5.

NOTE ON SPHERICAL HARMONICS.

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If for a moment we confine our attention to so-called "zonal" harmonics, and affect each element of a uniform spherical shell with a density varying as the product of two such harmonics of unequal degrees, we know that the mass of such shell is zero. A very slight consideration will serve to show that this is tantamount to affirming that if a given spherical surface be charged with a density inversely proportional to the product of the distances of each element from two fixed internal points lying in the same radius produced, then the mass of such shell will be a complete function of the product of the distances of the two points from the centre; and in fact, if we write dS for an element of a spherical surface, it is easy to find, by direct integration, that

$$\iint \frac{dS}{\sqrt{(c^2 - 2hx + h^2)}\,\sqrt{(c^2 - 2h'x + h'^2)}},$$

for the entire surface, is proportional to

$$\frac{1}{\sqrt{(hh')}} \log \frac{c^2 - \sqrt{(hh')}}{c^2 + \sqrt{(hh')}}.$$

In like manner, the truth of the more general theorem relating to the surface-integral of the product of any two harmonics of unequal degrees involves, and is involved in, the fact that the surface-integral $\iint \frac{dS}{R \,.\, R'}$, where

$$R^2 = (x-h)^2 + (y-k)^2 + (z-l)^2,$$
$$R'^2 = (x-h')^2 + (y-k')^2 + (z-l')^2$$

and $h^2 + k^2 + l^2$ and $h'^2 + k'^2 + l'^2$ are each less or each greater than the square of the radius of the sphere, is not merely a function (as we see *à priori* from the symmetry of the sphere must be the case) of the three quantities

$$h^2 + k^2 + l^2, \quad h'^2 + k'^2 + l'^2, \quad hh' + kk' + ll',$$

but, more definitely, is a complete function of the product of two of them, namely, $(h^2 + k^2 + l^2)(h'^2 + k'^2 + l'^2)$, and of the third. In other words, the fundamental law of spherical harmonics is exactly tantamount to the assertion that if each element of a sphere is charged with a density inversely proportional to the product of its distances from two internal or two external points, then the mass of the sphere will be a function only of the density at the centre and of the angle subtended at the centre by the line joining the given pair of points; or, venturing upon an irrepressible neologism, which explains its own meaning, the *Bipotential*, with respect to a given uniform sphere at any point-pair, is a function only of the Bipotential thereat with respect to a unit particle at the centre, and of the angle subtended at the centre by the line joining the two given points. Of course, if this is true for the volume of the sphere, it must be true for any shell of uniform thickness, or, in other words, for the surface, and *vice versâ*. In what immediately follows the volume of a spherical shell is to be understood. It is, I think, very noticeable that in that proof no process whatever of integration is employed; only the *idea* implied in integration is employed to acquire the fact that the integral in question cannot but be a function of three parts of the triangle, of which the centre of the sphere and the two given points are the apices. The rest of the proof follows as a matter of purely formal or algebraical necessity from the above fact, conjoined with that of each factor under the sign of integration being subject to Laplace's equation. In this feature of exemption from all use of integration as a process, this proof, I believe, stands alone.

It is further remarkable that its success depends on the proposition being stated as a whole; it would not be applicable, for example, to the simple case, taken *per se*, treated of at the beginning of this paper. It is by no means uncommon in mathematical investigation for this to happen, and (as regards the exigencies of reasoning) for the part to be in a sense greater than the whole—the groundwork of this wonder-striking intellectual phenomenon being that, for mathematical purposes, all quantities and relations ought to be considered (so experience teaches) as in a state of flux. In the particular case before us it is not difficult to see *à priori* why the general proposition should be more easily demonstrable than any special case of it, the reason being that more information as to the *form* of the function under consideration is made use of in dealing with the general than in dealing with any special case.

The integral under consideration is

$$\iiint \frac{dS}{RR'} \text{ (say } I\text{)},$$

where $R^2 = x^2 + y^2 + z^2 - 2hx - 2ky - 2lz + h^2 + k^2 + l^2$,

$$R'^2 = x^2 + y^2 + z^2 - 2h'x - 2k'y - 2l'z + h'^2 + k'^2 + l'^2.$$

Call $h^2 + k^2 + l^2 = r^2, \quad h'^2 + k'^2 + l'^2 = t^2, \quad hh' + kk' + ll' = s.$

Then $\frac{1}{RR'}$, expanded under the form of a converging series (x, y, z being for a moment regarded as constants), will be of the form $\frac{1}{rt}$ multiplied by a rational function of $\frac{1}{r}, \frac{h}{r^2}, \frac{k}{r^2}, \frac{l}{r^2}$ and of $\frac{1}{t}, \frac{h'}{t^2}, \frac{k'}{t^2}, \frac{l'}{t^2}$ when the two points are external, and (more simply) of h, k, l and of h', k', l' when they are both internal. I, we know, must turn out to be a complete function of r, s, t, and, when expressed in the form of a series derived from the above expansion, will be the sum of terms of the form $r^i . s^j . t^k$, where it is obvious that i and j must both be negative when the "pair-point" is exterior, both positive when it is interior to the shell, and one positive and one negative in the remaining case.

