we join the two fixed centres by a new link, and then unfix them, we obtain a linkage of eight bars, possessed of very remarkable properties, of one of which Peaucellier has availed himself to obtain a mechanical description of the Limaçon of Pascal, which is the inverse of a conic in respect to the focus as the origin of inversion.

By a combination of such linkages it is possible to cause any number of points, otherwise free, to remain always in a straight line with each other. The speaker believes that he is in possession of a *bonâ fide* valid proof of the proposition assumed on totally insufficient grounds by Peaucellier, namely, that every algebraical curve may be best described by link-work. The proof is founded on the union of the above statement (or still better, one founded on his own Kine-

The speaker illustrated these results by various models constructed in wood. By changing the length of the radial bar connecting one pole of the cell with a fixed point, the free pole was shown to describe arcs of circles convex or concave to the fulcrum, according as the ideal circle, in an arc of which the first-named pole moved, fell short of the fulcrum or contained that point within it; in the limiting case, when it passed through the fulcrum, the path was shown to be neither convex nor concave, but a straight line free from all curvature in either direction. This was further verified mechanically by connecting together at their free poles two perfectly equal and similarly mounted cells. If the tendency of either of these was to deviate from the straight path, the tendency of the other would be to deviate in the contrary direction; so that either the pair of mounted cells would become an

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matical Paradox subsequently referred to) with Grassmann's method of describing algebraical curves by means of an apparatus of fixed points and lines; this proposition, as far as concerns curves of the first nine genera (that is, of a *cursality*, or, so to say, circuit-*complexity* not transcending the 9th degree), and also for curves of the first six orders, or for any order where the degree of one of the variables in the representing equation is 5 or less, he had already demonstrated by a direct method. In using this method he found it necessary to prove that a general algebraical equation of the fifth degree could always be reduced to a trinomial form by *real* transformations, which, by Tschirnhausen's (the only method hitherto applied), as often as not, is incapable of being done. By an extension of the principle of Tschirnhausen's method he succeeded in establishing this important algebraical proposition. A very much more important conclusion relating to the representation of every algebraical function (that is, the function that one quantity is of another connected with it by any algebraical equation), under a quasi-explicit form, he believes he can show may be deduced from the transformed Grassmannian construction above alluded to: by quasi-explicit, meaning a form capable of being obtained by the elementary processes of addition, multiplication, change of sign, and reciprocation with that of general form-inversion superadded. Thus Peaucellier's discovery seems likely to throw open a new chapter in the highest summits of Analysis, no less important in the theoretical direction than its numerous applications to the mechanical arts in the direction of practice.

In the lineo-circular or parallel-motion adjustment imagine the connectors to be detached from the angles of the diamond, and joined on to the two sides of the diamond, which meet at the "parallel point," at equal distances from it. Then the motion of that point will no longer be in a straight line, but in a circle.

This method of producing one circular motion from another (which was first given by the speaker in the *Educational Times*) may probably be found to possess important practical advantages over the circulo-circular adjustment of the Peaucellier cell described in the text above.

The speaker exhibited another modification of the Peaucellier cell; like it consisting of six links, but having the property that the sum of the squares of the two arms (instead of their product) remains constant. This he calls a quadratic-binomial extractor.

By means of this cell, mounted with a suitable radius, a perfect lemniscate may be described; and what is very interesting, and flows from this construction (but was first observed by Dr Henrici), the same curve may be described by means of a binomial-extractor, of a certain kind, reduced to a link-work by the *ablative* method of fixing one of the links: in other words, a perfect lemniscate may be described throughout its complete extent by means of 5-bar motion. Peaucellier refers to, without specifying, a combination, "assez compliquée," of cells (or, as he terms them, compound compasses) by means of which a lemniscate may be traced; whereas, in the method above described the number of links employed is less by a pair than in the single mounted Peaucellier cell.

absolute fixture, or the two would crush or tear each other to pieces; but in the experiment exhibited the pair of mounted cells were seen to move together (as if in happy wedlock), without let or hindrance to each other's motion. The circular motion of the free pole of a single mounted cell in the general case was also verified experimentally, and even more simply than in the rectilinear case, by the addition of a second radial bar, taken of a suitable length, determined by previous mathematical calculation. As a general rule, the total number of bars in a link-work machine must be odd, but here there were eight bars, and yet the combination admitted of being set in free motion,—any one of the eight being, in fact, what may be termed a lazy-bar, and capable of being removed without disturbing the motion, very much in the same way as any one of the four legs of a table may be removed without disturbing the equilibrium*.

