3.

ON THE PLAGIOGRAPH *aliter* THE SKEW PANTIGRAPH.

[*Nature*, Vol. XII. (1875), pp. 168, 214—216;
also, *Archivo de Mat.* I. pp. 112—114.]

I HAVE been led by the study of linkages to the conception of a new instrument, or rather a simple modification of an old and familiar one, the Pantigraph, by means of which a figure in the act of being magnified or reduced may at the same time be slewed round the centre of similitude. Some of the readers of *Nature*, such possibly as my able and most ingenious friends, Messrs George Cayley and Francis Galton, may be able to pronounce with authority how far the invention is new and whether it is likely to be found in any way useful in practice as applied to the art of the designer or engine-turner. Already my invention of the Isagoniostat, or equal angle setter, which I shall take some other opportunity to communicate to this journal, has been deemed available in practice for working automatically the train of prisms of a spectroscope.

In Fig. 1, $AOBCQ$ represents an ordinary pantigraph. O is the fixed point, P is the tracer, and Q the corresponding follower; then, as everybody

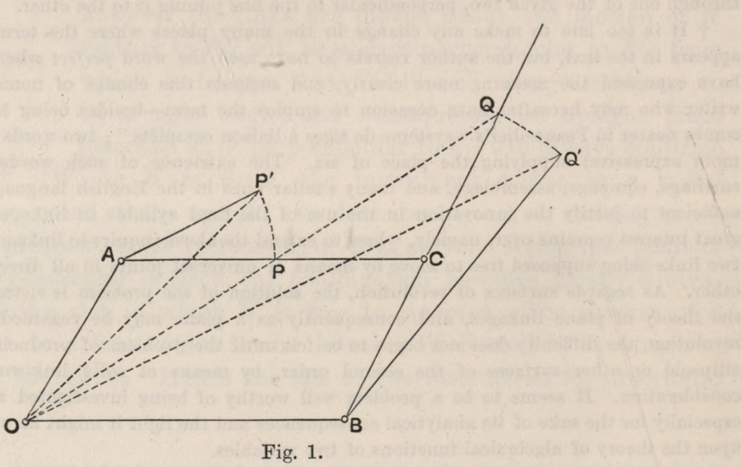


Fig. 1.

knows, any curve traced out by P will be imitated by Q, and the two curves will be similarly situated in respect to O. The point of addition is the following :—

Let P be moved through any angle, $P'AP$ round A, and Q through *an equal angle* QBQ' in the *opposite* direction round B, and let P' and Q' be supposed to be in any manner rigidly connected with the bars AC, BC respectively. Then it admits of an easy proof that in whatever way the jointed parallelogram $AOBC$ is deformed, OQ' will bear to OP' the constant ratio of AC to AP, and moreover the angle $P'OQ'$ will always remain equal to the angles $P'AP$, $Q'BQ$.

It follows that whilst P' is made to move upon any curve the follower Q' will trace out a similar curve altered in magnitude, and at the same time turned round the first point O.

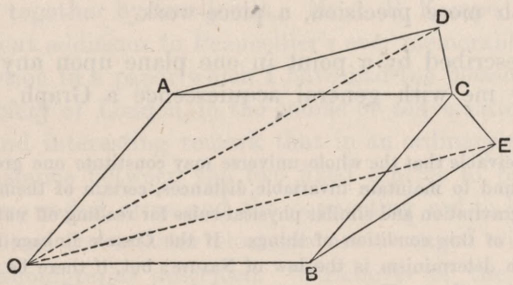


Fig. 2.

If, as in Fig. 2, we take AD equal to AC, BE equal to BC, and the angles CAD, CBE equal to each other, then the rays OD, OE will always remain equal and be inclined to each other at a constant angle. With this adjustment the instrument may be used to transfer a figure from one position in a sheet of drawing paper to any other position upon it, leaving its form and magnitude unaltered, but its position slewed round through any desired angle.

History of the Plagiograph.

I should like to add a few words to my description of the instrument called the Plagiograph* (the g to be pronounced soft, like j, as in Genesis

* It may be questioned whether a new-born child can have a history. Perhaps it might have been more correct to have used for my title, "History of the Birth of the Plagiograph," but this would have been long; moreover, the Plagiograph proves to be an unusually precocious child, having in its very cradle given birth to a greater than itself, the Quadruplane, a full-grown invention described in the sequel of the text.