Now we have identically

$$\left(h\frac{d}{dk} - k\frac{d}{dh}\right) r = 0,$$

$$\left(h'\frac{d}{dk'} - k'\frac{d}{dh'}\right) t = 0,$$

and

$$\left\{\left(h\frac{d}{dk} - k\frac{d}{dh}\right) + \left(h'\frac{d}{dk'} - k'\frac{d}{dh'}\right)\right\} s = 0.$$

Hence with respect to I as operand we have

$$\left(h\frac{d}{dk} - k\frac{d}{dh}\right) + \left(h'\frac{d}{dk'} - k'\frac{d}{dh'}\right) = 0.$$

Operate on this identity with

$$\left(h\frac{d}{dk} - k\frac{d}{dh}\right) - \left(h'\frac{d}{dk'} - k'\frac{d}{dh'}\right),$$

and we obtain

$$\left(h\frac{d}{dk} - k\frac{d}{dh}\right)^2 - \left(h\frac{d}{dh} + k\frac{d}{dk}\right) = \left(h'\frac{d}{dk'} - k'\frac{d}{dh'}\right)^2 - \left(h'\frac{d}{dh'} + k'\frac{d}{dk'}\right);$$

and there will be two other equations of like form. Adding all these together, changing all the signs, and remembering that in regard to I as operand

$$\left(\frac{d}{dh}\right)^2 + \left(\frac{d}{dk}\right)^2 = -\left(\frac{d}{dl}\right)^2,$$

$$\left(\frac{d}{dh'}\right)^2 + \left(\frac{d}{dk'}\right)^2 = -\left(\frac{d}{dl'}\right)^2,$$

we obtain

$$\left(h\frac{d}{dh} + k\frac{d}{dk} + l\frac{d}{dl}\right)^2 + 2\left(h\frac{d}{dh} + k\frac{d}{dk} + l\frac{d}{dl}\right)$$

$$= \left(h'\frac{d}{dh'} + k'\frac{d}{dk'} + l'\frac{d}{dl'}\right)^2 + 2\left(h'\frac{d}{dh'} + k'\frac{d}{dk'} + l'\frac{d}{dl'}\right).$$

In this formula $\left(h\frac{d}{dh}+k\frac{d}{dk}+l\frac{d}{dl}\right)^2$ stands for its algebraical value

$$h^2\left(\frac{d}{dh}\right)^2+2hk\frac{d}{dh}\frac{d}{dk}+\dots;$$

but if we write $$\left\{\left(h\frac{d}{dh}+k\frac{d}{dk}+l\frac{d}{dl}\right)*\right\}^2$$

to denote the operation twice repeated, then

$$\left(h\frac{d}{dh}+k\frac{d}{dk}+l\frac{d}{dl}\right)^2 = \left\{\left(h\frac{d}{dh}+k\frac{d}{dk}+l\frac{d}{dl}\right)*\right\}^2-\left(h\frac{d}{dh}+k\frac{d}{dk}+l\frac{d}{dl}\right),$$

and so for the like expressions with the accented letters. The formula thus is

$$\left\{\left(h\frac{d}{dh}+k\frac{d}{dk}+l\frac{d}{dl}\right)*\right\}^2+\left(h\frac{d}{dh}+k\frac{d}{dk}+l\frac{d}{dl}\right) = \left\{\left(h'\frac{d}{dh'}+k'\frac{d}{dk'}+l'\frac{d}{dl'}\right)*\right\}^2+\left(h'\frac{d}{dh'}+k'\frac{d}{dk'}+l'\frac{d}{dl'}\right);$$

or say $$\{(E*)^2+E-(E'*)^2-E'\}\,I=0,$$

or simply $$(F-F')\,I=0.$$

Let now $r^is^jt^k$ be any term in I; then since

$$Er=r,\quad Es=s,\quad Et=0,$$
$$E't=t,\quad E's=s,\quad E'r=0,$$

we have $$Fr^is^jt^k=\{(i+j)^2+(i+j)\}\,r^is^jt^k,$$
$$F'r^is^jt^k=\{(k+j)^2+(k+j)\}\,r^is^jt^k,$$

and thence $$(F-F')\,r^is^jt^k=(i^2+i+2ij-k^2-k-2kj)\,r^is^jt^k.$$

Hence $\Sigma\,(i^2+i+2ij-k^2-k-2kj)\,r^is^jt^k$ must be identically zero; therefore $i-k=0$, or $i+k+2j+1=0$.

But when the two points to which the Bipotential is referred (and which I shall hereafter call the points of *prise*) are both external or both internal, i and k have the same sign; therefore $i=k$, and the integral is a function only of rs and t, or say of

$$(h^2+k^2+l^2)(h'^2+k'^2+l'^2),\quad (hh'+kk'+ll')\dagger.$$

† When the point corresponding to r is external and that corresponding to t is internal, the equation $i+k+2j+1=0$ applies, which shows that each term is of the form $\frac{1}{r}\left(\frac{t}{r}\right)^\lambda.\left(\frac{s}{rt}\right)^\mu$; that is to say, the Bipotential multiplied by r is a complete function of $\frac{t}{r}$ and the cosine of the angle which the line joining the two fixed points subtends at the centre.