The speaker pointed out the important applications of the two kinds of motion above referred to (which he proposed to call the circulo-linear and the circulo-circular respectively) to various constructions in machinery, such as the steam-engine, planing and grinding machines, the construction of maps on the stereographic projection, millwrights' work, laying out of railway curves, dioptric apparatus for lighthouses, ornamental tracery, pendulum suspension to effect motion in a practically exact cycloidal arc, &c., &c., and referred to the use which, as he was informed by the authorities at Woolwich, might have been made of the circulo-circular adjustment in saving several weeks' work, inconvenience, and expense in cutting out the fish-bellied torpedo casings recently constructed in the laboratory department at the Royal Arsenal there, and the use contemplated to be made of the circulo-linear, or perfect parallel motion, for guiding a piston-rod in certain machinery connected with some new apparatus for the ventilation and filtration of the air of the Houses of Parliament, now under course of construction.

He next referred to the unlimited command over the motion of a point furnished by a combination of cells. Returning to the simple Peaucellier

* Suppose four circles to be given, and that it is proposed to inscribe upon them a quadrilateral whose four sides are given in length.

This is a determinate problem which will in general admit of a definite number of solutions. (The method of correspondence and of bipartite equations founded thereon seeming to indicate thirty-two as the total number of such solutions, some or all of which may be imaginary.) But now the question may be put, "Under what circumstances can the number of such solutions become infinite and the problem undeterminate?" It follows from what is stated in the text above that this may happen (other conditions being satisfied) when two of the circles coincide and the four given lengths are all equal. It remains to be ascertained whether with any new set of conditions a like undeterminateness can be brought about for the case of four circles all distinct. If so, a solution would be obtained of the problem of converting by link-work circular into circular and conceivably (as an extreme case) into linear motion by an arrangement radically distinct from Peaucellier's, and involving the use of three instead of two fixed centres, but with the same number of links.

cell, its use may be modified in a very remarkable manner by setting free the point of junction of the two connectors (termed, in what precedes, the fulcrum), and fixing one of the poles as a centre of rotation in its place. If now the liberated fulcrum be made to describe any curve, the free pole will describe a curve corresponding to it, according to a certain easily-statable mathematical law. Imagine the first-named curve to be part of a circle passing through the fixed point—it may be shown that in that case the free pole will describe the inverse of a conic section in respect to a vertex of the conic as the origin of the inversion; consequently, by combining with this cell a second, used as a Reciprocator, we may, mounting with a suitable radius a pair of Peaucellier cells duly adjusted, cause a point to move in a parabola, ellipse, or hyperbola.

The speaker exhibited a combination of this kind, and caused a point to describe portions of an ellipse, a parabola, and of the two branches of a hyperbola in succession; the traversing pole of the first cell, which might be termed the first follower, being seen to describe beautiful nodal cubics (or the inverses of the conics), whilst the free pole of the second cell or second follower described the conics themselves*.

* The nodal cubics or conic-inverses above described are for the parabola, the common cissoid, and for the ellipse and hyperbola curves which may be termed trans-cissoid, and cis-cissoid, or less barbarously and more euphoniously the hyper-cissoid and hypo-cissoid respectively. The common cissoid, as is well known, has a cusp which here coincides with the fulcrum. In the hyper-cissoid this becomes a detached, or, as it is ordinarily termed, a conjugate point, and in the hypo-cissoid a node on the curve, which in this case possesses a loop in addition to an infinite branch. When the first follower moves in this infinite branch, the second follower describes a portion of that branch of the hyperbola in which the fulcrum lies—but of course can never reach the vertex, which coincides with the fulcrum; when the first follower moves in the loop the second follower describes the opposite branch of the hyperbola, and can be made to pass through the vertex of that branch.