Plagiarist, Oxygen) in *Nature*, Vol. XII. p. 168, for the purpose of explaining the order of ideas in which it took its rise, and also a very beautiful extension of another recent kinematical invention to which it naturally leads the way, and which, thus generalised, I propose to term the Quadruplane.

The true view of the theory of *linkages** is to consider every link as carrying with it an indefinitely extended plane, and to look upon the question as one of relative† motion which may be put under this form: When a *complete* linkage (meaning thereby a combination of jointed planes capable of only a definite series of relative movements) is set in motion, *what is the curve which any point in one of these planes will describe upon any other?*

In this mode of stating the question, the lines joining the pivots round which the planes can turn correspond to the jointed rods of the common theory. Fix any one of the planes, and the linkage becomes a link-work, or, to speak with more precision, a piece-work.

The curve described by a point in one plane upon any other plane has been termed by me with general acquiescence a Graph, and to keep the

* It is quite conceivable that the whole universe may constitute one great linkage, that is, a system of points bound to maintain invariable distances, certain of them from certain others, and that the law of gravitation and similar physical rules for reading off natural phenomena may be the consequences of this condition of things. If the Cosmic linkage is of the kind I have called complete, then determinism is the law of Nature; but, if there be more than one degree of liberty in the system, there will be room reserved for the play of free-will. We should thus revert to the Aristotelian view under a somewhat wider aspect of circular (the most perfect because the simplest form of motion) being the primary (however recondite) law of cosmical dynamics. Speaking of cosmical laws brings to my mind a reflection I have made upon the new chemical theory of atomicity. Suppose it should turn out that the doctrine of *Valence* should be confirmed by experience, and that the consequent logico-mathematical theory of colligation containing the necessary laws of consecution, or if one pleases so to say of cause and effect, should plant its foot and introduce a firm basis of predictive science into chemistry, how beautiful will be the analogy between this and the physical law of inertia! which really merely affirms the fact of each atom or point of matter carrying about with it a certain number, denoting its communicative and inverse receptive faculty of motion; for in such case Valency, also affirming a numerical capacity for colligation, will be the exact analogue in chemistry to Inertia in the theory of mass motion, and might properly assume the name of chemical inertia. Social individuals differ as egregiously as Isomers in their capacity for forming multifarious attachments.

† I believe it is to Mr Samuel Roberts that we are indebted for the idea of passing from mere copulated links to planes associated with the links, and for the observation that the order of the corresponding Graphs is not thereby augmented. The substitution of the more general idea of linkage for link-work, and of isolating completely the conception of relative in lieu of absolute motion, is due to the author of these lines. Take the case of a Quadruplane in which the four joints in their natural order of sequence form a contra-parallelogram. It is well known (and the fact has been applied to machinery under the name of "the parallelogram of Reulleux") that the relative motion of an opposite pair of planes may be represented by causing two curves to roll upon each other; but I add that this may be done simultaneously for both pairs of planes, giving rise to a beautiful and previously unthought-of double motion of rolling (without slip) between two ellipses for one pair and two hyperbolas for the other pair of planes. This is an immediate deduction from the conception of purely relative motion.

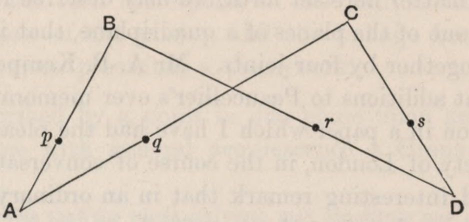
correlation closely in view, I have proposed to call the describing point the Gram*. We may further understand by canonigrams describing points taken in the lines connecting the joints and their corresponding curves, canonigraphs; Grams lying outside these lines and their appurtenant Graphs may be termed Epipedograms and Epipedographs; or, if these names are found too long for use, Planigrams and Planigraphs.

Now consider more particularly the generalised form of the linkage which corresponds to three-bar motion, of which Watt's parallel motion (so-called) offers a simple instance. If we were to revert to the old notion of link-work we should say that a three-bar motion is obtained by fixing one of the sides of a jointed quadrilateral of any form; but adhering to the more general conception of the matter here set forth, we may describe it as resulting from the fixation of any one of the planes of a quadriplane, that is, a system of four planes connected together by four joints. Mr A. B. Kempe, who has brought to light magnificent additions to Peaucellier's ever memorable discovery of an exact parallel motion in a paper which I have had the pleasure of presenting to the Royal Society of London, in the course of conversation with me made the very acute and interesting remark that in an ordinary three-bar motion, supposing the distance between the two fixed centres to be given, and the lengths of the two radial arms and the connecting rod to be also given, the order in which these three latter elements are arranged will not affect the nature of the canonigraphs described. Whichever of the three lengths is adopted as the length of the connector and the remaining two as the lengths of the radial arms, the very same system of curves will be described in all three cases so far as regards their form: every canonigram in the arrangement will have a canonigram corresponding to it in each of the other arrangements such that the corresponding curves described will be similar and similarly placed—a most remarkable, and, for the purposes of theory, an exceedingly important observation; but, as Prof. Cayley observed, when once stated, a self-evident deduction from the principle of the ordinary pantigraph†. It