Thus the desired theorem has been established by virtue of an algebraical necessity of form alone; and the proof is of course applicable to space in any number of dimensions, substituting for the sphere or spherical surface its analogue in such space, and for the reciprocal of distance the proper power necessary for the satisfaction of Laplace's equation, that is, the $(q-2)$th power of the reciprocal, where q is the number of dimensions (supposed to be greater than 2).

For the case of two dimensions, substituting the logarithm for the reciprocal, so that, for example, we are able to affirm that if each element of a circular ring be affected with a density proportional to the product of the logarithms of its distances from two fixed internal points, the mass of such ring will depend only on the product of their distances from the centre of the ring and the angle between these distances—for this case, writing $E = h\frac{d}{dh} + k\frac{d}{dk}$ and $E' = h'\frac{d}{dh'} + k'\frac{d}{dk'}$ in the equation $(F - F')\, I = 0$, $F = (E*)^2$ and $F' = (E'*)^2$; and if the two points are interior, every term in $\frac{1}{RR'}$ will be of the form $cr^i . s^j . t^k$, i and k being both positive, and we must have $i^2 + 2ij - k^2 - 2kj = 0$, and consequently $i = k$—the other solution, $i + k + 2j = 0$, being applicable to the case of one point being external and the other internal. If the points are both external there will be four sets of terms. One set will consist of the single term $A \log r \log t$; a second, of terms of the form $c \log r . r^i s^j t^k$; a third, of terms of the form $c \log t . r^i s^j t^k$; and the last set, of terms of the form $cr^i s^j t^k$: and it is easy to see that

$$F(\log r \log t) = 0, \quad F'(\log r \log t) = 0,$$

$$(F - F') \log r . r^i s^j t^k = \{(i+j)^2 \log r - (k+j)^2 \log r + 2(i-k)\}\, r^i s^j t^k,$$

and consequently $i = k$ for the second and third sets; as regards the fourth set, $i = k$ for the same reason as in the case of three dimensions. Hence

$$I = A \log r \log t + \log r\phi(rt, s) + \log t\psi(rt, s) + \omega(rt, s);$$

and as r and t are interchangeable, we must have $\phi = \psi$, and consequently

$$I - A \log r \log t = F(rt, s);$$

so that not now the mass of the ring, but the difference between it and the mass due to the density at the centre is invariable when rt and s are given.

For greater simplicity, and as bearing more immediately on the theory of spherical harmonics, I have hitherto regarded the points of the pair-point at which the "bipotential" is reckoned either both internal or both external. The results established in these two cases are not complementary, but mutually equivalent to each other, and to the theorem that the integral along a spherical surface of the product of two spherical harmonics of unequal degrees is zero. In the third case, where one point is internal and the other external, then for the case of space of three dimensions the equation between i and k will have to be satisfied, not by $i=k$ but by $i+k+2j+1=0$, as previously stated in a footnote; and for two dimensions the equation would have to be satisfied, not by $i=k$ but by $i+k+2j=0$.

The advantage of the method here indicated is that it is immediately applicable to space of any number of dimensions. I shall now proceed to show that it leads at once to the determination of the values of the surface-integral of the product of any two given types of spherical harmonics of equal degrees, and *mutatis mutandis* to the corresponding surface-integral in space of any order.

To prove that the degrees must be equal or else the integral will vanish, we have combined the two Laplacian operators applicable to R and R' respectively; to find the value of the integral in a series, I use either of these operators to act singly on the result acquired by their use in combination. For greater simplicity suppose the point-pair to be internal; then, calling

$$a+b+c=\mu=\alpha+\beta+\gamma,$$

the problem to be solved is in effect that of finding the value of the numerical coefficient of $h^a k^b l^c . h'^\alpha k'^\beta l'^\gamma$ in the integral I. Now we know by what precedes that the value, say I_μ, of that part of I which is of the μth order in the two sets h, k, l; h', k', l' respectively is a rational function of rt and s; and we may accordingly write

$$I_\mu = As^\mu + Bs^{\mu-2} . \theta + Cs^{\mu-4} . \theta^2 + \ldots,$$

where $\quad s = hh' + kk' + ll',$

and $\quad \theta = (h^2+k^2+l^2)(h'^2+k'^2+l'^2) = \rho\rho'.$

When $A, B, C, \ldots$ are determined, the problem is virtually solved, and we shall then know the coefficient of

$$h^a k^b l^c . h'^\alpha k'^\beta l'^\gamma$$

by mere binomial expansions.

Since $\quad \left(\frac{d}{dh}\right)^2 + \left(\frac{d}{dk}\right)^2 + \left(\frac{d}{dl}\right)^2,$

say ∇, operating on the whole of I gives the result zero, the same must obviously be true for each part I_μ.