The geometrical construction for the common cissoid, or cissoid proper, is well known to be as follows. Imagine a pencil of rays proceeding from one extremity of a diameter of a circle, and meeting a tangent to the circle drawn at the other extremity. Then if the portion of each ray intercepted between the circle and tangent be shifted along the ray until one point of it coincides with the centre of the pencil, the other point will mark out the cissoid. Now imagine everything to remain as above, with the exception that the tangent is moved parallel to itself and becomes fixed in a new position nearer to or further from the centre of the pencil than it was at first, then the curve marked out becomes the hypo-cissoid or hyper-cissoid respectively, a remark due to Mr Howard Elphinstone. The smoothness of the motion, and the facility with which the cissoidal curves and the corresponding curves were drawn was matter of general surprise and admiration to the audience. This circumstance, due in part to the skill of Dr Henrici in choosing the proportions of the parts, ably seconded by the mechanical experience and ingenuity of Mr Grant, modeller to University College, at the same time served to evince the extraordinary superiority of pure link-work motion, that is, motion due exclusively to the action of radiating bars about centres, over motion effected in whole or in part through the intervention of grooves and slides. It was the analogous superiority enjoyed by circular over linear construction for the purpose of graduating instruments of precision that actuated Mascheroni (the favourite geometer of the first Napoleon) in devising his admirable, most valuable, and most tedious exposition of the geometry of the compass. The superiority in question was still more

He next went on to state that by a combination of cells properly proportioned and suitably attached to each other in succession in a manner similar or analogous to that in which simple machines, as for example a number of levers, may be combined to produce a complex one, we are able to bring about any mathematical relation that may be desired between the

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strongly evinced in the triple-cell combinations employed in the instrument for the extraction of cube-roots and the trisection of an angle.

Clairaut has given a method of constructing an instrument for extracting the roots of an equation by means of linear measurements described in Borgni's *Traité de Mécanique Appliquée* (volume on *Machines Imitatives*, p. 226); but the author's method, founded on Peaucellier's discovery, is beyond all comparison superior in the range of algebraical operations which come within its scope, in the simplicity, homogeneity, and smaller number of its parts, in the facility of its application, and the smoothness of the resulting motion. His instrument for solving cubic equations is far less complicated than that of Clairaut for quadratics (which he does not suppose has ever been realized) and infinitely easier of application. For instance, in working his cube-root machine, one point of the instrument is fixed as the zero-point; a second point, called the setter, is drawn out to a division on a scale corresponding to any proposed number; a third point, called the finder, will then automatically place itself over the division on the same scale, corresponding to the cube root of that number. The zero, setter and finder points in the calculating linkage are identical (or, as in the transformation scene of a pantomime, may be said to change characters) with the fulcrum, power and weight (or driver and follower), points in the corresponding link-work used as a machine. In Clairaut's and other similar machines the calculations are made by means of measurements made upon curves described by the machines. The author's method is direct and does not involve the use of any such intermediary process.

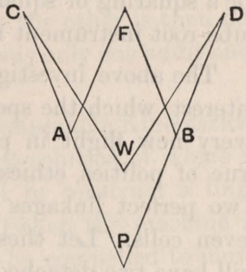
Returning to the subject, which has led to this digression, it will be noticed that by the method referred to in the text a mounted double reciprocating cell, that is, an apparatus of thirteen links, serves to describe a conic. Peaucellier's method, founded on the combination of what may be termed a collineator or radial protractor, with a mounted reciprocator, involves the use of fifteen links, besides a cross-piece rigidly attached to one of them, and, so far, is less simple, as well as less symmetrical than the author's method: but this must not be supposed to be said in derogation from the merit of the admirable invention of the collineator itself, by which Peaucellier has solved the beautiful and most important kinematical problem of devising a perfect linkage, enjoying the property, that however it is turned about, or drawn in and out, one point of it shall always remain upon or in the direction produced of a physical line rigidly attached to the linkage, but in different positions upon such line. It is believed that a conic-describing instrument (may one say conicograph?) on Peaucellier's plan has not been actually executed, and that a pure link-work for effecting conical motion was witnessed for the first time since the creation of the world in the lecture-room of the Royal Institution on the 23rd of January, 1874. Although it may be presumed that the Peaucellier conicograph would not work so simply as the one exhibited, it possesses a superiority in one respect, namely, that the fulcrum on this arrangement lying off the curve at the focus, the part of the curve described may be made to include the vertex of the parabola, which cannot be reached by the other method. It has been thought by competent judges, conversant with practical mechanism, that this (the writer's) method might be applied with advantage to constructing parabolic light-house reflectors; and as these, from the nature of the case, are made *without backs*, consisting of two paraboloids of revolution, situated *dos-à-dos*, having a common focus, at which the source of light is placed, from which the rays stream through the opening upon the surfaces of the two reflectors, the fact of the tracing or cutting or grinding instrument not being able to reach the vertex, would be no disadvantage in this case, since the portion of the surface in the neighbourhood of that point is not required, and, indeed, if formed would have to be subsequently cut away. But it should be added that by a generalized single-mounted cell an approximation to the parabolic form can be attained to a degree of precision far in excess of all practical needs.

