35.

NOTE ON AN EQUATION IN FINITE DIFFERENCES.

[*Philosophical Magazine*, VIII. (1879), pp. 120, 121.]

I GAVE* a great many years ago in this Magazine the integral of the equation in differences

$$u_x = \frac{u_{x-1}}{x} + u_{x-2},$$

which I obtained by observing that the equation could be solved by supposing each u of an odd order to be equal to the u of the order immediately superior, and also by supposing it to be equal to the u of the order immediately inferior. The upshot of the investigation expressed in the simplest language was to furnish two particular integrals of which one gives rise to the series

$$u_0 = 1, \quad u_1 = 1, \quad u_2 = \tfrac{1}{2}, \quad u_3 = \tfrac{1}{2}, \quad u_4 = \frac{1\,.\,3}{2\,.\,4}, \quad u_5 = \frac{1\,.\,3}{2\,.\,4} \ldots,$$

the other

$$u_0 = 1, \quad u_1 = 2, \quad u_2 = 2, \quad u_3 = \frac{2\,.\,4}{1\,.\,3}, \quad u_4 = \frac{2\,.\,4}{1\,.\,3}, \quad u_5 = \frac{2\,.\,4\,.\,6}{1\,.\,3\,.\,5} \ldots.$$

See also Boole's *Finite Differences*, 2nd Edition (edited by Mr Moulton), p. 235.

Now let ϕ, a function of any letter t, be the generating function of u_x. Then, since

$$xu_x - (x-2)\,u_{x-2} - u_{x-1} - 2u_{x-2} = 0,$$

we shall have

$$(1-t^2)\frac{d\phi}{dt} + (-1-2t)\,\phi = C;$$

and integrating we find

$$(1-t)^{\frac{3}{2}}(1+t)^{\frac{1}{2}}\,\phi = C\int dt\,\sqrt{\left(\frac{1-t}{1+t}\right)},$$

or

$$\phi = C'\,\frac{1+t}{(1-t^2)^{\frac{3}{2}}} + C\,\frac{\sin^{-1}t + \sqrt{(1-t^2)}}{(1-t)^{\frac{3}{2}}(1+t)^{\frac{1}{2}}}.$$

[* Vol. II. of this Reprint, p. 690.]

$\dfrac{1+t}{(1-t^2)^{\frac{3}{2}}}$ we see at a glance gives the values of u_x corresponding to the first particular integral; and since the two first terms of the function multiplied by C are $1+2t$, it follows that this function is the generatrix of the second particular integral—in other words, that

$$\frac{\sin^{-1}t+\sqrt{(1-t^2)}}{(1-t)^{\frac{3}{2}}(1+t)^{\frac{1}{2}}}=1+2t+2t^2+\frac{2\,.\,4}{1\,.\,3}t^3+\frac{2\,.\,4}{1\,.\,3}t^4+\frac{2\,.\,4\,.\,6}{1\,.\,3\,.\,5}t^5+\dots.$$

Hence

$$\frac{t\sin^{-1}t+\sqrt{(1-t^2)}}{(1-t^2)^{\frac{3}{2}}}=\frac{1}{1+t}\left\{t\left(\frac{\sin^{-1}t+\sqrt{(1-t^2)}}{(1-t)^{\frac{3}{2}}(1+t)^{\frac{1}{2}}}\right)+1\right\}$$

$$=1+\frac{2}{1}t^2+\frac{2\,.\,4}{1\,.\,3}t^4+\frac{2\,.\,4\,.\,6}{1\,.\,3\,.\,5}t^6+\dots;$$

and integrating

$$\frac{\sin^{-1}t}{\sqrt{(1-t^2)}}=t+\frac{2}{1}\cdot\frac{t^3}{3}+\frac{2\,.\,4}{1\,.\,3}\cdot\frac{t^5}{5}+\frac{2\,.\,4\,.\,6}{1\,.\,3\,.\,5}\frac{t^7}{7}+\dots.$$

Thus we have the remarkable identity

$$\left(1+\frac{1}{2}\tau+\frac{1\,.\,3}{2\,.\,4}\tau^2+\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\tau^3\dots\right)$$

$$\times\left(1+\frac{1}{2}\frac{\tau}{3}+\frac{1\,.\,3}{2\,.\,4}\frac{\tau^2}{5}+\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\frac{\tau^3}{7}\dots\right)$$

$$=1+\frac{2}{1}\frac{\tau}{3}+\frac{2\,.\,4}{1\,.\,3}\frac{\tau^2}{5}+\frac{2\,.\,4\,.\,6}{1\,.\,3\,.\,5}\frac{\tau^3}{7}\dots.$$

I do not recollect ever having met with these remarkable series before I discovered them by the preceding method; but on showing them to Dr Story of this University, he ascertained that they had been stated not long ago by Mr Glaisher in a paper in the *Mathematical Messenger*, and made the foundation there of various summations for calculating π; but where Mr Glaisher found these series, which are not given in the ordinary books on the Calculus, or (if new) how he lighted upon them, he has not stated, and it is desirable that he should do so.