Now ∇s^p is obviously equal to

$$(p^2-p)\,\rho' s^{p-2},$$

and $$\nabla\theta^q = 2q\,(2q+1)\,\rho'\theta^{q-1};$$

for
$$\frac{d^2}{dh^2}(h^2+k^2+l^2)^q = \Sigma\,\frac{d}{dh}\{2qh\,(h^2+k^2+l^2)^{q-1}\}$$
$$= \Sigma\,\{2q\rho^{q-1} + 4q\,(q-1)\,\rho\,.\,\rho^{q-2}\}$$
$$= \{6q + 4q\,(q-1)\}\,\rho^{q-1}.$$

Also $\nabla s^p \rho^q - \rho^q \nabla s^p - s^p \nabla \rho^q = 2pq\Sigma \left(\frac{d}{dh} s \frac{d}{dh} \rho\right) s^{p-1} . \rho^{q-1} = 4pqs^p . \rho^{q-1}$.

Therefore

$$\nabla s^p \theta^q = (p^2 - p) s^{p-2} \rho^q \rho'^{q+1} + (4pq + 4q^2 + 2q) s^p . \rho^{q-1} . \rho'^q,$$

or

$$\nabla s^{\mu-2j} \theta^j = (\mu - 2j)(\mu - 2j - 1) s^{p-2} (\rho\rho')^j \rho' + 2j(2\mu - 2j + 1) s^p (\rho\rho')^{j-1} . \rho'.$$

Hence, equating to zero the coefficients of the different combinations of ρ, ρ', s, we easily obtain by writing for j successively 0, 1, 2, 3, ...,

$$\mu(\mu-1) A + 2(2\mu - 1) B = 0,$$

$$(\mu - 2)(\mu - 3) B + 4(2\mu - 3) C = 0,$$

$$(\mu - 4)(\mu - 5) C + 6(2\mu - 5) D = 0,$$

....

$$B = -\frac{\mu(\mu-1)}{2(2\mu-1)} A,$$

$$C = \frac{\mu(\mu-1)(\mu-2)(\mu-3)}{2.4(2\mu-1)(2\mu-3)} A,$$

$$D = -\frac{\mu(\mu-1)(\mu-2)(\mu-3)(\mu-4)(\mu-5)}{2.4.6(2\mu-1)(2\mu-3)(2\mu-5)} A,$$

....

To find the value of A, I observe that when $k = 0$, $l = 0$, $k' = 0$, $l' = 0$, and $h' = h$, I_μ becomes

$$(A + B + C + \ldots) h^{2\mu}.$$

But in that case, taking the radius of the sphere equal to unity, I becomes the surface-integral of $\frac{1}{1 - 2hx + h^2}$, and is equal to

$$2\pi \int_{-1}^{1} \frac{dx}{1 - 2hx + h^2} = \frac{2\pi}{h} \log\left(\frac{1+h}{1-h}\right) = 4\pi\left(1 + \frac{h^2}{3} + \ldots \frac{h^{2\mu}}{2\mu+1} + \ldots\right).$$

Therefore

$$A + B + C + \ldots = \frac{4\pi}{2\mu+1},$$

or

$$S_\mu A = \frac{4\pi}{2\mu+1},$$

where

$$S_\mu = 1 - \frac{\mu(\mu-1)}{2(2\mu-1)} + \frac{\mu(\mu-1)(\mu-2)(\mu-3)}{2.4.(2\mu-1)(2\mu-3)} - \frac{\mu(\mu-1)(\mu-2)(\mu-3)(\mu-4)(\mu-5)}{2.4.6.(2\mu-1)(2\mu-3)(2\mu-5)} \ldots.$$

This series admits of summation. And I find

$$S_1 = 1, \quad S_2 = \frac{2}{3}, \quad S_3 = \frac{2}{5}, \quad S_4 = \frac{8}{35}, \quad S_5 = \frac{8}{63}, \quad S_6 = \frac{16}{3.7.11},$$

$$S_7 = \frac{16}{3.11.13}, \quad S_8 = \frac{128}{3.11.13.15}, \quad S_9 = \frac{120}{5.11.13.17},$$

$$S_{10} = \frac{256}{11.13.17.19} \ldots,$$

and in general

$$S_{2m}=\frac{2.4.6...(2m)}{(2m+1)(2m+3)(2m+5)...(4m-1)}$$

and $$S_{2m+1}=\frac{2m+1}{4m+1}S_{2m};$$

that is to say, S_μ is the reciprocal of the coefficient of h^μ in $(1-2h)^{-\frac{1}{2}}$.