distances of two of the poles of a linkage from a third, and are thus potentially in possession of a universal calculating machine. He exhibited and worked a cube-root extracting machine constructed on this principle, and claimed to have given the first really practical solution of the famous problem proposed by the ancients of the duplication or multiplication of the cube. This machine consisted of a combination of three cells; by changing the modulus of one of the three, he explained that it was also quite easy to solve the cubic equation involved in the analytical solution of the problem of the trisection of the angle; and a working model of an instrument of this kind executed in zinc was exhibited by Professor Henrici after the lecture. He concluded by expressing his great obligations to this gentleman, without whose aid he would have been able to do little more than adumbrate in general terms the results which, thanks to his friend's practical knowledge and skill, he had had the pleasure of exhibiting in a tangible form, and submitting before his audience to the test of actual experiment; and expressed his conviction that Peaucellier's unhoped-for discovery (even if viewed merely on its practical side as a new vital element of mechanism) was destined to produce lasting and important results through innumerable applications to the useful and ornamental arts, and would hand down the name of its inventor to posterity as one of the benefactors of mankind.

Postscript.

In some possibly forthcoming number of *Nature* a detailed account, which was expected to appear two months ago, will be given, illustrated with the necessary diagrams, of the cube-root extractor and angle-trisector: the materials for this purpose are in the hands of the editor of that journal, and have been entrusted by him to the most competent person to draw them out into form—the writer not feeling within himself the necessary energy for accomplishing this task. He thinks it, however, desirable (indeed almost a moral duty on his part) to supplement those materials by the desultory remarks which follow, in order that some results, which he believes to be important to the progress of mechanical and algebraical science, may be rescued from the chances of total oblivion and virtual annihilation.

The first question which presents itself relates to the square-root extractor. It is a remarkable fact that a cellular system for extracting square roots is much more complicated than what is required for the cube root; and so in general all even-degreed extractors require a more extensive apparatus of link-work than is required for the odd degrees. Such extractions may be performed in all cases by a system consisting of Peaucellier cells exclusively; but the process may be abridged in the case of even degrees by the interpolation of another form of cell, alluded to in a previous foot-note under the name of the quadratic-binomial extractor, which deserves a somewhat more detailed description. It is figured in the diagram below. $FAPB$ is a jointed rhomb or diamond; PC and PD are each doubles of the sides of the rhomb and CW, DW are two equal links. The difference between the squares of CP and CW is the modulus. FP, FW are the arms, and the difference between their squares is equal to the modulus. This is the instrument which, when F is fixed and P moves in a circle passing through W, describes a curve which may be called the Lemniscatoid, having the same general kind of relation to the Lemniscate that the Hypercissoid and Hypocissoid bear to the Cissoid proper. This Lemniscatoid becomes the Lemniscate when a certain simple arithmetical relation subsists between the modulus and the diameter of the circle described by P. If A as well as F be fixed, P will move in a circle passing through F, of which AP will be the radius, and consequently the five-bar link-work, consisting of the links CW, CP, DW, DP, FB (centred at F and A), will serve to describe the Lemniscate when the arithmetical relation above referred to subsists between CP and the modulus; that is, between CP and the difference of the squares of CP and CW; consequently, when the lengths CP, CW have a certain simple arithmetical proportion to each other, W will describe the Lemniscate: this proportion, it will be found, is such that when W comes to F the angle at P is a right angle. So much for the binomial-root extractor: obviously by aid of this kind of linkage when one arm is the tangent of any angle, the other arm may be made equal to the secant, and *vice versâ*. Again, it should be observed that, as in the Peaucellier cell (used as a reciprocator) the arms may be taken as x and $\frac{1}{x}$, by interchanging the fulcrum with one of the poles that is, reckoning the two arms as the distance between the fulcrum and one pole and from the other pole to the arm x, the new arm may be made to become $\frac{1}{x} - x$, which may be reciprocated into $\frac{2x}{1-x^2}$ by the use of a second Peaucellier cell. Hence by two Peaucellier cells an arm denoted by $\tan\theta$ may be, so to say, transformed into an arm $\tan 2\theta$. Thus we see that we may pass through the following series of transformations