* Gram is intended to suggest the notion of a *letter* discharging the duty of a point. In inventing new verbal tools of mathemathical thought, the following are the rules which I bear in mind:—1. The word must be transferable into the common currency of the mathematical centres of Europe, France, Germany, and Italy. 2. It must enter readily into combinations and be susceptible of inflexion fore and aft. 3. It should contain some suggestion of the function of the idea intended to be conveyed. 4. It should by similarity in quality or weight of sound conjure up association with the allied ideas. 5. When all these conditions are incapable of being simultaneously fulfilled, they should be observed as far as possible, and their relative importance estimated according to the order in which they are written above.

† Suppose AB, BC, CD to be three jointed rods fixed at A and D. Choose either of the fixed points, say A, and complete the parallelogram $ABCB'A$, regarding CB', $B'A$ as two additional jointed rods; through A draw any transversal, cutting the two indefinite straight lines CB, CB' in P and P' respectively; then whatever curve P describes when the system is set in motion, P' by the principle of the common Pantigraph will describe a curve similar and similarly situated

therefore occurred to me that a corresponding theorem ought to hold for all graphs whatever—for plagiographs just as well as for canonigraphs; and by a very simple application of the general double-algebra method of *Versors*, I found that this would be the case, the only difference being that now the corresponding graphs, instead of being similar and similarly situated, would be similar but not similarly situated; in other words, that the lines joining the centre of similitude with the corresponding points, instead of coinciding in direction, would make for each particular graph a constant angle with each other. Thus I passed from the conception of the common Pantigraph to that of the Quergraph, or Plagiograph, or Skew Pantigraph, as the new instrument described in the previous article may indifferently be called. I now come to



the second part of my story, and proceed to explain the remarkable extension a theorem analogous to and naturally suggested by the Plagiograph gives of Mr Hart's remarkable discovery of a *cell* consisting of only four jointed rods which possesses the same property of reciprocation as Peaucellier's six-sided cell.

This cell is exhibited in the figure above. The four jointed rods AB, AC, CD, BD are equal in pairs, AB and CD being equal, also AC and BD. In fact, the figure is nothing else but a jointed parallelogram twisted out of its position of combined parallelisms, and may be termed a contra-parallelogram. When the cell is in any position whatever, imagine a geometrical line to be drawn parallel to the lines joining A and D or B and C

thereto, A being the centre of similitude. Now, it will be noticed that $AB'CD$ is a system of four jointed rods in which the lengths AB', $B'C$ are the same as the lengths AB, BC in inverted order, namely, $AB'=BC$, and $B'C=AB$, and as we may proceed from the point D equally well as from A, it follows that all the six interchanges may be rung between the three lengths AB, BC, CD. This is the proof of Mr Kempe's admirable theorem; but does the simplicity of the principle involved take away in any degree from the beauty of the result, or rather, is not the interest of the conclusion enhanced by the simplicity of the means by which it is arrived at? In fact, as Kant has observed, the groundwork of all mathematical proof consists in putting things together by a free act of the imagination; and the essence of Euclid is to be sought in the constructions which antecede the formal proofs. The real proof is the construction, and no one has the right to call Mr Kempe's discovery "a truism."

(for these lines will obviously always remain parallel to each other), cutting the four links in the points p, q, r, s.