Hence the values of $A, B, C \ldots$ in I_μ are completely determined, and I_μ, and consequently the value of the complete integral of

$$\iint dS\left\{\left(\frac{d}{dx}\right)^a\left(\frac{d}{dy}\right)^b\left(\frac{d}{dz}\right)^c\cdot\frac{1}{r}\right\}\left\{\left(\frac{d}{dx}\right)^\alpha\left(\frac{d}{dy}\right)^\beta\left(\frac{d}{dz}\right)^\gamma\cdot\frac{1}{r}\right\},$$

is known for all values of a, b, c; α, β, γ—and this by a method which is applicable step by step to any number of variables, provided in place of $\frac{1}{r}$ we write $\frac{1}{r^{n-2}}$ when n exceeds 2, and $\log r$ when $n=2$, and consider dS to be the element of what in n dimensions corresponds to a spherical surface in three-dimensional space.

The method employed, of first using two Laplacian operators in combination to determine one property of the form under investigation and then a single one of them to act on the form thus partially determined, reminds one very much of the method for obtaining invariants of given orders from their two general partial differential equations. Combined, these two equations express the law of isobarism; then, assuming the isobarism, a single one of the two serves to determine the special values of the coefficients. The analogy between that process and the one here employed seems to me to be exact, although the subject-matter is so very unlike in the two problems—and is the more interesting on that very account.

The bipotential in the case where the two points of *prise* are both internal being known under the form $F\left(\frac{rr'}{a^2}, \cos\alpha\right)$, where a is the radius of the sphere, its value for the case where these points are both external, and for the case where they are one internal and the other external, may be assigned without any further calculation as follows:—

1. Suppose r greater than the radius of the sphere, but r' less. We know *à priori* from the result previously obtained (and stated in a footnote), that the bipotential for this case is of the form $\frac{1}{r}G\left(\frac{r'}{r}, \cos\alpha\right)$. Now in place of r, r' substitute $a, \frac{ar'}{r}$; then the bipotential becomes $\frac{1}{a}G\left(\frac{r'}{r}, \cos\alpha\right)$.

But we may by an easily justifiable application of the principle of continuity now regard a $\left(\text{as well as } \frac{ar'}{r}\right)$ as the distance of an internal point from the centre. Hence we have

$$\frac{1}{a} G\left(\frac{r'}{r}, \cos\alpha\right) = F\left(\frac{r'}{r}, \cos\alpha\right),$$

or
$$\frac{1}{r} G\left(\frac{r'}{r}, \cos\alpha\right) = \frac{a}{r} F\left(\frac{r'}{r}, \cos\alpha\right),$$

which is the value of the bipotential of a spherical surface cut by the line of *prise*, r being the distance of the external point of *prise* from the centre.

2. Suppose r and r' to be each greater than the radius, and $r > r'$; we know the bipotential is of the form $H\left(\frac{rr'}{a^2}, \cos\alpha\right)$. For r, r' substitute respectively a, $\frac{rr'}{a}$. Then we may regard the case as that of an exterior and interior point of *prise*, and consequently from the last case we have

$$H\left(\frac{rr'}{a^2}, \cos\alpha\right) = \frac{a^2}{rr'} F\left(\frac{a^2}{rr'}, \cos\alpha\right).$$

If we compare the two expressions

$$F\left(\frac{rr'}{a^2}, \cos\alpha\right) \text{ and } \frac{a^2}{rr'} F\left(\frac{a^2}{rr'}, \cos\alpha\right)$$

respectively applicable to two internal and two external points of *prise*, it will easily be seen that it leads to the following theorem. Let there be two concentric spheres, and let any two radii cut the first and second surfaces in the points P, Q and P', Q' respectively; then the bipotential of the first surface with respect to P', Q' as the points of *prise*, is to the bipotential of the second surface with respect to P, Q as the points of *prise* in the ratio of the squares of the radii of the two surfaces to each other.

This is a theorem of precisely the same kind as Ivory's for the comparison of the attractions (or, if we please, the potentials) of two confocal ellipsoids in the particular case when they become two concentric spheres, and may be verified by precisely the same geometrical method. For we have only to divide the two spherical surfaces into corresponding elements m, m' by radii drawn in all directions to meet the two surfaces, and it is evident that we shall have the distances mP' and $m'P$ equal, as also mQ' and $m'Q$. And, moreover, the ratio of any two corresponding elements m, m' will be as the square of the radii, which evidently establishes the theorem in question. It may further be noticed that the relations between the bipotentials in the three several cases considered may be deduced from the fact that each such radical as

$$\frac{1}{\sqrt{(1-2hx-2ky-2lz+h^2+k^2+l^2)}},$$

where $h^2+k^2+l^2$ is greater than unity, may be put under the form

$$\frac{1}{\sqrt{(h^2+k^2+l^2)}}\;\frac{1}{\sqrt{(1-2h_1x-2k_1y-2l_1z+h_1{}^2+k_1{}^2+l_1{}^2)}},$$

where h_1, k_1, l_1 and h, k, l are the coordinates of two points the inverses (or electrical images) of each other in regard to the origin, and consequently $h_1{}^2+k_1{}^2+l_1{}^2$ less than unity. This is going to the heart of the matter. So I may observe that if we would go to the root of the relation between positive- and negative-degreed solid spherical harmonics, the more logical mode of proceeding is not (as is usually done) to infer this by a lengthy *à posteriori* process, but immediately from the fact that since