$$\cos\theta,\ \sec\theta,\ \tan\theta,\ \tan 2\theta,\ \sec 2\theta,\ \tfrac{1}{2}\cos 2\theta$$

by means of a P.C., a Q.B.E., a pair of P.C.'s, a Q.B.E., and a P.C.,—that is, by an apparatus containing four Peaucellier cells and two cells of the new kind—making a linkage of six cells or 36 links in all. In other words, by means of such a linkage the arm x may be, so to say, converted into $x^2 - \frac{1}{2}$.

If, therefore, by a Q.B.E. we first convert x into the square root of $x^2 + \frac{1}{2}$ by superadding to this the linkage last named, that is, by a linkage of seven cells

2—2

or 42 links, x becomes converted into x^2. Thus, then, seven cells are required for a squaring or square-root extractor instrument analogous to the cubing or cube-root instrument for which only three cells are required*.

The above investigation leads to a further construction of extraordinary interest, which the speaker is wont to describe as the Kinematical Paradox: every new flight in physics and mathematics, and the same seems equally true of politics, ethics, and philosophy†, is apt to commence with a paradox. Two perfect linkages have been described above, one of six, the other of seven cells. Let these linkages both be constructed simultaneously; they will have two detached points of the one (namely, the two extremities of the arm x) coincident with two of the other: their union will itself (according to a general principle) form a perfect linkage. In this linkage of 13 cells two points will lie in the same straight line with the original zero point from which the arms are measured, one at the distance x^2, the other at the distance $x^2 - \frac{1}{2}$ therefrom. Hence there will be two points in this linkage which are disconnected, but in whatever way the other links are drawn in and out, retain an invariable distance from each other! Any other two points of the apparatus may be made to vary their distances from each other, but no force that can be applied at these two points to force them nearer to or separate them further from each other can be of any effect. There is no immediate rigid connection between them, and yet they are as good as rigidly connected. Imagine now that they become connected by a material link: the linkage will not be a fixture, but a perfect linkage as before, consisting, however, of an odd number, namely, 79 links; any one of these may be regarded as a lazy-bar, and may be removed without affecting the motion of which the apparatus is susceptible. Returning to the original state of things, where there are 13 cells, if we fix the two points of invariable distance the instrument will not become a fixture (as would be the case if any two other disconnected points in it were fixed), but a free link-work with a superfluous or lazy-bar,

* The much simpler scheme for converting x into x^3, which explains the principle of the cube root machine, is as follows:

First conversion, $x - \frac{1}{x}$, that is, $\frac{x^2-1}{x}$.

Second conversion, $\frac{x}{x^2-1} - \frac{1}{x}$, that is, $\frac{1}{x^3-x}$.

Third conversion, $(x^3 - x) + x$, that is, x^3.

For the trisection of the angle it is necessary to solve kinematically the equation between $\cos 3\theta$ and $\cos\theta$, to effect which it is only necessary to replace the third conversion above by

$$4(x^3 - x) + x, \text{ that is, } 4x^3 - 3x.$$

† As for example Cramer's paradox (the foundation of the highest modern geometry) the πoῦ στῶ of Archimedes and the hydrostatic paradox, "The king can do no wrong," "It is better to suffer than to do wrong," "All proof is reducible to syllogisms, and the syllogism can prove nothing," "A heavy body begins to fall with no velocity." The Kantian antinomies. Helmholtz's vortices. A variable function which never varies, that is, an Invariant as distinguished from a Constant.

represented by any of the links at will; for by fixing these particular two points, not *four*, but only *three* degrees of liberty are abstracted. By fixing one of them two such degrees are taken away; but as the other is then not free, but compelled to move in a circle, fixing *it* takes away only one additional degree of liberty of motion.