Now take up the cell and manipulate it in any manner you please so as to change its form, the same four points p, q, r, s will always remain in the same straight line, the distances pq and rs will always remain equal to one another, and the product of pq by pr, or, which is the same thing, of sr by sq, will never vary, so that pr remains (so to say) a constant inverse of pq, and sr of sq—the actual value of the constant product (called the modulus of the cell) being the difference between the squares of the unequal sides of the contra-parallelogram, multiplied by the product of the segments into which any one of the links is separated by the points p, q, r, or s, and divided by the square of such link. Now Mr Kempe and myself—he by the free play of his vivacious geometrical imagination, I by the sure and fatal march of algebraical analysis—have arrived at the following beautiful generalisation of Mr Hart's discovery. On AB, CA, BD, DC describe a chain of four similar triangles, the angles of which are arbitrary, but looking towards the same parts, and so placed that the equal angles in any two contiguous triangles are adjacent—call the vertices of these triangles P, Q, R, S (which will be in fact the analogues of the points p, q, r, s before mentioned): then it will be found that the figure $PQRS$ will be a parallelogram whose angles are invariable, and the product of whose unequal sides is constant; in a word, a parallelogram of constant area and constant obliquity*.

The modulus, or constant product of the sides, follows the same rule as in the special case, except that for the product of the segment of a link divided by the square of its entire length, must be substituted the product of the sines of the angles adjacent to any link divided by the square of the sine of the angle subtended by it.

* I early noticed the analogy between M. Peaucellier's six-linked reciprocator and the primitive form of the pantigraphic proportionator formed by two parallelograms having an angle and the directions of its two containing sides in common, also therefore consisting of six links; and indeed pointed out that, starting (to fix the ideas) from a negative Peaucellier-cell (such as is in successful use in the Houses of Parliament for ventilating the brains of our representative and hereditary legislators), we have only to unfix the two interior links from the angles to which they are attached, and attach them instead to two sides of the containing lozenge, so as to be parallel to the other two sides; and we pass from a Reciprocator to the comparatively barren Proportionator. Now as a Proportionator (the Pantigraph in common use) exists with only four sides, it ought to have been inferred as fairly probable that a Reciprocator also might be discovered having only four sides, that is, by analogy, the probable existence might have been inferred of a Hart cell—the contra-parallelogram first imagined by Mr Samuel Roberts, but rediscovered and hugged with the affection of a supposed original discoverer, and warmed into new and unsuspected uses by its foster-parent Mr Hart. I shall have no difficulty in finding a generalisation of the Peaucellier-cell standing to it in the same relation as the Quadruplane does to the Hart-cell, and similarly for the Proportionator, so that we shall have the fourfold proportion—Peaucellier-cell : Hart-cell : Quadruplane : New Peaucellier-cell :: Old Pantigraph : Common Pantigraph : Plagiograph : Oblique Old Pantigraph; but, except as completing a chain of analogies, the last terms in each quatrain will probably not prove of any practical importance.

Just as in the first case $pq \cdot pr$ and $sr \cdot sq$ are constant, so now $PQ \cdot PR$ and $SR \cdot SQ$ are constant; but whereas pq coincided in direction with pr and sr with sq, PQ and PR, like SR and SQ, remain inclined to each other at a constant angle; in a word, as the Plagiograph is to the Pantigraph, so is the Sylvester-Kempe Inverter or Reciprocator to Mr Hart's*. Do not let it be supposed that this new reciprocator is to be consigned to the limbo of barren mathematical generalities—very far from it; it is very likely indeed to find a most valuable application to mechanical practice, and to subserve the purposes of that immediate "Utilitarianism†" so dear to the Philistine mind; for, as by means of Mr Hart's Quadrilateral, when one of the four named points, say p, is absolutely fixed, and one of its non-conjugate points,

* In the case of a three-piece motion whose fundamental linkage (that is, the quadrilateral formed by the lines joining the pivots and the fixed points in their natural order of succession) is subject to the condition that either the two pairs of opposite sides or two pairs of contiguous sides are equal for each pair, the Planigraph (leaving out of account its circular portion) is the inverse of a conic. In the first case (that of the contra-parallelogram) the position of this node is seen immediately to be the opposite to the Planigram in respect to the centre of the figure in its untwisted (that is, parallelogrammatic) form. In the second case, that of the so-called kite-form, it was found to be far from easy to determine its position. Even our Cayley did not quite succeed in determining it from the analytical equations, and it was reserved for M. Manheim to deduce it geometrically by a most elegant but very elaborate construction given in a paper inserted in the *Proceedings of the Mathematical Society of London*. By the aid of the reciprocity established by me above we pass at once from the case of the contra-parallelogram to that of the kite-form, and the problem literally solves itself as easily as a musical passage can be transposed from one key to another. It is to that profound mathematician, Mr Samuel Roberts, that we are indebted for bringing to light these two cases of three-bar motion, in which the general three-bar sextic Graph breaks up into a circle, and the inside of a conic, and I have proved that no other such cases exist. Mr Roberts's papers are inserted in the *Proceedings of the London Mathematical Society*, which is indebted for its existence, at least in its present form (being originally little more than a juvenile mathematical debating society among the students of University College), to the organising talents of Mr Hirst, who has reason to be proud of his progeny. Similar societies on a precisely similar basis, and adopting the rules of its elder sister, have been subsequently founded in Paris, Warsaw, and, I believe, other capitals in Europe, and, it is safe to predict, are destined to play no unimportant part in the further evolution of our time-honoured yet ever young, ever fresh, and self-renovating science—Othello, Hamlet, and Romeo all in one. Meanwhile, in the University supposed to be peculiarly dedicated to the advance of mathematical science, a young and very promising mathematician (whose name shall not be divulged) *àpropos* of a movement kindly attempted, without my being previously consulted, to place me in a position where, in the vicinity of our central luminary, I might have been in my proper place, and helped to reflect some portion of his rays upon surrounding bodies, wrote to me lately: "You cannot imagine the bitter prejudice that prevails here against pure mathematics, &c." I freely forgive those, "the bigots of a narrow creed," who entertain such sentiments, knowing that "they know not what they do."