$$\frac{1}{\sqrt{\{(x^2+y^2+z^2)-2(hx+ky+lz)+(h^2+k^2+l^2)\}}}$$

is nullified by the operator

$$\left(\frac{d}{dx}\right)^2+\left(\frac{d}{dy}\right)^2+\left(\frac{d}{dz}\right)^2,$$

so also must the same operator nullify the radical

$$\frac{1}{\sqrt{\{1-2(hx+ky+lz)+(h^2+k^2+l^2)(x^2+y^2+z^2)\}}}.$$

Before proceeding further, I ought to observe that I_μ in the above series for the bipotential may easily be shown to be $\frac{4\pi}{2\mu+1}$ multiplied into the coefficient of t^μ in the expansion of $\frac{1}{\sqrt{(1-2st+\theta t^2)}}$*; or, in other words,

* s, it will be remembered, is $hh'+kk'+ll'$, and θ is the product $(h^2+k^2+l^2)(h'^2+k'^2+l'^2)$. The statement in the text follows as a consequence from the fact that $(1-2st+\theta t^2)^{-\frac{1}{2}}$ obeys Laplace's law, and, when expanded according to powers of t, is of the form found for I_μ, and must consequently be identical with it to a factor *près*, that factor being a function of μ, whose value is easily found by making $h=h'$ and k, l; k', l' each zero. In like manner it may be shown that in higher space of n dimensions the corresponding value of I_μ is a function of μ multiplied by the coefficient of t^μ in $\{1-2t\Sigma hh'+\Sigma(h^2)\,\Sigma(h'^2)\,t^2\}^{1-\frac{1}{2}n}$; and [writing m for μ] I find† that this function, say $\phi(m, n)$, (as will be shown in a sequel to this paper) is always a rational function in m, containing in the denominator, when n is odd, one factor of the form $2m+j$, all the others being of the form $m+i$—and when n is even, factors all of the form $m+i$. Whatever the form of these linear factors had been for even numbers, we could see *à priori* that the Bipotential for space of even dimensions could contain only algebraic and inverse circular or logarithmic functions. But as regards the case of space of odd dimensions, the fact of there being no factors except of the form $m+i$, $2m+j$, is prepotent in determining the form of the result. For space of two dimensions the Bipotential does not appear readily to yield to summation in finite

[† See below, p. 51.]

if the distances from the centre of a spherical surface of two points *in the interior* be r, t, and the angle which the line joining them subtends at the centre be ω, then [for a sphere of radius c] the value of the bipotential of the surface at this point-pair is the elliptic integral

$$\int_0^{\frac{\sqrt{(rt)}}{c}} \frac{4\pi\, dx}{\sqrt{(1-2x^2\cos\omega+x^4)}},$$

which I take leave to call the Cardinal Theorem of Spherical Harmonics; for it is the theorem from which spring all the properties relating to the "surface-integral" of the product of any two rational forms of Laplace's coefficients.

Since every spherical harmonic of integral degree is a linear function of the differential derivatives of $(x^2+y^2+z^2)^{-\frac{1}{2}}$, the whole theory of the diplo-spherical-harmonic-surface integral is contained in the annexed equation,

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terms. Thus at one blow the theory of spherical harmonics has been extended to "globoidal" harmonics in general; and the chief cases of statical distribution of electricity heretofore solved may be regarded as virtually solved *mutatis mutandis* for space of any number of dimensions, of course with the proviso that the law of attraction (in consonance with the hypothetical principle of force-emanation to which the English school of physicists seem to be returning) is always to be supposed to vary as the $(i-1)$th power of the distance in space of i dimensions.

The actual expression for $\phi(m, n)$ when n is 3 we know is $\frac{4\pi}{2m+1}$. In general when n is any other odd number, I find that its value is

$$\frac{2(2\pi)^{\frac{n-1}{2}}}{(2m+n-2)(m+n-3)(m+n-4)\ldots\left(m+\frac{n-1}{2}\right)}.$$

As this expression may be split up into partial fractions, it is obvious that the value of the Bipotential may be expressed by means of the sum of integrals of the form

$$\int_\infty^1 \frac{u^j\, du}{\{\sqrt{(u^2+Au+B)}\}^{n-2}},$$

and one of the form

$$\int_\infty^1 \frac{du}{\{\sqrt{(u^4+Au^2+B)}\}^{n-2}};$$

so that it involves no transcendents of a higher order than an ordinary elliptic function. I think also that it follows from the limits to the value of j that the other integrals are mere algebraical functions. The less interesting case when n is an even number (being very much pressed for time and within twenty-four hours of steaming back to Baltimore) I have not taken the trouble to work out in detail.

