43.

ON THE EXACT RELATION WHICH RESULTANTS AND DISCRIMINANTS BEAR TO THE PRODUCT OF DIFFERENCES OF ROOTS OF EQUATIONS.

[*Messenger of Mathematics*, IX. (1880), pp. 164—166.]

First, for Resultants.

Let there be two rational integral functions in x of the degrees r, s respectively; and, for greater simplicity, let the coefficients of x^r, x^s in these functions be each made equal to unity. Call ρ the roots of the one, σ of the other; and denote the product of the differences found by subtracting each σ from each ρ by $D_{\rho,\sigma}$.

Also, by the resultant $R_{r,s}$ understand that irreducible rational integral function of the coefficients, vanishing when the functions have a root in common, in which the highest power of the last coefficient of the "s" equations enters with the positive sign.

We must then have $R_{r,s} = \mu D_{\rho,\sigma}$; and it only remains to determine μ as a function of r, s.

To do this let the r function become x^r, and the s function $x^s + 1$.

For greater distinctness, suppose $r = 4$, $s = 2$.

Then, obviously, $R_{r,s}$ becomes the dialytic resultant of

$$\begin{array}{cccccc}
x^5 & & & & & \\
 & x^4 & & & & \\
x^5 & & +x^3 & & & \\
 & x^4 & & +x^2 & & \\
 & & x^3 & & +x & \\
 & & & x^2 & & +1
\end{array}$$

which is equal to 1.

And in like manner for all values of r, s,

$$R_{r,s} = 1.$$

Again, $$D_{\rho,\sigma} = \{0 - (-1)^{\frac{1}{s}}\}^{rs} = (-)^{rs+r}.$$

Hence μ, which is a function of r, s exclusively, $= (-)^{rs+r}$.

Next, for Discriminants.

By the discriminant of fx of the order n, and where, for greater simplicity, the coefficient of x^n is supposed to be unity, I mean the resultant of fx and $f'x$; or, which is the same thing, of $\frac{df(x, 1)}{dx}$ and $\frac{df(x, 1)}{d1}$, when the term in which the highest power of the last coefficient in fx appears is made positive. Let this be called R_n, and the product of the squared differences of the roots Z_n; we have then $R_n = \mu Z_n$, where μ is a function of n to be determined. To find it let us take $fx = x^n - 1$.

R_n is then the resultant of nx^{n-1}, $-ny^{n-1}$, that is, is equal to

$$(-)^{n-1} n^{2n-2}.$$

Again,
$$\left.\begin{array}{l} Z_n = (1-\rho) \quad (1-\rho^2) \quad \ldots (1-\rho^{n-1}) \\ \quad\quad\; (\rho-\rho^2) \quad (\rho-\rho^3) \quad \ldots (\rho-1) \\ \quad\quad\; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \quad\quad\; \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \quad\quad\; (\rho^{n-1}-1)(\rho^{n-2}-\rho) \ldots (\rho^{n-1}-\rho^{n-2}) \end{array}\right\} \div (-)^{\frac{1}{2}(n \cdot n-1)},$$

ρ representing $\cos\frac{2\pi}{n} + i \sin\frac{2\pi}{n}$.

Hence
$$Z_n = n^n . (-)^{\frac{1}{2}n(n-1)} . \{\rho^{\frac{1}{2}n(n-1)}\}^{n-1}$$
$$= (-)^{\theta} n^n,$$

where
$$\theta = -\tfrac{1}{2}\{n(n-1)\} + (n-1)^2 = \tfrac{1}{2}\{(n-1)(n-2)\},$$

and
$$\theta + (n-1) = \tfrac{1}{2}\{(n-1)n\}.$$

Hence $R_n = (-)^{\frac{1}{2}(n-1)n} n^{n-2} Z_n$, or $\mu = (-)^{\frac{1}{2}(n-1)n}$, which was to be found.

For ordinary algebraical investigations the determination of μ has little importance, which may account for its value being omitted in the ordinary text books; but for certain investigations concerning the numerical divisors of cyclotomic functions, with which I am occupied, I found it necessary to pay attention to the numerical part at least of this factor, and I have thought that the publication of the result might save others some unnecessary trouble.