By this link-work of 78 bars (one supererogatory) a remarkable Kinematical problem has been solved (and it is probably the simplest solution of which it admits), which may be stated as follows:—"Required to construct a link-work fixed or centred at two of its points, such that (when the machine is set in motion) some other point or points therein shall be compelled to move in the line of centres."

There are some similar questions to this, which ought, in a strict logical order, to have preceded it, which we may now take into consideration. By a single mounted Peaucellier cell fixed at two centres, one point is made to move perpendicular to the line of centres. Suppose now it were required to devise a link-work such that a point should move parallel to such line.

The motion perpendicular to the line of centres is due to the fact that by the Peaucellier cell the radius vector $C \cos \theta$ is transformed into $C \sec \theta$; in like manner to get the parallel direction a means must be found of passing from the cosine to the cosecant. Now although a single cell serves to change the tangent into the secant, *or vice versâ,* and consequently a single *imaginary* cell will serve to change the cosine into the sine (which of course could then be immediately Peaucellierized into the cosecant), he is not aware of any direct real process simpler than that about to be stated by which this can be effected. His actual law of deduction is as follows: Cosine; secant; tangent; cotangent; cosecant, involving the use of two Peaucellier cells and two quadratic-binomial extractors.

With one cell more, that is, with five in all, the cosine becomes converted into the sine, and consequently by introducing a pantigraphic cell $\cos \theta$ may be converted into $\cos (\theta + \alpha)$, and this reciprocated into $\sec (\theta + \alpha)$. Thus it seems (at all events after the present method) that four cells are required to obtain by link-work rectilinear motion parallel to the line of centres, and seven cells to convert it into motion oblique to the line of centres; or taking into account the mounting radius 7, 25, 43 links are required to obtain motions respectively perpendicular, parallel, and oblique to that line. In the Kinematical Paradox it will have been seen that there are 13 cells employed, that is, 78 links, of which any one is liable to removal at will, so that for motion in the very line of centres 77 links are requisite. Consider this system in its entirety. In a straight line with the two fixed points there will be 13 other medial points; and two parallel ranks on both sides, each also containing 13 points. The whole apparatus admits of being moved with a sort of see-saw motion backwards and forwards; and it may assist the imagination of the reader if he will conceive such an instrument armed with 13 picks in the line of centres, each at work to remove the asphalt of a pavement under repair; an idea suggested by a member or visitor at a soirée of the Amateur Mechanical Society of London, of which the ingenious and accomplished "Senior Member for Greenwich" acts as honorary secretary. Or we might describe the Kinematical Paradox as a kind of compound saw. If the "two points of invariable distance" be set free, and some other of the medial points be fixed as a fulcrum, the instrument may be used like Peaucellier's second invention referred to in a previous foot-note as a radial protractor to change the curve

$$\rho = \text{a given function of } \theta$$

into the curve $$\rho + c = \text{the same function of } \theta;$$

as, for instance, to pass from the circle to the limaçon of Pascal, or from a straight line to a conchoid. For while one of the two points of constant distance described any curve, the other would describe the same curve with all its radii vectores reckoned from the fixed point lengthened or shortened by a constant quantity. The Kinematical Paradox ought not to be regarded in the light of a mere luxury of speculation; it serves to represent a constant as a Kinematical function of the independent variable (corresponding to the use of the zero power of x to represent unity in algebra), without which the general analytical theory of linkages, and the very important theory of algebraical functions founded thereon, would fall to the ground, or rather be incapable of being constructed.

It would be difficult to quote any other discovery which opens out such vast and varied horizons as this of Peaucellier—in one direction, as has been shown, descending to the wants of the workshop, the simplification of the steam-engine, the revolutionizing of the millwright's trade, the amelioration of garden-pumps, and other domestic conveniences (the sun of science glorifies all it shines upon), and in the other soaring to the sublimest heights of the most advanced doctrines of modern analysis, lending aid to, and throwing light from a totally unsuspected quarter on the researches of such men as Abel, Riemann, Clebsch, Grassmann, and Cayley. Its head towers above the clouds, while its feet plunge into the bowels of the earth.