The determination of the Bipotential constitutes in itself a vast accession to the theory of definite integrals, and promises to be fruitful in yielding whole new families of such when subjected to the usual processes performed under the sign of integration. But does the theory stop here? The success of my method for the Bipotential depends solely upon the discovery that, as regards internal points of *prise*, it may be regarded as a function of only two variables, rr' and $\cos\omega$. Now a Tripotential will obviously at first sight be a function of not more than six variables, viz. the three quantities r, r', r'' and the cosines of the angles between them; but it becomes a question whether this number also may not be reduced to be less than six, themselves simple functions of the six parts of a tetrahedron; and so for a multipotential of any order the question arises, Is it a function of $\frac{1}{2}m(m+1)$ quantities or of a smaller number? and if so, of what number of what variables?

which springs immediately from the expression found above for the bi-potential of a spherical surface at two internal points (slightly modified by taking $-h, -k, -l; -h', -k', -l'$ for the coordinates of the points) by means of the simple and familiar principle that any differential derivative with respect to x, y, z of a function of x, y, z is identical with what the corresponding derivative with respect to h, k, l of the like function of $x+h$, $y+k$, $z+l$ becomes when h, k, l are made to vanish.

Let U stand for $u^4 - 2u^2\Sigma hh' + \Sigma h^2 . \Sigma h'^2$, and let

$$V(h, k, l;\, h', k', l') = \int_\infty^1 \Phi\left(\frac{d}{dh}, \frac{d}{dk}, \frac{d}{dl}\right)\Psi\left(\frac{d}{dh'}, \frac{d}{dk'}, \frac{d}{dl'}\right)\frac{du}{\sqrt{U}},$$

where Φ and Ψ are forms of function which denote series, whether finite or infinite, containing only positive integer powers of the variables. Then, if $\rho = (x^2+y^2+z^2)^{-\frac{1}{2}}$ and dS is the element of a spherical surface of unit radius, the complete integral

$$\iint dS\left\{\Phi\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right)\rho\Psi\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right)\rho\right\} = 4\pi V(0, 0, 0;\, 0, 0, 0).$$

When Φ and Ψ are homogeneous forms of function, each of the degree i, if we write

$$T = 1 - 2\Sigma hh' + \Sigma h^2 . \Sigma h'^2,$$

and make

$$\Omega\,(h, k, l;\, h', k', l') = \Phi\left(\frac{d}{dh}, \frac{d}{dk}, \frac{d}{dl}\right)\Psi\left(\frac{d}{dh'}, \frac{d}{dk'}, \frac{d}{dl'}\right)\frac{1}{\sqrt{T}},$$

the value of the corresponding harmonic surface integral becomes

$$\frac{4\pi}{2m+1}\,\Omega\,(0, 0, 0;\, 0, 0, 0).$$

I am not aware that a rule for finding such integral so simple in form and of such absolute generality in operation as the one above has been given before; the interesting rule furnished by Professor Clerk Maxwell, *Electricity and Magnetism* (vol. I. p. 170), assumes that Φ and Ψ have been each reduced to the form of the product of linear functions of $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$—a reduction which cannot practically be effected, as it involves the solution of systems of equations of a high order—not, however, so high as might at first sight be inferred from Professor Maxwell's statement that, for the case of i factors, it depends on the solution of a system of $2i$ equations of the ith degree, as the equations referred to (evidently those obtained by the use of the method of indeterminate coefficients in its crude form) would be of a special character: thus, for example, when $i = 2$, the order of the system of the four quadratic equations sinks down from $4 . 2^3$ or 32 (its value in the general case) to be only 3, as will presently be seen.

The method of poles for representing spherico-harmonics, devised or developed by Professor Maxwell, really amounts to neither more nor less than the choice of an apt canonical form for a ternary quantic, subject to the condition that the sum of the squares of its variables (here differential operators) is zero; and I am quite at a loss to understand how it can at all assist "in making the conception of the general spherical harmonic of an integral degree perfectly definite," or what want of definiteness apart from the use of this canonical form can be said to exist in the subject.

Since $\left(\frac{d}{dx}\right)^2 + \left(\frac{d}{dy}\right)^2 + \left(\frac{d}{dz}\right)^2$ retains its form when any orthogonal linear substitutions are impressed on x, y, z, we recognize *à priori* that a harmonic distribution on the surface of a sphere is invariantive in the sense that it bears no intrinsic relation to the particular set of axes which may happen to be used to express the value of the harmonic at each point of the surface; and the great merit, it seems to me, of Professor Maxwell's beautiful conception of harmonic poles is that it puts this fact in evidence: for it is easy to see at a glance, from the use of successive linear operators, that the harmonic at any variable point on the surface for any given degree (n) will depend in an absolutely determinate manner (save as to an arbitrary constant factor) on the cosines of the arcs joining it with n arbitrarily assumed fixed points on the sphere, and of the arcs joining those n points with one another (being in fact a symmetrical function of each of the two sets of cosines), so that intrinsic poles are substituted for extrinsic Cartesian axes. I am a little surprised that this distinguished writer should not have noticed that there is always one, and only one, *real* system of poles appertaining to any given harmonic, and that to find this system it is not necessary, as he has stated, to employ a system of n equations each of the order $2n$, but one single equation of that order. For calling $\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}$ by the names ξ, η, ζ, then any given harmonic of the nth degree may be reduced by the use of mere linear equations to the form $(\xi, \eta, \zeta)^n \frac{1}{r}$, and the problem to be solved in order to find its poles is the purely algebraical one of converting the quantic