† What would our English statesmen say to the conduct of the proverbially parsimonious Prussian Government and the nineteenth century Richelieu, "der tolle Bismarck," in appropriating a million and a half of marks (75,000*l.* sterling) placed at the free disposal of the modern Aristotle, Helmholtz, for constructing the bare shell alone of the new Physical Laboratory at Berlin! If such an appropriation were proposed at home, would there not run through the land a frantic shriek or muttered growl of disapprobation at such a wanton waste of the public funds on mere speculative science?

say r, is attached to the end of a radius so centred and of such a length that the path of r is a circle which, *geometrically* completed, would pass through p, the remaining conjugate point q will be forced to describe a straight line perpendicular to the line joining the two fixed points—so by means of our Quadruplane, when P is fixed and R made to move in the arc of a circle passing through P, the point Q may be made to describe a straight line having any *desired obliquity* to the line of centres, the amount of such obliquity depending on the magnitude of the supplemental equal angles P, Q, R, S. Thus the Plagiograph (and in the first instance Mr Kempe's notice of the homœographic commutability of the lengths of the connecting rod and the radial bars in ordinary three-bar motion) has led by a devious path to the construction of a three-piece-work giving the most general and available solution of the problem of exact parallel motion that has been discovered or that can exist—I say the most available, for it is evident, in general, that piece-work must possess the advantage of greater firmness and steadiness from the more equal distribution of its strain over ordinary link-work.

The Peaucellier and Hart cells, duly mounted, afford the means by obvious methods of adjustment to cut straight lines at any distance from either of the fixed centres, but confined to lying perpendicular to the line of centres; whereas the Quadruplane puts it into our power with one and the same instrument affected with simple means of adjustment to make straight cuts (and, if desired, two parallel ones simultaneously) in all directions as well as at all distances in the plane of motion. So again the Plagiograph enables us to apply the principle of angular repetition (as, for instance, in making an ellipse with dimensions either fixed or varying at will, successively turn its axis to all points of the compass) to produce designs of complicated and captivating symmetry from any simple pattern or natural form, such as a flower or sprig; and as the head of a house at Oxford in the good old port-wine days was heard to complain about the electro-magnetic machine, that "he feared it would place a new weapon in the hands of the incendiary" (the power of Swing being then rampant in the land), so, but with better reason and upon the highest authority, it may be predicted that this simple invention will be found to place a new and powerful experimentative and executory implement in the hand of the engine-turner, the pattern-designer, and the architectural decorator.

P.S.—I rejoice to be able to state that the Institute of France has quite recently adjudged its great mechanical prize, the "Prix Montyon," to Col. Peaucellier for his discovery of an exact parallel motion when a lieutenant in 1864. The first practical application of this discovery, made by Mr Prim under the sanction of Dr Percy, may be seen daily at work in the Ventilating

Department of our Houses of Parliament. The workmen there, who never tire of admiring its graceful and silent action, have given it the pet name of the "Octopus," from some fancied resemblance between its backward and forward motion and that of the above-named distinguished Cephalopod. I feel a strong persuasion that when the inertia of our operative classes shall have been overcome, this application will prove to be but the signal, the first stroke of the tocsin, of an entire revolution to be wrought in every branch of construction; and that machinery is destined eventually to merge into a branch of the science of Linkage in the sense in which that word is used in the text above.