Prophetic and well-timed were the parting words to the speaker of the illustrious Tchebicheff: "Take to Kinematics, it will repay you; it is more fecund than geometry; it adds a fourth dimension to space." So also said Lagrange.

In the course of the foregoing exposition, incidental reference has been made to the addition of perfect linkages to each other*. This gives rise to

* Namely, by pivoting together two disconnected points of the one with two disconnected points of the other, each with each. The sum of two perfect linkages so connected will satisfy the same numerical linear equation between joints and links as its two constituents, and thus will itself constitute a perfect linkage.

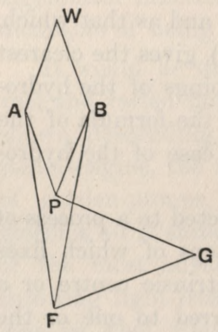
the important distinction of all perfect linkages into prime and composite—prime ones being such as can be resolved into the sum of two others, and composite those for which no two such components can be found. As an example of one kind, imagine an octagon with its four pairs of opposite angles (or, which will do as well, its four pairs of opposite sides) connected by links. There will then be 12 links and 16 joints; and since $3 \times 12 - 2 \times 16 = 4$, the linkage will be perfect. Such a linkage is prime, for it will be found impossible to resolve it into two others. Whereas, every cell previously described is capable of being formed by the successive accretions of single pairs of links, thereby justifying in a new and specialized sense the title of Compound Compass, used by Peaucellier to designate his cell. Moreover, cells belong to a very special class of compound linkages, those namely which by successive processes of decomposition can eventually be reduced to depend on sets of link-pairs, and which may accordingly be termed Dyadisms. Dyadisms, again, require to be classed according to their order. A dyadism of the first order is one that can be obtained by successive additions of single duads at a time. A dyadism of the second order is one that can be formed by successive additions of single dyadisms of the first order at a time, and so on; and it is very essential to notice that the addition together of two dyadisms of a given order will not in general be a dyadism of the same order. Thus we see that a pure tactical theory of colligation underlies the subject of linkages, a theory of the same nature as that which is known to underlie the doctrine of crystallography and polyhedra; and as that which, under the name of ramification (proposed by the speaker), gives the clearest notion of the modern chemical doctrine of the atom-groupings of the hydro-carbons, and in a manner supplies an *à priori* ground for the formula of the saturated hydro-carbons $C_n H_{2n+2}$, which, for the simpler case of the hydro-borons (if such series existed), would become $C_n B_{n+2}$.

It may be shown that every ramification may be subjected to a process of reduction (a sort of divulsion process, the number of steps of which fixes its genus, or order), which leads eventually to a single intrinsic centre or a pair of intrinsic centres, and consequently may be referred to one or the other of two great classes of forms which may be termed central and axial respectively; and it seems only reasonable to anticipate that the physical properties of such chemical compounds as the hydrocarbons will eventually be found to correspond to this distinction between their representative ramifications; and that they will accordingly range themselves under one or the other of two great families distinguished by properties at least as important and specific as those which serve to distinguish the crystalloidal and colloidal states of matter. The theory of ramification is one of pure colligation, for it takes no account of magnitude or position; geometrical lines are used, but have no more real bearing on the matter than those employed in genealogical tables have in explaining the laws of procreation.

The sphere within which any theory of colligation works is not spatial but logical—such theory is concerned exclusively with the necessary laws of antecedence and consequence, or in one word of *connection* in the abstract, or in other terms is a development of the doctrine of the compound parenthesis. M. Camille Jordan, independently of and anteriorly to the author, discovered and published in a memoir, the title of which would never suggest the notion of ramification, the existence of the intrinsic centre and centres here referred to—without having any suspicion of its bearing on modern chemical doctrine. He has moreover discovered the existence of another kind of intrinsic centre of ramification which was unknown to the author of these lines.

A ramification, it ought to be added, is a rootless tree, that is, one in which the root only ranks the same as the terminal of a branch, and saturated hydrocarbons are typified by ramifications in which every joint is trifurcated, meaning thereby that in tracing the wood outwards from any terminal assumed as the root, it splits and splits again, so that trifurcation takes place at each joint, or in other words, *four* lines radiate out from each joint*; the joints are supposed to adumbrate the carbon atoms and the terminal points the hydrogens.