$$(\xi, \eta, \zeta)^n + \Lambda (\xi^2 + \eta^2 + \zeta^2),$$

where Λ is a quantic of the order $(n-2)$, into a product of linear factors. Now this again is merely the problem of finding a pencil of rays that shall pass through the intersections of the curve $(\xi, \eta, \zeta)^n$ with the curve $(\xi^2 + \eta^2 + \zeta^2)$; that is to say, any dispersal of the $2n$ intersections into n sets of two each will give a system of n polar factors in Professor Maxwell's problem. We have therefore only to find the values of $\xi : \eta : \zeta$ in the two simultaneous equations $(\xi, \eta, \zeta)^n = 0$, $\xi^2 + \eta^2 + \zeta^2 = 0$, and this leads to a resolving equation of the $2n$th order. From the form of the second equation we see that the values $x : y : z$ are *all imaginary*; consequently there will be one, and but one, system of real rays, that is, real polars corresponding to the distribution of the $2n$ roots of the resolving equation into n *conjugate pairs*. The remaining systems (there are in all $1 . 3 . 5 \ldots (2n-1)$ of them) will each contain imaginary elements, so that all or some of the poles become imaginary.

In the case of $n = 2$, the problem becomes the familiar one of finding the principal axes of a cone of the second order; and instead of employing a biquadratic resolvent we make the discriminant of $(\xi, \eta, \zeta)^2 + (\xi^2 + \eta^2 + \zeta^2)$ vanish, which of course only requires the solution of a cubic equation; but as subsequently (when the pair is to be divided into its elements) a new quadratic surd is introduced, we are virtually solving a biquadratic, in accordance with the general rule that, to find the poles of a spherical harmonic of the degree n, it is necessary to solve an equation of the degree $2n$.

To put the coping-stone to Professor Clerk Maxwell's method of poles, I think it would be desirable to find an intrinsic definition of spherical harmonics to correspond with their representation referred to intrinsic axes: I mean we ought to be able to dispense with the Laplacian operator altogether, and to define a Harmonic with sole reference to some algebraical or geometrical (but certainly not physical) condition which it satisfies in regard to its poles. With all possible respect for Professor Maxwell's great ability, I must own that to deduce purely analytical properties of spherical harmonics, as he has done, from "Green's theorem" and the "principle of potential energy" (*Electricity and Magnetism*, vol. I. p. 168), seems to me a proceeding at variance with sound method, and of the same kind and as reasonable as if one should set about to deduce the binomial theorem from the law of virtual velocities, or make the rule for the extraction of the square root flow as a consequence from Archimedes' law of floating bodies.

POSTSCRIPT. NOTE ON SPHERICAL HARMONICS.

The value of $\phi(m, n)$ is stated inaccurately in the long footnote at pp. [46, 47]. If

$$\Omega_i = \frac{(2\pi)^i}{1.3.5\ldots(2i-1)}$$

and

$$R = \sqrt{(1 - 2\Sigma hh'.t^2 + \Sigma h^2.\Sigma h'^2.t^4)}$$

then I find

$$\phi(m, 2i+1) = \frac{(2i-1)\,\Omega_i}{2m+2i-1};$$

and accordingly the Bipotential in space of $2i+1$ dimensions is

$$\int_1^0 \frac{\Omega_i dt^{2i-1}}{R^{2i-1}}.$$

Also I find that in space of $2i+2$ dimensions the prospherical Bipotential is

$$\frac{2\pi^i}{1.2.3\ldots i}\int_1^0 \frac{dt^i}{(1-2\Sigma hh'.t+\Sigma h^2.\Sigma h'^2.t^2)^i}.$$

The above results may be extended to general quadric surfaces and prosurfaces. Thus, for example, if an indefinitely thin ellipsoidal shell be contained between two concentric surfaces, the equation to one of which is $G(x, y, z) = 1$, where G is a general quadric, and if the squared density at x, y, z is the reciprocal of

$$G(x-h,\ y-k,\ z-l).G(x-h',\ y-k',\ z-l'),$$

then the mass of the shell divided by its volume is

$$\int_1^0 \frac{dt}{\sqrt{(1-At^2+Bt^4)}},$$

where

$$A = \Sigma\left(h\frac{d}{dx}\right).\Sigma\left(h'\frac{d}{dx}\right)G(x, y, z),$$

and

$$B = G(h, k, l).G(h', k', l').$$

It is further noticeable that if F and G are contravariantive forms, each numerator of the fractions expressing the differential derivatives of $\frac{1}{\sqrt{\{G(x, y, z)\}}}$ is nullified by the operator

$$F\left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right);$$

and conversely, every rational integer function of x, y, z so nullifiable is a linear function of such numerators. And so in general the Theory of Spherical and Prospherical Harmonics merges in a theory of Conicoidal and Proconicoidal Harmonics.