To conclude, as he has begun, with the principal personage of his story, the author thinks it will be useful to several of his readers to have before their eyes the figure which contains the property of the admirable linkage which lies at the root of Peaucellier's conicograph.



In the given figure $APBW$ is a rhomb. PA is equal to PB, GP to GF, and G' is a point lying on FG, or FG produced such that $FG'W$ is a right angle. Then, however the links are moved about, the motion of W *relative* to FG will be always perpendicular to FG, from which it follows that $FG'W$ will always continue to be a right angle, and consequently an upright piece attached at G' perpendicular to FG will always continue to point to W. When W is fixed, the instrument serves as a radial protractor. One point of the upright can describe any curve, and any other point a radial protraction (or retraction) of that curve. When one point of the upright perpendicular is fixed, the combination becomes ideally equivalent to a revolving slot, in which W is free to traverse. The inverse of a conic in respect to a focus (that is, the Limaçon of Pascal) is a protraction or retraction of the circle. Hence the use of the instrument for describing conics.

* Observe that if there were *no* splitting, as in a bamboo cane, *two* lines would issue from each joint.

In the above linkage let a pair of equal links *GP*, *GW* be *substituted* for the pair *GP*, *GF*. It is easy to prove that if *O* be the intersection of the diagonals of the rhomb, *GO* and *FO* will then be at right angles to each other, and the sum of their squares will be a constant. If now any one link of the rhomb is transferred parallel to itself so as to pass through *O*, and is jointed on to the sides at the points where it meets them, and *O* is fixed, and *F* made to move in a circle containing *O*, the path of *G* will be the *inverse in respect to O of a conic* of which *O* is the centre, so that by the aid of a radius and a reciprocator in addition to the transformed linkage above described, a point may be made to move in any conic round its *centre* as a fixed point*. This is rather a simpler construction than Peaucellier's for motion in a conic round the *focus* as a fixed point, for the number of links is no greater, and the ungainly cross-piece disappears. Moreover, it possesses all the advantages of Peaucellier's method arising from the fulcrum lying off the curve to be described. Finally, as regards the most general motion that can be produced by a Peaucellier-mounted cell in its generalized form, if *F* be the junction of two links on which *FA*, *FB* are two equal segments, and *FC*, *FD* two other equal segments, and *PA*, *PB* and *WC*, *WD* be two pairs of equal links in the same plane with the first pair, such combination of three pairs is the generalized form of cell in question. In applying it to draw curves, *F* may be fixed, and a mounting radius of any length attached to *P* or *W*, or *P* or *W* may be fixed, and the mounting radius attached to *W* or *P*, or *P* or *W* be fixed, and the mounting radius attached to *F*. In a résumé of this general kind it would be out of place to enter into a discussion of the forms thus generated†.

* It follows as a particular case of the above, that an apparatus of nine links moving round two fixed centres will serve to generate motion in a circle whose centre is in a right line drawn through one of the given two, perpendicular to the line joining it to the other.

† It is too late to make any change in the many places where the term perfect linkage appears in the text, but the author regrets to have used the word *perfect* when *complete* would have expressed the meaning more clearly, and suggests this change of nomenclature to any writer who may hereafter have occasion to employ the term—besides being better in itself, it comes nearer to Peaucellier's "système de tiges à liaison complète"; two words (and those much more expressive) supplying the place of six. The existence of such words as surplusages, curtilage, equipage, assemblage, and many similar ones in the English language, appears quite sufficient to justify the innovation in the use of the final syllable in linkage. A question of great interest remains over, namely, "how to extend the above inquiry to linkages in space"; any two links being supposed free to move by means of universal joints in all directions round each other. As regards surfaces of revolution, the solution of the problem is virtually contained in the theory of plane linkages, and consequently as a plane may be regarded as a surface of revolution, the difficulty does not begin to be felt until the problem of producing motion in an ellipsoid or other surfaces of the second order, by means of solid link-work, comes under consideration. It seems to be a problem well worthy of being investigated and thought out, especially for the sake of its analytical consequences and the light it might be expected to throw upon the theory of algebraical functions of two variables.